Multivariate Birnbaum-Saunders Distribution Based on Multivariate Skew Normal Distribution

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Abstract

Birnbaum-Saunders distribution has received some attention in the statistical literature since its inception. Univariate Birnbaum-Saunders distribution has been used quite effectively in analyzing positively skewed data. Recently, bivariate and multivariate Birnbaum-Saunders distributions have been introduced in the literature. In this paper we propose a new generalization of the multivariate (p-variate) Birnbaum-Saunders distribution based on the multivariate skew normal distribution. It is observed that the proposed distribution is more flexible than the multivariate Birnbaum-Saunders distribution, and the multivariate Birnbaum-Saunders distribution can be obtained as a special case of the proposed model. We obtain the marginal, reciprocal and conditional distributions, and also discuss some other properties. The proposed p-variate distribution has total $3p + \binom{p}{2}$ parameters. We use the EM algorithm to compute the maximum likelihood estimators of the unknown parameters. One data analysis has been performed for illustrative purposes.

Keywords: Birnbaum-Saunders distribution; joint probability density function; conditional probability density function; maximum likelihood estimators; skew normal distribution; multivariate normal distribution.

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1 Introduction

Birnbaum and Saunders (1969a, 1969b) introduced a two-parameter lifetime distribution which has been used to analyze positively skewed data. The Birnbaum-Saunders (BS) distribution was derived through a monotone transform of the normal distribution. Since then a considerable amount of work has taken place on the development of the different aspects of this distribution, see for example Chang and Tang (1993, 1994), Dupis and Mills (1998), From and Li (2006), Ng *et al.* (2003, 2006), Leiva *et al.* (2008), Lemonte *et al.* (2007, 2008) and the references cited therein.

A random variable T is said to have a two-parameter BS distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, if it has the cumulative distribution function (CDF) as follows:

$$F_T(t; \alpha, \beta) = \Phi(a(t; \alpha, \beta)); \quad t > 0,$$

where $\Phi(\cdot)$ is the CDF of a standard normal distribution function and

$$a(t; \alpha, \beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right). \tag{1}$$

Kundu *et al.* (2010) introduced a bivariate Birnbaum-Saunders (BBS) distribution by using the same monotone transformation. A bivariate random vector $(T_1, T_2)^T$ is said to have a BBS distribution, if the joint CDF can be written as follows;

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2\left[a(t_1; \alpha_1, \beta_1), a(t_2; \alpha_2, \beta_2); \rho)\right]; \quad t_1 > 0, t_2 > 0,$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $-1 < \rho < 1$, and $\Phi_2(u, v; \rho)$ is the CDF of a standard normal random vector $(Z_1, Z_2)^T$ with correlation coefficient ρ. The authors discussed different properties of the BBS distribution and also addressed inferential issues. In a subsequent paper the authors, Kundu *et al.* (2013), proposed a multivariate Birnbaum-Saunders (MBS) model and discussed different properties.

Several generalizations of the BS distribution have been proposed by different authors, see for example Diaz-Garcia and Leiva (2005), Leiva *et al.* (2008), Gomez *et al.* (2009) and

Vilca *et al.* (2011). Vilca and Leiva (2006) introduced a new univariate BS distribution based on skew normal distribution. The skew normal distribution has been proposed by Azzalini (1985). It is more flexible than the normal distribution, and normal distribution can be obtained as a special case. Moreover, the skew normal distribution can have heavier tail than the normal distribution. The proposed generalized multivariate Birnbaum-Saunders (GMBS) distribution is obtained by taking the same monotone transform as the BS distribution, by replacing the multivariate normal distribution with the multivariate skew normal distribution.

The random variable T is said to have a generalized Birnbaum-Saunders (GBS) distribution based on skew normal distribution, if it has the PDF

$$f_T(t) = 2\phi(a(t;\alpha,\beta))\Phi(\lambda a(t;\alpha,\beta))A(t;\alpha,\beta); \quad t > 0.$$

Here $\alpha > 0$, $\beta > 0$, $a(t;\alpha,\beta)$ is same as defined in (1), and

$$A(t;\alpha,\beta) = \frac{d}{dt}a(t;\alpha,\beta) = \frac{1}{2\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right\} = \frac{t+\beta}{2\alpha\sqrt{\beta}t^{3/2}}.$$

It is observed that the GBS model is quite a flexible model, and the BS distribution can be obtained as a special case. Moreover, it can have a heavy tail depending on the parameter λ. Some recent development on GBS distribution can be obtained in Leiva *et al.* (2008) and Vilca *et al.* (2011).

The aim of this paper is to introduce a multivariate Birnbaum-Saunders distribution based on multivariate skew normal distribution using the same monotone transformation as the multivariate BS distribution with replacing the multivariate normal distribution with the multivariate skew normal distribution. Multivariate skew normal distribution was introduced by Azzalini and Dalla-Valle (1996), and it is a more flexible distribution than the multivariate normal distribution.

Different properties of the generalized p-variate Birnbaum-Saunders (GMBS_p) distribution based on the multivariate skew normal (MSN) distribution have been established. It is observed that the multivariate BS distribution can be obtained as a special case of the proposed GMBS distribution. Marginal and conditional distributions are also provided. It is quite simple to generate samples from a GMBS_p distribution, hence simulation experiments can be performed very easily.

The proposed GMBS_p model has $3p + \binom{p}{2}$ unknown parameters. The maximum likelihood estimators (MLEs) cannot be obtained in explicit forms, as expected. They can be obtained by solving $3p+\binom{p}{2}$ non-linear equations simultaneously. We use the EM algorithm to compute the MLEs of the unknown parameters, which involves solving one p dimensional non-linear equation at each 'M'step of the EM algorithm. Therefore, the implementation of the EM algorithm becomes quite straight forward. The observed Fisher information matrix can be used to construct the asymptotic confidence intervals of the unknown parameters. Finally we address some testing of hypotheses issues also. We perform the analysis of one data set for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries. GMBS_p is introduced and different properties are discussed in Section 3. The use of EM algorithm is provided in Section 4. The analysis of one data set has been presented in Section 5 and finally conclude the paper in Section 6.

2 Preliminaries

2.1 Multivariate BS Distribution

Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^p$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_p)^T$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)^T$, with $\alpha_i > 0$, $\beta_i > 0$, for $i = 1, \ldots, p$. Let $\boldsymbol{\Gamma}$ be a $p \times p$ positive definite correlation matrix. The random vector $\boldsymbol{T} = (T_1, \cdots, T_p)^T$ is said to have a p-variate BS distribution with parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma})$, if it has the PDF

$$P(\boldsymbol{T} \le \boldsymbol{t}) = P(T_1 \le t_1, \ldots, T_p \le t_p) = \Phi_p(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}); \boldsymbol{\Gamma}), \tag{2}$$

where $\boldsymbol{t} = (t_1, \ldots, t_p)^T$, $t_1 > 0, \ldots, t_p > 0$, and

$$\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = (a(t_1; \alpha_1, \beta_1), \ldots, a(t_p; \alpha_p, \beta_p))^T.$$

Here $\boldsymbol{u} = (u_1, \ldots, u_p)^T$ and $\Phi_p(\boldsymbol{u}; \boldsymbol{\Gamma})$ denotes the joint CDF of a standard normal vector $\boldsymbol{Z} = (Z_1, \ldots, Z_p)^T$, with mean zero and correlation matrix $\boldsymbol{\Gamma}$.

