Weibull Step-Stress Model with a Lagged Effect

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Abstract

In survival analysis and reliability, researchers are often interested in assessing the effects of different stress factors on the lifetime of experimental units. The model introduced in the article is motivated by a study of the effects of altitude and other risk factors on decompression sickness, a condition encountered when individuals are exposed to significant changes in environmental pressure. Unlike standard life-testing experiments, in this study, the levels of the stress factor, viz. altitude, are changed during the exposure duration. This is known as a step-stress test, a class of accelerated testing, widely used in material testing. Recently Kannan, Kundu and Balakrishnan (2010) introduced the cumulative risk model as an alternative to the widely used cumulative exposure model. The new model allows for the inclusion of a lag period in the hazard function, a more realistic assumption in most applications. In this paper we consider the cumulative risk model assuming that the lifetime distributions of the experimental units follow Weibull distributions at the different levels of the risk factor. It is assumed that the level of the stress factor is changed only once during the exposure duration at a pre-fixed time τ_1. The maximum likelihood and the least squares methods have been used to estimate the unknown parameters. Monte Carlo simulations are performed to compare the performances of the two different methods. We further propose the Bayes estimators of the unknown parameters of the model. To evaluate the performance of the model, one data set from the altitude decompression sickness experiment has been analyzed and it is observed that the proposed model fits the data quite well.

Key Words and Phrases: Step-stress model; hazard function; cumulative exposure model; maximum likelihood estimators; least squares estimators.

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1 Introduction

The experiment that motivated this particular research relates to altitude decompression sickness (DCS), a condition frequently observed in pilots flying at high altitudes, mountaineers, and astronauts performing extravehicular activities in space. To assess the effects of different risk factors on DCS, researchers at Brooks Airforce Base performed experiments involving human subjects in a hypobaric chamber. The study over a 20 year period involved exposures to different altitudes, varying preoxygenation routines, and different levels of exercise while at altitude. While physiologists agreed that rates of DCS increased with increasing altitude, they were interested in determining whether a staged ascent, wherein subjects were exposed to a lower altitude for a specified period of time prior to exposure to the higher altitude, would reduce the incidence when compared to subjects directly exposed to the higher altitude. In the staged ascent experiments, researchers changed the levels of the risk factor (altitude) at pre-specified times during the exposure.

The experimental scenario described above is known as a step-stress test, a particular type of accelerated life testing experiment. Accelerated testing has been extensively used in reliability and life-testing for units/items that are highly reliable. In accelerated life tests, the experimental units are exposed to higher than normal levels of the stress factors, affecting the underlying lifetime distribution and resulting in early failures. Assuming a model that relates stress and the lifetime distributions, the data from the accelerated test may be used to estimate the lifetime distribution under standard levels of the stress factor. For an excellent review on accelerated life testing, readers may refer to the books by Nelson (1990) and Bagdonavičius and Nikulin (2002).

In a standard step-stress experiment, all individuals or items are subject to an initial stress level s_1. The stress is gradually increased at pre-specified times during the exposure. The stress factor may refer to the dose of a drug, elevation of the treadmill, altitude, temperature, voltage, and pressure. In a simple step-stress experiment, the stress is increased only once at a pre-determined time τ_1. All units that have survived up to τ_1 will be exposed to the new stress level s_2 for the remainder of the experiment. The most popular model used to analyze data from a step-stress experiment is the cumulative exposure model (CEM) introduced by Seydyakin (1966). In recent years, there has been extensive work related to the CEM; key references include Xiong (1998), Xiong and Milliken (1999), Bai et al. (1989), Balakrishnan et al. (2007), Sha and Pan (2014), Kateri and Kamps (2015), Ismail (2016), Kundu and Ganguly (2017) and the references cited therein. Recently, Balakrishnan (2009) provided an excellent review of step-stress models focusing primarily on the exponential distribution. One major disadvantage of changing the stress level at a fixed time point is the possibility that no failures may occur before τ_1. Under this scenario, inference of the unknown parameters associated with the lifetime distribution at the stress level s_1 may not be possible. Kundu and Balakrishnan (2009), Wang and Yu (2009) and Balakrishnan et al. (2012) considered step-stress models where the stress changes after a fixed number of failures are observed. Assuming the underlying lifetime distributions to be exponential, they have obtained the point and interval estimators of the unknown parameters based on the data obtained under different censoring schemes.