The joint PDF of $\boldsymbol{T} = (T_1, \ldots, T_p)^T$ can be obtained from (2) as

$$f_{\boldsymbol{T}}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}) = \phi_p(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}); \boldsymbol{\Gamma}) \prod_{i=1}^{p} A(t_i; \alpha_i, \beta_i), \tag{3}$$

for $t_1 > 0, \ldots, t_p > 0$, and for $\boldsymbol{u} = (u_1, \ldots, u_p)^T$

$$\phi_p(\boldsymbol{u}; \boldsymbol{\Gamma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Gamma}|^{1/2}} e^{-\frac{1}{2}\boldsymbol{u}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{u}},$$

is the PDF of a standard normal vector with mean zero and correlation matrix $\boldsymbol{\Gamma}$. From now on, the p-variate BS distribution with joint PDF (3) will be denoted by $\mathrm{BS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma})$. We will be further using the following notation

$$\phi_p(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Gamma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Gamma}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Gamma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}.$$

2.2 Multivariate Skew Normal Distribution

The multivariate skew normal distribution was introduced by Azzalini and Dalla Valle (1996). A p-dimensional random vector $\boldsymbol{X} = (X_1, \ldots, X_p)^T$ is said to have a multivariate skew normal (SN_p) distribution with parameter $\boldsymbol{\Gamma}$, a $p \times p$ positive definite correlation matrix, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)^T \in \mathbb{R}^p$, if $\boldsymbol{X}$ has the PDF

$$f_{SN_p}(\boldsymbol{x}; \boldsymbol{\lambda}, \boldsymbol{\Gamma}) = 2\phi_p(\boldsymbol{x}; \boldsymbol{\Gamma})\Phi(\boldsymbol{\lambda}^T\boldsymbol{x}); \quad \boldsymbol{x} \in \mathbb{R}^p. \tag{4}$$

A multivariate skew normal distribution with PDF (4) will be denoted by $\mathrm{SN}_p(\boldsymbol{\Gamma}, \boldsymbol{\lambda})$. In the special case when $\boldsymbol{\lambda} = \mathbf{0}$, the PDF (4) reduces to $\phi_p(\boldsymbol{x}, \boldsymbol{\Gamma})$, that is $\mathrm{SN}_p(\boldsymbol{\Gamma}, \mathbf{0}) = \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Gamma})$. Let us use the following notations.

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{bmatrix}. \tag{5}$$

Here the vectors $\boldsymbol{X}_1$ and $\boldsymbol{\lambda}_1$ are of the order q and the matrix $\boldsymbol{\Gamma}_{11}$ is of the order $p \times p$. Rest of the quantities are defined so that they are compatible. The following lemma provides the marginal of $\boldsymbol{X}$.

Lemma 1:

$$\boldsymbol{X}_1 \sim \mathrm{SN}_q\left(\boldsymbol{\Gamma}_{11}, \frac{\boldsymbol{\lambda}_1 + \boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{\Gamma}_{12}\boldsymbol{\lambda}_2}{\sqrt{1 + \boldsymbol{\lambda}_2^T \boldsymbol{\Gamma}_{22.1}\boldsymbol{\lambda}_2}}\right) \quad \text{and} \quad \boldsymbol{X}_2 \sim \mathrm{SN}_{p-q}\left(\boldsymbol{\Gamma}_{22}, \frac{\boldsymbol{\lambda}_2 + \boldsymbol{\Gamma}_{22}^{-1}\boldsymbol{\Gamma}_{21}\boldsymbol{\lambda}_1}{\sqrt{1 + \boldsymbol{\lambda}_1^T \boldsymbol{\Gamma}_{11.2}\boldsymbol{\lambda}_1}}\right).$$

Here $\boldsymbol{\Gamma}_{22.1} = \boldsymbol{\Gamma}_{22} - \boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{\Gamma}_{12}$ and $\boldsymbol{\Gamma}_{11.2} = \boldsymbol{\Gamma}_{11} - \boldsymbol{\Gamma}_{12}\boldsymbol{\Gamma}_{22}^{-1}\boldsymbol{\Gamma}_{21}$.

The following definition will be useful to provide the conditional distribution of $\boldsymbol{X}_2$ given $\boldsymbol{X}_1$ or vice versa.

A p-dimensional random vector $\boldsymbol{X} = (X_1, \cdots, X_p)^T$ is said to have a multivariate extended skew normal distribution with parameters $\boldsymbol{\Gamma} \in \mathbb{R}^{p \times p}$ ($\boldsymbol{\Gamma}$ is a positive definite correlation matrix), $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p)^T \in \mathbb{R}^p$ and $\tau \in \mathbb{R}$, denoted by $\mathrm{ESN}_p(\boldsymbol{\Gamma}, \boldsymbol{\lambda}, \tau)$, if its PDF is

$$f_{ESN_p}(\boldsymbol{x}; \boldsymbol{\Gamma}, \boldsymbol{\lambda}, \tau) = \frac{\phi_p(\boldsymbol{x}; \boldsymbol{\Gamma})\, \Phi\left(\boldsymbol{\lambda}^T \boldsymbol{x} + \tau\right)}{\Phi\left(\tau / \sqrt{1 + \boldsymbol{\lambda}^T \boldsymbol{\Gamma} \boldsymbol{\lambda}}\right)}, \qquad \boldsymbol{x} \in \mathbb{R}^p, \tag{6}$$

see for example Arnold and Beaver (2000).

Lemma 2: Suppose $\boldsymbol{X}$ follows $\mathrm{SN}_p(\boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and $\boldsymbol{X}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\lambda}$ are partitioned as in (5). Then for $\boldsymbol{x}_1 \in \mathbb{R}^p$,

(a) $\quad [diag(\boldsymbol{\Gamma}_{22.1})]^{-\frac{1}{2}} \left(\boldsymbol{X}_2 - \boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{x}_1\right) | (\boldsymbol{X}_1 = \boldsymbol{x}_1) \sim$

$$\mathrm{ESN}_{p-q}\left([diag(\boldsymbol{\Gamma}_{22.1})]^{-\frac{1}{2}} \boldsymbol{\Gamma}_{22.1} [diag(\boldsymbol{\Gamma}_{22.1})]^{-\frac{1}{2}}, [diag(\boldsymbol{\Gamma}_{22.1})]^{\frac{1}{2}} \boldsymbol{\lambda}_2, \left(\boldsymbol{\lambda}_1^T + \boldsymbol{\lambda}_2^T \boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\right) \boldsymbol{x}_1\right).$$

(b) The PDF of $\mathbf{X}_2$ given $\boldsymbol{X}_1 = \boldsymbol{x}_1$, is

$$f_{\boldsymbol{X}_2|(\boldsymbol{X}_1=\boldsymbol{x}_1)}(\boldsymbol{x}_2) = \frac{\phi_{p-q}\left(\boldsymbol{x}_2; \boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{x}_1, \boldsymbol{\Gamma}_{22.1}\right)\Phi\left(\boldsymbol{\lambda}_2^T\boldsymbol{x}_2 + \boldsymbol{\lambda}_1^T\boldsymbol{x}_1\right)}{\Phi\left(\left(\boldsymbol{\lambda}_1^T + \boldsymbol{\lambda}_2^T\boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\right)\boldsymbol{x}_1/\sqrt{1+\boldsymbol{\lambda}_2^T\boldsymbol{\Gamma}_{22.1}\boldsymbol{\lambda}_2}\right)},$$

where $\phi_{p-q}\left(\cdot; \mathbf{\Gamma_{21}\Gamma_{11}^{-1}x_1}, \boldsymbol{\Gamma}_{22.1}\right)$ is the PDF of $N_{p-q}\left(\mathbf{\Gamma_{21}\Gamma_{11}^{-1}x_1}, \boldsymbol{\Gamma}_{22.1}\right)$.

Proof: The above results can be obtained directly, see Azzalini and Capitanio (1999).

Lemma 3: If $\boldsymbol{X} \sim \mathrm{SN}_p(\boldsymbol{\Gamma}, \boldsymbol{\lambda})$, then

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y} + \boldsymbol{\delta}H, \tag{7}$$

where $\boldsymbol{Y} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T)$, and $H \sim \mathrm{HN}(0,1)$, with

$$\boldsymbol{\delta} = \frac{\boldsymbol{\Gamma}\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^T\boldsymbol{\Gamma}\boldsymbol{\lambda}}}.$$

Here HN(0,1) denotes the half normal distribution with parameters 0 and 1 respectively, $H = |Z|$, where $Z \sim \mathrm{N}(0,1)$, and the PDF of H is as follows:

$$f_H(h) = \sqrt{\frac{2}{\pi}}e^{-h^2/2}; \quad h > 0, \tag{8}$$

see for example Azzalini and Dalla-Valle (1996).

Lemma 4: If $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$ are defined above, then there is a one to one correspondence between $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$, if $\boldsymbol{\Gamma}$ is non-singular.

Proof: By simple algebraic calculations, it can be seen that

$$\boldsymbol{\delta} = \frac{\boldsymbol{\Gamma}\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^T\boldsymbol{\Gamma}\boldsymbol{\lambda}}} \quad \Leftrightarrow \quad \boldsymbol{\lambda} = \frac{\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}}{\sqrt{1-\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}}},$$

therefore, the result follows. ∎

3 Generalized Multivariate BS Distribution Based on Multivariate SN distribution

In this section, we define generalized multivariate BS distribution based on multivariate SN distribution, and discuss its different properties.