Although the CEM is extremely popular for modeling data from step-stress tests, it suffers from one major disadvantage that limits its use for certain types of applications. The hazard function of the underlying distribution is discontinuous at the point at which the stress is changed. This implies the effect of the change in the level of the stress factor is instantaneous, which may not be reasonable. Because of this drawback, Kannan et al. (2010) proposed a new step-stress model in which the effect of the change of stress is not felt immediately. They assume that there is a latency or lag period, δ, before the effect of the increased stress is completely observed. They refer to this as the cumulative risk model

(CRM). Kannan et al. (2010) developed inference for the unknown parameters of the CRM, based on the assumption that the lifetime distributions of the experimental units follow exponential distribution with different scale parameters at two different stress levels. The authors also showed that the CEM may be obtained as a limiting case of the Cumulative Risk Model. Beltrani (2015) recently extended the results of Kannan et al. (2010) in case of competing risks model, when the competing causes of failures follow exponential distribution. The results were further extended by Huang et al. (2015) in case of masked data. See also Yao and Luo (2013) for some applications of this model in electronic products.

While the exponential distribution has many attractive features, the assumption of a constant hazard function may not always be useful in practice. In this article, we consider the Weibull model, one of the most widely used distribution in the survival and reliability literature. We consider inference of the unknown parameters based on the assumptions that the lifetime distributions are Weibull with different scale and shape parameters at the two stress levels. Having different scale and shape parameters provides tremendous flexibility in modeling the hazard function for the step-stress experiment. We propose to use the maximum likelihood and least squares methods to estimate the unknown parameters. Monte Carlo simulations are performed to compare the performances of the two different methods in terms of the biases and mean squared errors (MSEs) of the estimates. The Bayes estimators of the model parameters are also obtained. One data set obtained from the DCS experiment has been analyzed using the proposed model, and it is observed that the new model provides a better fit than the existing Weibull hazard model.

The main contribution about the present paper compared to the existing papers is that the present paper deals with the lagged model, which have not been considered by most of the researchers in this area. Most of the work deal with the model where the hazard function changes instantaneously as the stress changes, which may not be very reasonable. In fact in this paper we have shown that the lagged model provides a better fit to the DCS data set than the very popular Khamis-Higgins model where there is no lagged effect. Moreover, we have provided both classical and Bayesian inferences of the unknown parameters and compared their performances by using simulation experiments and by analyzing one real data set.

The rest of the article is organized as follows. In Section 2, we provide the model formulation. The maximum likelihood estimators (MLEs) and the least squares estimators (LSEs) of the unknown parameters are provided in Section 3. In Section 4, we present the Monte Carlo simulation results comparing the performances of the MLEs and the LSEs. In Section 5, we provide the Bayes estimators and the associated credible intervals. The analysis of a data set is presented in Section 6. Finally we conclude the paper in Section 7.

2 Model Description

A two-parameter Weibull random variable with shape parameter α and scale parameter $\lambda > 0$ will be denoted by WE(α, λ). The probability density function (PDF) is given by

$$f(x; \alpha, \lambda) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}}; \quad x > 0. \tag{1}$$

A two-parameter gamma random variable with shape parameter $\alpha > 0$, and scale parameter $\lambda > 0$ will be denoted by GA(α, λ) with the following PDF

$$f(x; \alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}; \quad x > 0. \tag{2}$$

Consider, a simple step-stress experiment, where the stress level changes only once during the experiment. Assume n identical items are placed on a life test at stress level s_1. The lifetime distribution of each experimental unit follows a Weibull distribution with the scale parameter λ_1 and shape parameter α_1. Subjects are monitored continuously until failure.

Let $t_1 < t_2 < \ldots < t_r < \tau_1 < t_{r+1} < \ldots < t_n$ denote the failure times. At the time τ_1 (pre-fixed), the stress level is increased to s_2. When the stress changes the hazard function of the surviving experimental units also changes, although the effects of the increased stress level is not seen immediately. It is assumed that there is a lag period δ, that may be unknown, before the effects are completely observed. After the time point $\tau_1 + \delta = \tau_2$, the lifetime of the experimental units follows a Weibull distribution with the scale parameter λ_2, and the shape parameter α_2. In the interval $[\tau_1, \tau_2]$, the hazard function increases and for simplicity it is assumed to be linear. The piecewise hazard function takes the following form:

$$h(t) = \begin{cases} \alpha_1 \lambda_1 t^{\alpha_1 - 1} & \text{if} \quad 0 < t < \tau_1 \\ \beta_0 + \beta_1 t & \text{if} \quad \tau_1 < t < \tau_2 \\ \alpha_2 \lambda_2 t^{\alpha_2 - 1} & \text{if} \quad \tau_2 < t < \infty. \end{cases} \tag{3}$$

The parameters β_0 and β_1 are chosen to ensure that the hazard function is continuous, *i.e.*

$$\begin{aligned} \beta_0 + \beta_1 \tau_1 &= \alpha_1 \lambda_1 \tau_1^{\alpha_1 - 1} \qquad \text{and} \\ \beta_0 + \beta_1 \tau_2 &= \alpha_2 \lambda_2 \tau_2^{\alpha_2 - 1}. \end{aligned} \tag{4}$$

By solving (4) for $\tau_1 \neq \tau_2$ we obtain,

$$\begin{aligned} \beta_0 &= \frac{\alpha_1 \lambda_1 \tau_2 \tau_1^{\alpha_1 - 1} - \alpha_2 \lambda_2 \tau_1 \tau_2^{\alpha_2 - 1}}{\tau_2 - \tau_1} \qquad \text{and} \\ \beta_1 &= \frac{\alpha_2 \lambda_2 \tau_2^{\alpha_2 - 1} - \alpha_1 \lambda_1 \tau_1^{\alpha_1 - 1}}{\tau_2 - \tau_1}. \end{aligned} \tag{5}$$

Kannan, Kundu and Balakrishnan (2010) defined the cumulative risk model (CRM) with hazard function given by

$$h_o(t) = \begin{cases} \lambda_1 & \text{if} \quad 0 < t < \tau_1 \\ \beta_0 + \beta_1 t & \text{if} \quad \tau_1 < t < \tau_2 \\ \lambda_2 & \text{if} \quad \tau_2 < t < \infty, \end{cases}$$

where β_0 and β_1 satisfy

$$\begin{aligned} \beta_0 + \beta_1 \tau_1 &= \lambda_1 \qquad \text{and} \\ \beta_0 + \beta_1 \tau_2 &= \lambda_2. \end{aligned}$$

It is noted that when $\alpha_1 = \alpha_2 = 1$, the proposed model (3) reduces to the CRM of Kannan, Kundu and Balakrishnan (2010). Moreover, for the proposed model, when $\tau_2 = \tau_1 = \tau$ (say), then (3) takes the form

$$h(t) = \begin{cases} \alpha_1 \lambda_1 t^{\alpha_1 - 1} & \text{if} \quad 0 < t \leq \tau \\ \\ \alpha_2 \lambda_2 t^{\alpha_2 - 1} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{6}$$

Note that (6) is the hazard function of the cumulative exposure model; the case of exponential hazards corresponds to $\alpha_1 = \alpha_2 = 1$, (see for example Balakrishnan; 2009) and the hazard function of the Khamis and Higgins (1998) model when $\alpha_1 = \alpha_2 = \alpha$, in case of Weibull hazards with equal shape parameters. The proposed model can be seen as a generalization of the tampered failure rate (TFR) model of Bhattacharyya and Soejoeti (1989).

Based on the hazard function $h(t)$ in (3), the corresponding cumulative hazard function becomes;

$$H(t) = \begin{cases} \lambda_1 t^{\alpha_1} & \text{if} \quad 0 < t < \tau_1 \\ \lambda_1 \tau_1^{\alpha_1} + \beta_0 (t - \tau_1) + \frac{\beta_1}{2}(t^2 - \tau_1^2) & \text{if} \quad \tau_1 < t < \tau_2 \\ \lambda_1 \tau_1^{\alpha_1} + \beta_0 (\tau_2 - \tau_1) + \frac{\beta_1}{2}(\tau_2^2 - \tau_1^2) + \lambda_2 (t^{\alpha_2} - \tau_2^{\alpha_2}) & \text{if} \quad t \geq \tau_2. \end{cases} \tag{7}$$

The survival function $S(t)$ is given by

$$S(t) = e^{-H(t)} = \begin{cases} e^{-\lambda_1 t^{\alpha_1}} & \text{if} \quad 0 < t < \tau_1 \\ e^{-\lambda_1 \tau_1^{\alpha_1} - \beta_0 (t - \tau_1) - \frac{1}{2}\beta_1 (t^2 - \tau_1^2)} & \text{if} \quad \tau_1 < t < \tau_2 \\ e^{-\lambda_1 \tau_1^{\alpha_1} - \beta_0 (\tau_2 - \tau_1) - \frac{1}{2}\beta_1 (\tau_2^2 - \tau_1^2) - \lambda_2 (t^{\alpha_2} - \tau_2^{\alpha_2})} & \text{if} \quad t \geq \tau_2. \end{cases} \tag{8}$$

The corresponding PDF becomes;

$$f(t) = -\frac{d}{dt} S(t) = \begin{cases} \alpha_1 \lambda_1 t^{\alpha_1 - 1} e^{-\lambda_1 t^{\alpha_1}} & \text{if} \quad 0 < t < \tau_1 \\ (\beta_0 + \beta_1 t) e^{-\lambda_1 \tau_1^{\alpha_1} - \beta_0 (t - \tau_1) - \frac{1}{2}\beta_1 (t^2 - \tau_1^2)} & \text{if} \quad \tau_1 < t < \tau \\ \alpha_2 \lambda_2 t^{\alpha_2 - 1} e^{-\lambda_2 t^{\alpha_2}} e^{-\lambda_1 \tau_1^{\alpha_1} - \beta_0 (\tau_2 - \tau_1) - \frac{1}{2}\beta_1 (\tau_2^2 - \tau_1^2) + \lambda_2 \tau_2^{\alpha_2}} & \text{if} \quad t \geq \tau_2. \end{cases} \tag{9}$$

3 Maximum Likelihood and Least Squares Estimators

In this section, we obtain the maximum likelihood and least squares estimators of α_1, λ_1, α_2 and λ_2 assuming that δ (the lag period) is fixed. It may be mentioned that in practice

δ is usually unknown, in which case we obtain an estimate of δ by maximizing the profile likelihood function. Details are provided in the data analysis section.