3.1 Definition

Definition 1: A p-variate random vector $\boldsymbol{T} = (T_1, \ldots, T_p)^T$ is said to have a generalized multivariate BS distribution based on multivariate SN distribution with parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\lambda}$ if the CDF of $\boldsymbol{T}$ is

$$F_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda}) = F_{SN_p}(\boldsymbol{a}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta});\boldsymbol{\Gamma},\boldsymbol{\lambda}); \quad \boldsymbol{t} \in \mathbb{R}_+^p, \tag{9}$$

here the parameters $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda}$ are same as defined before and $F_{SN_p}(\cdot;\boldsymbol{\Gamma},\boldsymbol{\lambda})$ denotes the CDF of $\text{SN}_p(\boldsymbol{\Gamma},\boldsymbol{\lambda})$. The PDF of $\boldsymbol{T} = (T_1, \ldots, T_p)^T$ becomes

$$f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda}) = f_{SN_p}(\boldsymbol{a}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta});\boldsymbol{\Gamma},\boldsymbol{\lambda}) \prod_{i=1}^{p} A(t_i;\alpha_i,\beta_i); \quad \boldsymbol{t} \in \mathbb{R}_+^p. \tag{10}$$

From now it will be denoted by $\text{GMBS}_p(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda})$. It is immediate that when $\boldsymbol{\lambda} = \boldsymbol{0}$, (10) coincides with the PDF of the multivariate BS distribution as defined by Kundu *et al.* (2013). Clearly, because of the presence of the parameter $\boldsymbol{\lambda}$, it is more flexible than the multivariate BS distribution.

In particular when $p = 2$, the PDF of $\boldsymbol{T} = (T_1, T_2)^T$, has the following form;

$$\begin{aligned} f_{\boldsymbol{T}}(t_1,t_2) &= 2\phi_2\left(\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}}\right), \frac{1}{\alpha_2}\left(\sqrt{\frac{t_2}{\beta_2}} - \sqrt{\frac{\beta_2}{t_2}}\right); \rho\right) \\ &\quad \times\Phi\left(\frac{\lambda_1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}}\right) + \frac{\lambda_2}{\alpha_2}\left(\sqrt{\frac{t_2}{\beta_2}} - \sqrt{\frac{\beta_2}{t_2}}\right)\right) \\ &\quad \times\frac{1}{2\alpha_1\beta_1}\left\{\left(\frac{\beta_1}{t_1}\right)^{1/2} + \left(\frac{\beta_1}{t_1}\right)^{3/2}\right\} \times \frac{1}{2\alpha_2\beta_2}\left\{\left(\frac{\beta_2}{t_2}\right)^{1/2} + \left(\frac{\beta_2}{t_2}\right)^{3/2}\right\}, \end{aligned}$$

where

$$\phi_2(u, v; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2 + v^2 - 2\rho uv)\right\}.$$

We provide the surface plot of the joint PDF of GMBS_2 for different parameter values in Figure 1. It is clear that it can take variety of shapes, depending on the parameter values.

3.2 Stochastic Representation and Simulation Algorithm

If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, then it has the following stochastic representation:

$$\boldsymbol{T} \stackrel{d}{=} \left(\frac{\beta_1}{4}\left[\alpha_1 X_1 + \sqrt{(\alpha_1 X_1)^2 + 4}\right]^2, \ldots, \frac{\beta_p}{4}\left[\alpha_p X_p + \sqrt{(\alpha_p X_p)^2 + 4}\right]^2\right)^T,$$

where $\boldsymbol{X} = (X_1, \ldots, X_p)^T \sim \text{SN}_p(\boldsymbol{\Gamma}, \boldsymbol{\lambda})$. Therefore, using Lemma 3, we immediately obtain;

$$\boldsymbol{T} \stackrel{d}{=} \left(\frac{\beta_1}{4}\left[\alpha_1(Y_1 + \delta_1 H) + \sqrt{(\alpha_1(Y_1 + \delta_1 H)^2 + 4}\right]^2, \ldots, \frac{\beta_p}{4}\left[\alpha_p(Y_p + \delta H) + \sqrt{(\alpha_p(Y_p + \delta H))^2 + 4}\right]^2\right)^T, \tag{11}$$

Here $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_p)^T$, $\boldsymbol{Y} = (Y_1, \ldots, Y_p)^T$, H are same defined in Lemma 3. Therefore, the following steps can be adopted to generate $\boldsymbol{T} = (T_1, \ldots, T_p)^T$ from $\text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$.

Step 1: Make a Cholesky decomposition of $\boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T = \boldsymbol{A}\boldsymbol{A}^T$ (say).

Step 2: Generate $p + 1$ independent standard normal random variables say, $U, U_1, \ldots, U_p$.

Step 3: Compute $\boldsymbol{Y} = (Y_1, \ldots, Y_P)^T = \boldsymbol{A}(U_1, \ldots, U_p)^T$.

Step 4: Make the following transformation:

$$T_i = \frac{\beta_i}{4}\left[\alpha_i(Y_i + \delta_i|U|) + \sqrt{(\alpha_i(Y_i + \delta_i|U|)^2 + 4}\right]^2, \quad \text{for} \quad i = 1, \ldots, p. \tag{12}$$

Then, $\boldsymbol{T} = (T_1, \ldots, T_p)^T$ has the required $\text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$ distribution.

3.3 Marginal, Conditional and Reciprocals Distributions

In this section we provide the marginal and conditional distributions of $\text{GMBS}_p(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda})$ distribution.

Theorem 1: If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda})$, and let $\boldsymbol{T},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda}$ be partitioned as follows

$$\boldsymbol{T}=\begin{pmatrix}\boldsymbol{T}_1\\ \boldsymbol{T}_2\end{pmatrix},\quad \boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\alpha}_1\\ \boldsymbol{\alpha}_2\end{pmatrix},\quad \boldsymbol{\beta}=\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix},\quad \boldsymbol{\lambda}=\begin{pmatrix}\boldsymbol{\lambda}_1\\ \boldsymbol{\lambda}_2\end{pmatrix},\quad \boldsymbol{\Gamma}=\begin{bmatrix}\boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12}\\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22}\end{bmatrix}, \tag{13}$$

where $\boldsymbol{T}_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\boldsymbol{\lambda}_1$ are all $q\times 1$ vectors, $\boldsymbol{\Gamma}_{11}$ is a $q\times q$ matrix and the remaining elements are suitably defined. We have the following results.

(a) $\boldsymbol{T}_1 \sim \text{GMBS}_q\left(\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\boldsymbol{\Gamma}_{11},\frac{\boldsymbol{\lambda}_1+\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{\Gamma}_{12}\boldsymbol{\lambda}_2}{\sqrt{1+\boldsymbol{\lambda}_2^T\boldsymbol{\Gamma}_{22.1}\boldsymbol{\lambda}_2}}\right)$

(b) $\boldsymbol{T}_2 \sim \text{GMBS}_{p-q}\left(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\boldsymbol{\Gamma}_{22},\frac{\boldsymbol{\lambda}_2+\boldsymbol{\Gamma}_{22}^{-1}\boldsymbol{\Gamma}_{21}\boldsymbol{\lambda}_1}{\sqrt{1+\boldsymbol{\lambda}_1^T\boldsymbol{\Gamma}_{11.2}\boldsymbol{\lambda}_1}}\right)$

(c) For $\boldsymbol{t}=(t_1,...,t_p)^T=\left(\boldsymbol{t}_1^T,\boldsymbol{t}_2^T\right)^T \in \mathbb{R}^{+^p}$, where $\boldsymbol{t}_1\in\mathbb{R}^{+^q}$ and $\boldsymbol{t}_2\in\mathbb{R}^{+^{p-q}}$, and $\boldsymbol{a}\left(\boldsymbol{t}:\boldsymbol{\alpha},\boldsymbol{\beta}\right)=\left(\boldsymbol{a}_1^T\left(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1\right),\boldsymbol{a}_2^T\left(\boldsymbol{t}_2;\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2\right)\right)^T$, we have the conditional PDF of $\boldsymbol{T}_2$ given $\boldsymbol{T}_1=\boldsymbol{t}_1$, as

$$\begin{aligned} f_{\boldsymbol{T}_2|\boldsymbol{T}_1=\boldsymbol{t}_1}(\boldsymbol{t}_2) &= \frac{\phi_{p-q}\left(\boldsymbol{t}_2;\boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\mathbf{a}_1\left(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1\right),\boldsymbol{\Gamma}_{22.1}\right)\Phi\left(\boldsymbol{\lambda}_2^T\mathbf{a}_2\left(\boldsymbol{t}_2;\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2\right)+\boldsymbol{\lambda}_1^T\mathbf{a}_1\left(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1\right)\right)}{\Phi\left(\frac{\left(\boldsymbol{\lambda}_1^T+\boldsymbol{\lambda}_2^T\boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\right)\mathbf{a}_1\left(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1\right)}{\sqrt{1+\boldsymbol{\lambda}_2^T\boldsymbol{\Gamma}_{22.1}\boldsymbol{\lambda}_2}}\right)} \\ &\quad \times \prod_{i=q+1}^{p} A\left(t_i;\alpha_i,\beta_i\right) \end{aligned}$$

(d) The random variables $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are independent if and only if $\boldsymbol{\Gamma}_{12}=\boldsymbol{\Gamma}_{21}=\mathbf{0}$, and $\boldsymbol{\lambda}_1=\mathbf{0}$ or $\boldsymbol{\lambda}_2=\mathbf{0}$.

Proof: (a) It can be obtained by letting $t_{q+1}\to\infty,\ldots,t_p\to\infty$, in (9) and using part (i) of Lemma 1. The proof (b) follows along the same line.

To prove (c), observe that

$$f_{\boldsymbol{T}_2|\boldsymbol{T}_1=\boldsymbol{t}_1}(\boldsymbol{t}_2)=\frac{f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda})}{f_{\boldsymbol{T}_1}(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\boldsymbol{\Gamma}_{11},\boldsymbol{\lambda}_1)}=\frac{f_{SN_p}(\boldsymbol{a}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\beta});\boldsymbol{\Gamma},\boldsymbol{\lambda})}{f_{SN_q}(\boldsymbol{a}_1(\boldsymbol{t}_1;\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1);\boldsymbol{\Gamma}_{11},\boldsymbol{\lambda}_1)}\times\prod_{i=q+1}^{p}A\left(t_i;\alpha_i,\beta_i\right).$$

Now the result follows using Lemma 2.

Proof of (d) follows from the result (c). ∎

THEOREM 2: If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and let $\boldsymbol{T}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda}$ be partitioned as in (13). We further use the following notation. If the vector $\boldsymbol{a} = (a_1, \ldots, a_p)^T$, then $\boldsymbol{a}^{-1} = (a_1^{-1}, \ldots, a_p^{-1})^T$. We have the following results.

(a) $\begin{pmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2^{-1} \end{pmatrix} \sim \text{GMBS}_p \left\{ \boldsymbol{\alpha}, \begin{pmatrix} \boldsymbol{\beta}_1^{-1} \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{bmatrix} \boldsymbol{\Gamma}_{11} & -\boldsymbol{\Gamma}_{12} \\ -\boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{bmatrix}, \begin{pmatrix} \boldsymbol{\lambda}_1 \\ -\boldsymbol{\lambda}_2 \end{pmatrix} \right\},$

(b) $\begin{pmatrix} \boldsymbol{T}_1^{-1} \\ \boldsymbol{T}_2 \end{pmatrix} \sim \text{GMBS}_p \left\{ \boldsymbol{\alpha}, \begin{pmatrix} \boldsymbol{\beta}_1^{-1} \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{bmatrix} \boldsymbol{\Gamma}_{11} & -\boldsymbol{\Gamma}_{12} \\ -\boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{bmatrix}, \begin{pmatrix} -\boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} \right\},$

(c) $\boldsymbol{T}^{-1} \sim \text{GMBS}_p \left(\boldsymbol{\alpha}, \boldsymbol{\beta}^{-1}, \boldsymbol{\Gamma}, -\boldsymbol{\lambda} \right)$

PROOF: (a) Let us denote

$$\widetilde{\boldsymbol{\Gamma}} = \begin{pmatrix} \boldsymbol{\Gamma}_{11} & -\boldsymbol{\Gamma}_{12} \\ -\boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}^{-1} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}.$$

We have, see Rao (1973),

$$|\boldsymbol{\Gamma}| = |\widetilde{\boldsymbol{\Gamma}}| \quad \text{and} \quad \widetilde{\boldsymbol{\Gamma}}^{-1} = \begin{pmatrix} \boldsymbol{A}_{11} & -\boldsymbol{A}_{12} \\ -\boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}.$$

Consider $S_{q+1} = T_{q+1}^{-1}, \ldots, S_p = T_p^{-1}$. We use the following notation; $\boldsymbol{S}_2 = (S_{q+1}, \ldots, S_p)^T$. To compute the joint PDF of $(\boldsymbol{T}_1^T, \boldsymbol{S}_2^T) = (T_1, \ldots, T_q, S_{q+1}, \ldots, S_p)$ first observe the following facts:

$$a(t^{-1}; \alpha, \beta) = -a(t; \alpha, \beta^{-1}) \tag{14}$$

and

$$\phi_p(u_1, \ldots, u_q, -u_{q+1}, \ldots, -u_p; \boldsymbol{\Gamma}) = \phi_p(u_1, \ldots, u_q, u_{q+1}, \ldots, u_p; \widetilde{\boldsymbol{\Gamma}}). \tag{15}$$

Therefore, the joint PDF of $(\boldsymbol{T}_1, \boldsymbol{S}_2)$ obtained from (10) as

$$f_{(\boldsymbol{T}_1, \boldsymbol{S}_2)}(\boldsymbol{t}_1, \boldsymbol{s}_2; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda}) = f_{\boldsymbol{T}}(\boldsymbol{t}_1, \boldsymbol{s}_2^{-1}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})|\boldsymbol{J}|.$$

Since $|\boldsymbol{J}| = 1$, using the PDF of $\boldsymbol{T}$ from (10) and the relations (14) and (15) the result follows. The proofs of (b) and (c) can be obtained along the same line. ∎

THEOREM 3: If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and H is same as defined in (11), then the conditional PDF of $\boldsymbol{T}$ given $H = h > 0$, is

$$f_{\boldsymbol{T}|H=h}\left(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda}\right) = \phi_p\left(\mathbf{a}\left(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}\right); h\boldsymbol{\delta}, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T\right) \times \prod_{i=1}^{p} A\left(t_i; \alpha_i, \beta_i\right), \quad \text{for} \quad \boldsymbol{t} = (t_1, ..., t_p)^T \in \mathbb{R}_+^p.$$

PROOF: From (11) it is immediate that

$$\{\boldsymbol{T}|H = h\} \stackrel{d}{=} \left(\frac{\beta_1}{4}\left[\alpha_1 V_1 + \sqrt{(\alpha_1 V_1)^2 + 4}\right]^2, \ldots, \frac{\beta_p}{4}\left[\alpha_p V_p + \sqrt{(\alpha_p V_p)^2 + 4}\right]^2\right)^T, \quad (16)$$

where $\boldsymbol{V} = (V_1, \ldots, V_p)^T \sim \text{N}_p(h\boldsymbol{\delta}, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T)$. Using one to one correspondence between $\boldsymbol{T}$ and $\boldsymbol{V}$, and using (2), it follows that

$$P(\boldsymbol{T} \leq \boldsymbol{t}|H = h) = P(T_1 \leq t_1, \ldots, T_p \leq t_p|H = h) = \Phi_p(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}); \boldsymbol{\delta}h, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T). \quad (17)$$

Therefore, the result follows. ∎

THEOREM 4: If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and H is same as defined in (11). Let us define the random vector $\boldsymbol{U} = (U_1, \ldots, U_P)^T$, where

$$U_1 = \frac{1}{\alpha_1}\left(\sqrt{\frac{T_1}{\beta_1}} - \sqrt{\frac{\beta_1}{T_1}}\right), \ldots, U_p = \frac{1}{\alpha_p}\left(\sqrt{\frac{T_p}{\beta_p}} - \sqrt{\frac{\beta_p}{T_p}}\right).$$

(a) The PDF of $\boldsymbol{U}$ is

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = f_{SN_p}(\boldsymbol{u}; \boldsymbol{\lambda}, \boldsymbol{\Gamma}) \quad \text{for} \quad \boldsymbol{u} = (u_1, ..., u_p)^T \in \mathbb{R}.$$

(b) The conditional PDF of $\boldsymbol{U}$, given $H = h > 0$ is

$$f_{\boldsymbol{U}|H=h}\left(\boldsymbol{u}; \boldsymbol{\Gamma}, \boldsymbol{\lambda}\right) = \phi_p\left(\boldsymbol{u}; h\boldsymbol{\delta}, (\boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T)\right) \quad \text{for} \quad \boldsymbol{u} = (u_1, ..., u_p)^T \in \mathbb{R}.$$

PROOF: (a) It can be obtained by using the transformation. (b) It immediately follows from Theorem 3. ∎

THEOREM 5 If $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and H is same as defined in (11), then the conditional PDF of $H = h > 0$, given $\{\boldsymbol{T} = \boldsymbol{t} = (t_1, \ldots, t_p)^T\}$ is

$$\{H\,|\boldsymbol{T} = \boldsymbol{t}\} \stackrel{d}{=} U\,|(U > 0)\,,$$

where $U \sim \text{N}\big(\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})\,, 1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\big)$.

PROOF: Note that for $\boldsymbol{t} = (t_1, ..., t_p)^T \in \mathbb{R}_+^p$ and $h > 0$,

$$f_{H|\boldsymbol{T}=t}(h) = \frac{f_{\boldsymbol{T}|H=h}(\boldsymbol{t}) f_H(h)}{f_{\boldsymbol{T}}(\boldsymbol{t})} = \sqrt{\frac{2}{\pi}}\ \frac{\phi_p(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}); h\boldsymbol{\delta}, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T) e^{-h^2/2}}{f_{SN_p}(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}); \boldsymbol{\Gamma}, \boldsymbol{\delta})}.$$

Using the fact, see Rao (1973),

$$\left(\boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T\right)^{-1} = \boldsymbol{\Gamma}^{-1} + \frac{\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}}{1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}},$$

it can be seen after some simplification that

$$f_{H|\boldsymbol{T}=t}(h) = K \exp\left\{-\frac{h^2}{2(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta})} + \frac{h(\boldsymbol{a}(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta})}{(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta})}\right\},$$

where K is independent of h. Now the result follows after completing the squares. ∎

If we use the following notations $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$ and $r\,(t) = \dfrac{\phi\,(t)}{\Phi\,(t)}$, for $t \in \mathbb{R}$, then using Theorem 5, the following can be easily obtained.

$$E_{\boldsymbol{\theta}}\,(H\,|\boldsymbol{T} = \boldsymbol{t}) = \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t};\boldsymbol{\alpha}, \boldsymbol{\beta}) + \sqrt{\left(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right)}\ \ r\left(\frac{\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})}{\sqrt{\left(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right)}}\right) \tag{18}$$

$$\begin{aligned} E_{\boldsymbol{\theta}}\left(H^2\,|\boldsymbol{T} = \boldsymbol{t}\right) &= \left(\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})\right)^2 + \left(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right) + \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta}) \\ &\quad \times\sqrt{\left(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right)}\ \ r\left(\frac{\boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})}{\sqrt{\left(1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right)}}\right). \end{aligned} \tag{19}$$

The conditional PDF of H given $\boldsymbol{T} = \boldsymbol{t}$, for $\boldsymbol{t} = (t_1, ..., t_p)^T \in \mathbb{R}_+^p$, is

$$f_{H|\boldsymbol{T}=\boldsymbol{t}}\,(h) = \frac{\phi\left(h; \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})\,, 1 - \boldsymbol{\delta}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}\right)}{\Phi\left(\boldsymbol{\lambda}^T\mathbf{a}\,(\boldsymbol{t}; \boldsymbol{\alpha}, \boldsymbol{\beta})\right)}, \quad h > 0.$$

4 Inference

4.1 Estimation

In this section we consider the estimation of the unknown parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\lambda}$ based on a random sample of size n, $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n\}$, from $\text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$. We will be using the following notations;

$$\boldsymbol{t}_1^T = (t_{11}, \ldots, t_{1p}), \ldots, \boldsymbol{t}_n^T = (t_{np}, \ldots, t_{np}).$$

The log-likelihood function of the observations without the additive constant becomes

$$\begin{aligned} l(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda}) &= \sum_{i=1}^{n} \ln \phi_p(\boldsymbol{a}(\boldsymbol{t}_i; \boldsymbol{\alpha}, \boldsymbol{\beta}); \boldsymbol{\lambda}, \boldsymbol{\Gamma}) + \sum_{i=1}^{n} \ln \Phi(\boldsymbol{\lambda}^T \boldsymbol{a}(\boldsymbol{t}_i; \boldsymbol{\alpha}, \boldsymbol{\beta})) \\ &\quad + \sum_{i=1}^{n} \sum_{j=1}^{p} \ln A(t_{ij}; \alpha_j, \beta_j). \end{aligned} \tag{20}$$

The maximum likelihood estimators (MLEs) of the unknown parameters can be obtained by maximizing the log-likelihood function (20) with respect to unknown parameters. It involves solving $3p + p(p-1)/2$ non-linear equations. To avoid that we use the EM algorithm which involves maximizing a $2p$ dimensional optimization problem, at each step of the EM algorithm.

The following observations will be useful to understand the basic idea of the EM algorithm. Since $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ have a one to one to correspondence, we mainly restrict to estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\delta}$ only for the EM algorithm. Let us assume that the complete data is as follows;

$$\boldsymbol{t}_1^{(c)} = (\boldsymbol{t}_1^T, h_1)^T, \ldots, \boldsymbol{t}_n^{(c)} = (\boldsymbol{t}_n^T, h_n)^T, \tag{21}$$

where $\{\boldsymbol{t}_1^{(c)}, \ldots, \boldsymbol{t}_n^{(c)}\}$ is a random sample of size n from $(\boldsymbol{T}, H)$, where $\boldsymbol{T} \sim \text{GMBS}_p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\lambda})$, and H is same as defined in (11). We will show that based on the complete observations (21), the MLEs of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\delta}$ can be obtained by solving $2p$ dimensional optimization problem. The log-likelihood function of the complete data without the additive constant becomes

$$l_c(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\delta}) = \sum_{i=1}^{n} \ln \phi_p(\boldsymbol{a}(\boldsymbol{t}_i; \boldsymbol{\alpha}, \boldsymbol{\beta}); h_i\boldsymbol{\delta}, \boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T) + \sum_{i=1}^{n}\sum_{j=1}^{p} \ln A(t_{ij}; \alpha_j, \beta_j). \tag{22}$$

We maximize profile log-likelihood function to compute the MLEs of the unknown parameters, for the complete data set. First consider the following transformation of the data;

$$\boldsymbol{u}_1^T = (u_{11}, \ldots, u_{1p}), \ldots, \boldsymbol{u}_n^T = (u_{n1}, \ldots, u_{np}),$$

where

$$u_{ij} = \frac{1}{\alpha_j}\left(\sqrt{\frac{t_{ij}}{\beta_j}} - \sqrt{\frac{\beta_j}{t_{ij}}}\right); \quad i = 1, \ldots, n, \; j = 1, \ldots, p. \tag{23}$$

Now using Theorem 4, the log-likelihood function of the transformed data without the additive constant becomes

$$l_{ct}(\boldsymbol{\delta}, \boldsymbol{\Gamma}) = \sum_{i=1}^{n} \ln \phi_p(\boldsymbol{u}_i; h_i\boldsymbol{\delta}, (\boldsymbol{\Gamma} - \boldsymbol{\delta}\boldsymbol{\delta}^T)). \tag{24}$$

The MLEs of $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$ are as follows

$$\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\sum_{i=1}^{n} \boldsymbol{u}_i h_i}{\sum_{i=1}^{n} h_i^2} \quad \text{and} \quad \widehat{\boldsymbol{\Gamma}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{S} + \widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta})^T, \tag{25}$$

where

$$\boldsymbol{S} = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{u}_i - h_i\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta}))(\boldsymbol{u}_i - h_i\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta}))^T.$$

The MLEs of the unknown parameters can be obtained by maximizing the profile log-likelihood function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, namely

$$\begin{aligned} l_{cp}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \sum_{i=1}^{n} \ln \phi_p(\boldsymbol{a}(\boldsymbol{t}_i; \boldsymbol{\alpha}, \boldsymbol{\beta}); h_i\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \widehat{\boldsymbol{\Gamma}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\widehat{\boldsymbol{\delta}}^T(\boldsymbol{\alpha}, \boldsymbol{\beta})) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{p} \ln A(t_{ij}; \alpha_j, \beta_j). \end{aligned} \tag{26}$$

Suppose we denote the MLEs of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which can be obtained by maximizing (26) are denoted by $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$, respectively, then the MLEs of $\boldsymbol{\Gamma}$ and $\boldsymbol{\delta}$ become

$$\widehat{\boldsymbol{\Gamma}} = \widehat{\boldsymbol{\Gamma}}(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}) \quad \text{and} \quad \widehat{\boldsymbol{\delta}} = \widehat{\boldsymbol{\delta}}(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}),$$

respectively. Therefore, the MLEs of the unknown parameters can be obtained by solving $2p$ dimensional optimization problem.

Now we propose the following method to compute the MLEs of the unknown parameters of the GMBS$_p$ model. The method is mainly based on maximizing the profile log-likelihood function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where for given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the MLEs of $\boldsymbol{\Gamma}$ and $\boldsymbol{\delta}$ are performed using EM algorithm.

Algorithm

Step 1: Assume some initial estimates of $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$, say $\boldsymbol{\delta}^{(0)}$ and $\boldsymbol{\Gamma}^{(0)}$, respectively.

Step 2: Now obtain $E(H|\boldsymbol{T}=t)$ and $E(H^2|\boldsymbol{T}=t)$ from (18) and (19), respectively, by replacing $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$ with $\boldsymbol{\delta}^{(0)}$ and $\boldsymbol{\Gamma}^{(0)}$, respectively. Note that the 'pseudo log-likelihood' function of the transformed data obtained from (24) involves $E(H|\boldsymbol{T}=t)$ and $E(H^2|\boldsymbol{T}=t)$.

Step 3: Obtain $\boldsymbol{\delta}^{(1)}$ and $\boldsymbol{\Gamma}^{(1)}$ from (25) by replacing h_i and h_i^2 with $E(H|\boldsymbol{T}=\boldsymbol{t}_i)$ and $E(H^2|\boldsymbol{T}=\boldsymbol{t}_i)$

Step 4: Go back to Step 1, and continue the process until converges, and obtain $\widehat{\boldsymbol{\delta}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\widehat{\boldsymbol{\Gamma}}(\boldsymbol{\alpha},\boldsymbol{\beta})$.

Step 5: Now maximize the profile log-likelihood function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $l(\boldsymbol{\alpha},\boldsymbol{\beta},\widehat{\boldsymbol{\Gamma}}(\boldsymbol{\alpha},\boldsymbol{\beta}),\boldsymbol{\delta}(\boldsymbol{\alpha},\boldsymbol{\beta}))$ as given in (22), to compute the MLEs of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Now we discuss the asymptotic properties of the MLEs when all the parameters are unknown.

Theorem 6: If $\boldsymbol{\theta}=(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Gamma},\boldsymbol{\lambda})^T$ is the parameter vector, and $\widehat{\boldsymbol{\theta}}$ denotes the corresponding MLE, then

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \xrightarrow{d} N_m(\boldsymbol{0},\boldsymbol{I}^{-1}),$$

with $m = 3p + p(p-1)/2$ being the dimension of the vector $\boldsymbol{\theta}$. Here, $\xrightarrow{d}$ denotes the convergence in distribution while $N_m(\mathbf{0}, \boldsymbol{I}^{-1})$ denotes the m-variate normal distribution with mean vector $\mathbf{0}$, and the dispersion matrix $\boldsymbol{I}^{-1}$, with $\boldsymbol{I}$ being the Fisher information matrix.

Proof: Since GMBS_p model satisfies all the regularity conditions for the MLEs to be consistent and asymptotically normally distributed, the result follows from the known asymptotic properties of the MLEs.

4.2 Testing of Hypothesis

In this subsection we discuss the likelihood ratio tests for some testing of hypotheses problems which will of interest. We will be considering the following testing problem which might be useful in practice.

Test I: $H_0 : \boldsymbol{\lambda} = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{\lambda} \neq \mathbf{0}$.

This is an important testing problem, as it tests whether the data are coming from multivariate Birnbaum-Saunders distribution or not? Since $\boldsymbol{\lambda} = \mathbf{0} \Leftrightarrow \boldsymbol{\delta} = \mathbf{0}$, the MLEs of the unknown parameters can be obtained as follows. For a given $\boldsymbol{\beta}$, the MLEs of $\boldsymbol{\alpha}$ and $\boldsymbol{\Gamma}$ become

$$\widehat{\alpha}_j(\boldsymbol{\beta}) = \left(\frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{\frac{t_{ij}}{\beta_j}} - \sqrt{\frac{\beta_j}{t_{ij}}} \right)^2 \right)^{1/2}; \quad j = 1, \ldots, p, \tag{27}$$

and

$$\widehat{\Gamma}(\boldsymbol{\beta}) = \boldsymbol{P}(\boldsymbol{\beta}) \boldsymbol{Q}(\boldsymbol{\beta}) \boldsymbol{P}^T(\boldsymbol{\beta}); \tag{28}$$

here $\boldsymbol{P}(\boldsymbol{\beta})$ is a diagonal matrix given by $\boldsymbol{P}(\boldsymbol{\beta}) = \text{diag}\{1/\widehat{\alpha}_1(\boldsymbol{\beta}), \ldots, \widehat{\alpha}_p(\boldsymbol{\beta})\}$, and the elements $q_{jk}(\boldsymbol{\beta})$ of the matrix $\boldsymbol{Q}(\boldsymbol{\beta})$ are given by

$$q_{jk}(\boldsymbol{\beta}) = \frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{\frac{t_{ij}}{\beta_j}} - \sqrt{\frac{\beta_j}{t_{ij}}} \right) \left(\sqrt{\frac{t_{ik}}{\beta_k}} - \sqrt{\frac{\beta_k}{t_{ik}}} \right); \quad \text{for} \quad j, k = 1, \ldots, p. \tag{29}$$

Finally the MLE of $\boldsymbol{\beta}$ can be obtained by maximizing the profile log-likelihood function of $\boldsymbol{\beta}$, see Kundu et al. (2013) for details. If we denote $\widetilde{\boldsymbol{\alpha}}$, $\widetilde{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{\Gamma}}$ as the MLEs of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\Gamma}$, respectively under H_0, then under H_0, for large n,

$$-2\{l(\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\Gamma}}, \mathbf{0}) - l(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Gamma}}, \widehat{\boldsymbol{\lambda}}\} \sim \chi^2_p. \tag{30}$$

In Table 1 we present the critical values based on 5% level of significance of the likelihood ratio test (30) for different parameter values. The critical values are obtained based on 1000 replications. We have taken $p = 2$, and we denote the matrix $\boldsymbol{\Gamma} = ((\gamma_{ij}))$, for $i, j = 1, 2$, where $\gamma_{11} = \gamma_{22} = 1$, and $\gamma_{12} = \gamma_{21} = \rho$. The value of the likelihood ratio test does not depend on the scale parameter. Hence, we take $\beta_1 = \beta_2 = 1$. We have considered six different parameter sets namely (i) Set 1: $\alpha_1 = 1$, $\alpha_2 = 1$, $\rho = 0.0$, (ii) Set 2: $\alpha_1 = 1$, $\alpha_2 = 1$, $\rho = 0.5$, (iii) Set 3: $\alpha_1 = 1$, $\alpha_2 = 1$, $\rho = 0.90$, (iv) Set 4: $\alpha_1 = 2$, $\alpha_2 = 2$, $\rho = 0.0$, (v) Set 5: $\alpha_1 = 2$, $\alpha_2 = 2$, $\rho = 0.5$, (vi) Set 6: $\alpha_1 = 2$, $\alpha_2 = 2$, $\rho = 0.90$,

n	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
20	9.012	9.219	9.481	8.967	9.018	9.213
25	8.786	8.978	9.116	8.314	8.564	8.997
30	7.564	7.786	8.013	7.154	7.321	7.675
35	6.786	6.984	7.005	6.453	6.654	6.878
40	6.012	6.058	6.111	6.001	6.021	6.078

Table 1: Critical values of the test statistic (30) for different parameter values.

In Tables 2 to 4 we present the size and powers of the test $H_0 : \boldsymbol{\lambda} = \mathbf{0}$ vs. $H_1 : \boldsymbol{\lambda} \neq \mathbf{0}$, for different parameter values.

5 Real Data Analysis

In this section we present the analysis of a bivariate data set to see the effectiveness of the proposed model. The data set has been obtained from Johnson and Wichern (1999), and it

n	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = \lambda_2 = 2$	$\lambda_1 = \lambda_2 = 3$	$\lambda_1 = \lambda_2 = 4$	$\lambda_1 = \lambda_2 = 5$	$\lambda_1 = \lambda_2 = 6$
20	0.065	0.167	0.536	0.575	0.601	0.687
25	0.065	0.198	0.657	0.687	0.700	0.743
30	0.061	0.215	0.667	0.701	0.712	0.789
35	0.057	0.278	0.710	0.745	0.765	0.809
40	0.051	0.356	0.723	0.777	0.801	0.901

Table 2: Size and power of the test for parameter Set 1 for different sample sizes.

n	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = \lambda_2 = 2$	$\lambda_1 = \lambda_2 = 3$	$\lambda_1 = \lambda_2 = 4$	$\lambda_1 = \lambda_2 = 5$	$\lambda_1 = \lambda_2 = 6$
20	0.061	0.143	0.323	0.356	0.415	0.467
25	0.061	0.151	0.354	0.389	0.421	0.472
30	0.061	0.162	0.375	0.412	0.465	0.513
35	0.053	0.175	0.412	0.478	0.511	0.549
40	0.051	0.212	0.453	0.512	0.543	0.598

Table 3: Size and power of the test for parameter Set 2 for different sample sizes.

represents two different measures of stiffness of 30 different boards. The first measurement involves sending a shock wave down the board, and the second measurement is determined while vibrating the board. The data set has been presented below in Table 5.

Before progressing further, we compute the basic statistics of the data vector, and they are reported in Table 6. We present the mean (ME), standard deviation (SD), median (Q_2), first quartile (Q_1), third quartile (Q_3) for both T_1 and T_2. Histograms of T_1 and T_2 are also provided in Figure 2. From Q_1, Q_2 and Q_3, it is immediate that T_1 and T_2 are not symmetric, both T_1 and T_2 are right skewed. The histograms of T_1 and T_2 also suggest that. We perform the test of symmetry for both the marginals. We have used the distribution free test suggested by Randles *et al.* (1980). The test statistics for T_1 and T_2 are 1.71 and 1.77, and the associated p values are 0.0436 and 0.0384, respectively. Therefore, it suggests that the marginals are not from symmetric distributions.

n	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = \lambda_2 = 2$	$\lambda_1 = \lambda_2 = 3$	$\lambda_1 = \lambda_2 = 4$	$\lambda_1 = \lambda_2 = 5$	$\lambda_1 = \lambda_2 = 6$
20	0.058	0.078	0.115	0.214	0.275	0.311
25	0.057	0.101	0.154	0.254	0.298	0.323
30	0.057	0.115	0.198	0.287	0.354	0.398
35	0.050	0.165	0.221	0.301	0.375	0.401
40	0.050	0.198	0.254	0.376	0.401	0.412

Table 4: Size and power of the test for parameter Set 3 for different sample sizes.

T_1	T_2	T_1	T_2	T_1	T_2
1889	1651	2403	2048	2119	1700
1645	1627	1976	1916	1712	1712
1943	1685	2104	1820	2983	2794
1745	1600	1710	1591	2046	1907
1840	1841	1867	1685	1859	1649
1954	2149	1325	1170	1419	1371
1828	1634	1725	1594	2276	2189
1899	1614	1633	1513	2061	1867
1856	1493	1727	1412	2168	1896
1655	1675	2326	2301	1490	1382

Table 5: Two different measures of stiffness of 30 boards.

The sample correlation coefficient between T_1 and T_2 is 0.932, which is very high. To get an idea about the shape of the empirical hazard function of the marginal, we provide the scaled TTT plots of T_1 and T_2 in Figure 3. It indicates that both of them have increasing empirical hazard functions.

We want to fit the proposed GMBS_2 distribution to the above data set. First we fit the bivariate Birnbaum-Saunders distribution to the above data set, and we obtain the estimates of the unknown parameters as follows:

$$\widetilde{\alpha}_1 = 3.7011, \quad \widetilde{\beta}_1 = 53.8357, \quad \widetilde{\alpha}_2 = 2.7314, \quad \widetilde{\beta}_2 = 87.0131, \quad \widetilde{\rho} = 0.9997.$$

The associated log-likelihood value becomes -350.2114.

Variable	ME	SD	Q_2	Q_1	Q_3
T_1	1906.1	319.5	1863.0	1718.5	2053.5
T_2	1749.5	313.3	1680.0	1597.0	1881.5

Table 6: Descriptive statistics of the data vector.

To perform the EM algorithm, we have used the above values as the starting values of α_1, β_1, α_2, β_2 and ρ. Further we have used the staring values of λ_1 and λ_2 to be 0. The final estimates are as follows:

$$\widehat{\alpha}_1 = 3.4420, \quad \widehat{\beta}_1 = 65.5531, \quad \widehat{\alpha}_2 = 3.0844, \quad \widehat{\beta}_2 = 86.6142, \quad \widehat{\rho} = 0.9203,$$

$$\widehat{\lambda}_1 = -5.3737, \quad \widehat{\lambda}_2 = 5.7942.$$

The associated log-likelihood value becomes -295.8717. The 95% confidence intervals of α_1, β_1, α_2, β_2, ρ, λ_1 and λ_2 are

$$3.4420(\mp 0.7513), \;\; 65.5531(\mp 9.2391), \;\; 3.0844(\mp 0.6978), \;\; 86.6142(12.7765), \;\; 0.9203(\mp 0.1523),$$

$$-5.3737(\mp 1.6574), \quad 5.7942(\mp 1.6624),$$

respectively. We perform the following testing of hypothesis

Test: $H_0 : (\lambda_1, \lambda_2) = (0, 0)$ vs. $H_1 : (\lambda_1, \lambda_2) \neq (0, 0)$.

Based on the likelihood ratio test as suggested in Section 4.2, the $p-$value of the test statistic is less than 0.01, hence we reject the null hypothesis. The confidence intervals of λ_1 and λ_2 also suggest the same. It seems that the proposed GMBS_2 provides a better fit than the bivariate BS distribution to the above stiffness data set.

For comparison purposes we have also fitted (a) bivariate normal and (b) bivariate skew normal to this data set. We present the MLEs and the associated log-likelihood values in each case.

Bivariate Normal:

$\widehat{\mu}_1 = 1749.5332$, $\widehat{\mu}_2 = 1507.9000$, $\widehat{\sigma}_1 = 313.2514$, $\widehat{\sigma}_2 = 298.6844$, $\widehat{\rho} = 0.7886$,

log-likelihood $= -394.3062$

Bivariate skew normal:

$\widehat{\mu}_1 = 1660.0479$, $\widehat{\mu}_2 = 1591.4816$, $\widehat{\sigma}_1 = 378.3129$, $\widehat{\sigma}_2 = 316.9348$, $\widehat{\rho} = 0.8943$,

log-likelihood $= -303.6883$.

It is clear that based on the log-likelihood values, we prefer to use $GMBS_2$ model to analyze this data set.

6 Conclusions

In this paper we have proposed a new multivariate distribution based on the multivariate skew normal and multivariate Birnbaum-Saunders distribution, and we name it as the generalized multivariate Birnbaum-Saunders. The proposed distribution is more flexible than the multivariate Birnbaum-Saunders distribution, and the later can be obtained as a special case of the proposed distribution. We derive different properties of the proposed distribution, and use EM algorithm to compute the MLEs of the unknown parameters. One data set has been analyzed, and it is observed that the proposed distribution provides a better fit than the multivariate Birnbaum-Saunders distribution.

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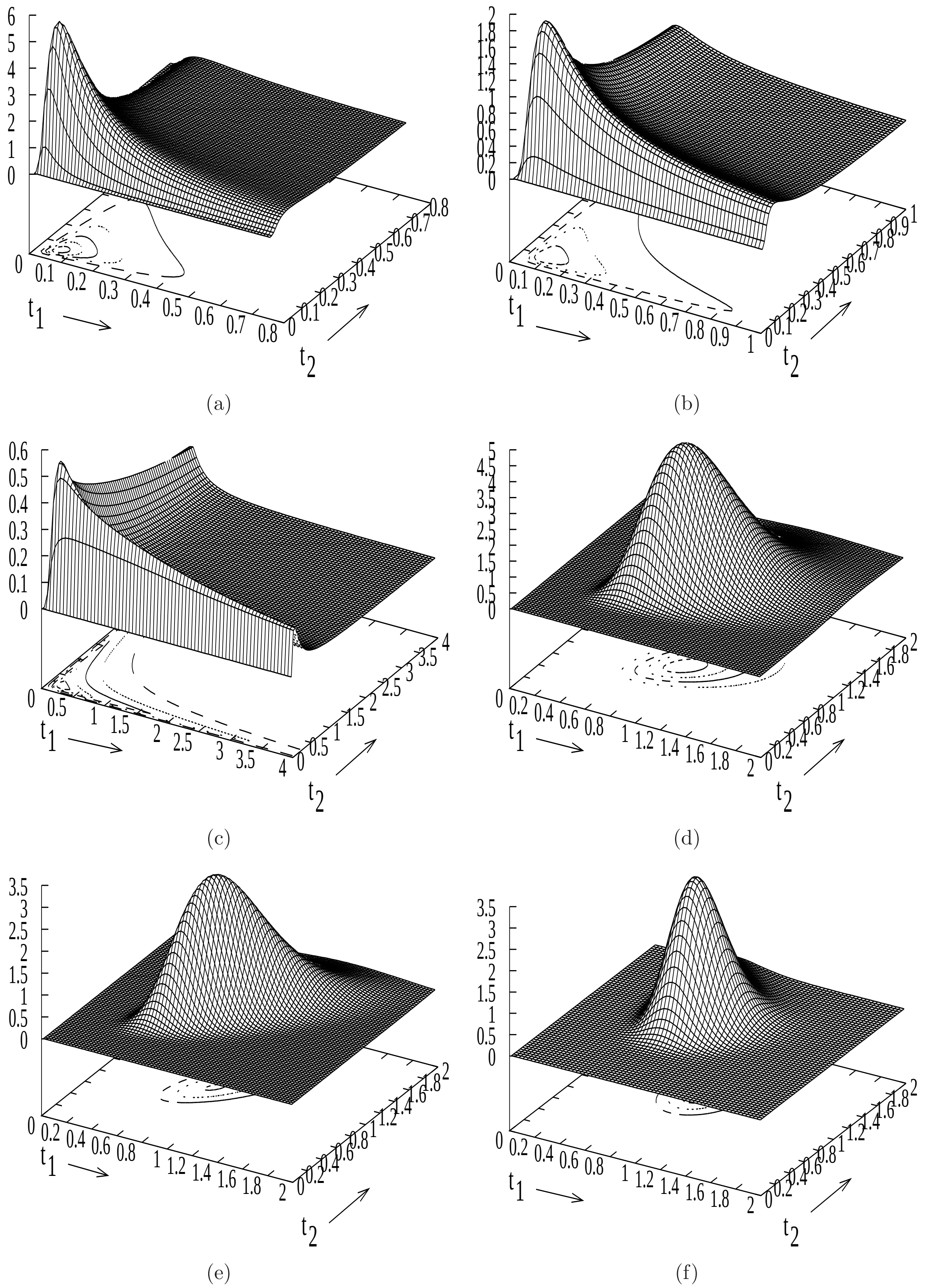
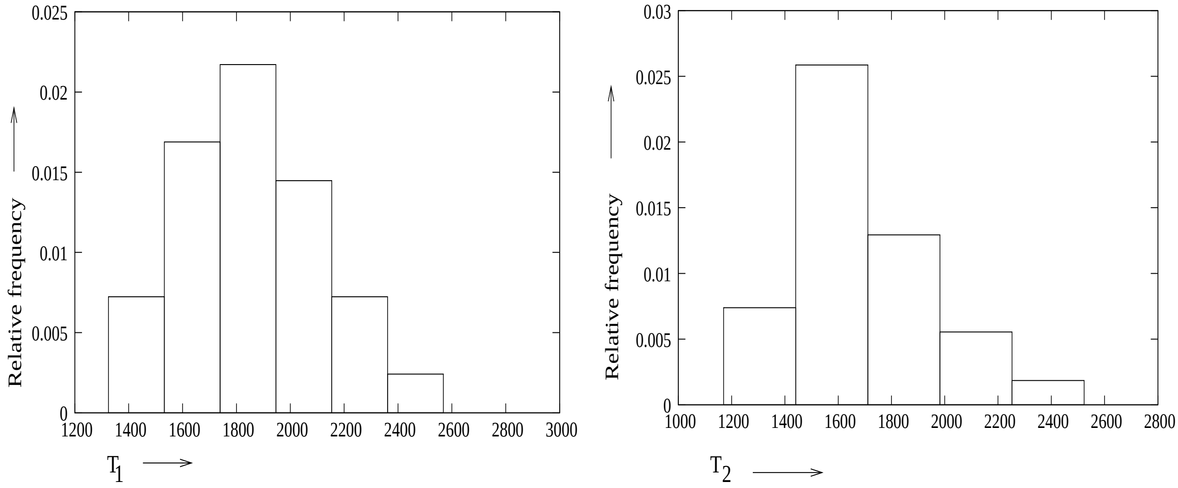
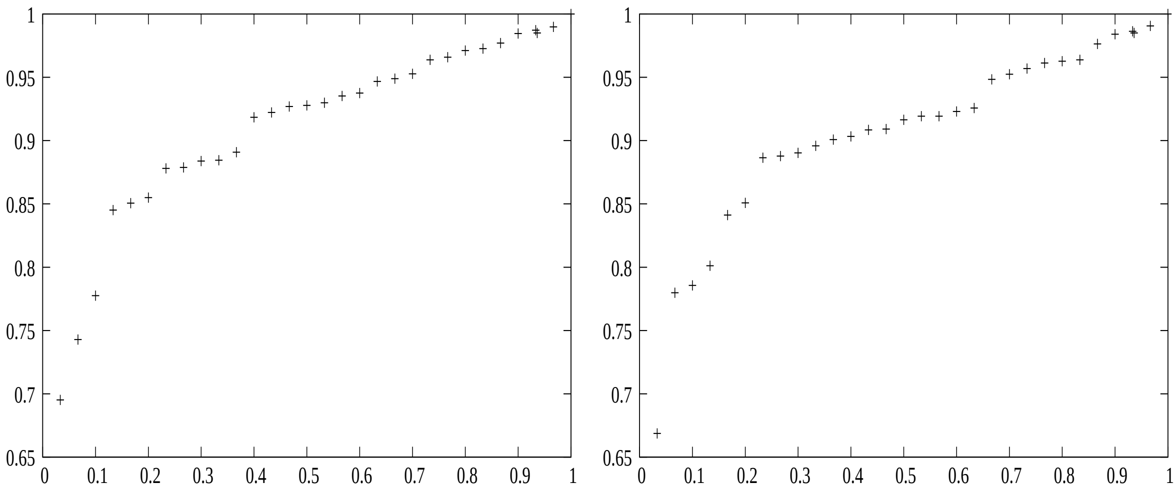


Figure 1: The surface plot of BS-SN$_2$ for different parameter values when $\beta_1 = \beta_2 = 1$,and (a) $\alpha_1 = 2 = \alpha_2$, $\lambda_1 = \lambda_2 = 1$, $\rho = 0.5$, (b) $\alpha_1 = 2 = \alpha_2$, $\lambda_1 = \lambda_2 = 1$, $\rho = 0.0$, (c) $\alpha_1 = 2 = \alpha_2$, $\lambda_1 = \lambda_2 = 1$, $\rho =$ -0.5, (d) $\alpha_1 = \alpha_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\rho = 0.5$, (e) $\alpha_1 = \alpha_2 = 0.3$, $\lambda_1 =$ -5.0, $\lambda_2 = 5.0$, $\rho = 0.5$, (f) $\alpha_1 = \alpha_2 = 0.3$, $\lambda_1 =$ -5.0, $\lambda_2 = 5.0$, $\rho =$ -0.5.



(a) (b)

Figure 2: Histogram of (a) T_1 and (b) T_2.



(a) (b)

Figure 3: Scaled TTT plots of (a) T_1 and (b) T_2.