We have the following set of observations:

$$t_1 < \cdots < t_r < \tau_1 < t_{r+1} < \cdots t_{r+m} < \tau_2 < t_{r+m+1} < \cdots < t_{r+m+k} < T. \qquad (10)$$

We are assuming that there are r failures before τ_1, m failures between τ_1 and τ_2 and k failures beyond τ_2, and before the experiment is terminated at T. At the time point T, we assume that $s = n - (r + m + k)$ observations are censored. It is assumed that $r > 0$, $m > 0$ and $k > 0$.

3.1 Maximum Likelihood Estimators

Based on (10), the likelihood function becomes

$$\begin{aligned} L(\beta_0, \beta_1, \alpha_1, \alpha_2) \;=\; & \alpha_1^r \lambda_1^r \prod_{i=1}^{r} t_i^{\alpha_1 - 1} e^{-\lambda_1 \sum_{i=1}^{r} t_i^{\alpha_1}} \times \prod_{i=r+1}^{r+m} (\beta_0 + \beta_1 t_i) e^{-\lambda_1 \tau_1^{\alpha_1} - \beta_0 (t_i - \tau_1) - \beta_1 (t_i^2 - \tau_1^2)/2} \times \\ & \alpha_2^k \lambda_2^k e^{-k\lambda_1 \tau_1^{\alpha_1} - k\beta_0(\tau_2 - \tau_1) - k\beta_1(\tau_2^2 - \tau_1^2)/2 + k\lambda_2 \tau_2^{\alpha_2}} \prod_{i=r+m+1}^{r+m+k} t_i^{\alpha_2 - 1} e^{-\lambda_2 \sum_{i=r+m+1}^{r+m+k} t_i^{\alpha_2}} \times \\ & e^{-s\lambda_2 T^{\alpha_2}}. \end{aligned} \qquad (11)$$

Using (4), we obtain the log-likelihood function in terms of $\alpha_1, \alpha_2, \beta_0, \beta_1$:

$$\begin{aligned} l(\beta_0, \beta_1, \alpha_1, \alpha_2) \;=\; & r \ln(\beta_0 + \beta_1 \tau_1) + (\alpha_1 - 1) \sum_{i=1}^{r} (\ln t_i - \ln \tau_1) - (\beta_0 + \beta_1 \tau_1) \times \frac{1}{\alpha_1} \sum_{i=1}^{r} \frac{t_i^{\alpha_1}}{\tau_1^{\alpha_1 - 1}} \\ & + \sum_{i=r+1}^{r+m} \ln(\beta_0 + \beta_1 t_i) - m(\beta_0 + \beta_1 \tau_1) \frac{\tau_1}{\alpha_1} - \beta_0 \sum_{i=r+1}^{r+m} (t_i - \tau_1) \\ & - \frac{\beta_1}{2} \sum_{i=r+1}^{r+m} (t_i^2 - \tau_1^2) + k \ln(\beta_0 + \beta_1 \tau_2) + (\alpha_2 - 1) \sum_{i=r+m+1}^{r+m+k} (\ln t_i - \ln \tau_2) \\ & - (\beta_0 + \beta_1 \tau_2) \times \frac{1}{\alpha_2} \sum_{i=r+m+1}^{r+m+k} \frac{t_i^{\alpha_2}}{\tau_2^{\alpha_2 - 1}} - k(\beta_0 + \beta_1 \tau_1) \frac{\tau_1}{\alpha_1} - k\beta_0 (\tau_2 - \tau_1) \\ & - \frac{k\beta_1}{2} (\tau_2^2 - \tau_1^2) + k(\beta_0 + \beta_1 \tau_2) \frac{\tau_2}{\alpha_2} - s(\beta_0 + \beta_1 \tau_2) \frac{T^{\alpha_2}}{\alpha_2 \tau_2^{\alpha_2 - 1}}. \end{aligned} \qquad (12)$$

The MLEs of the unknown parameters can be obtained by maximizing (12) with respect to the unknown parameters. The normal equations can be written as follows:

$$\begin{aligned}\frac{\partial l(\beta_0,\beta_1,\alpha_1,\alpha_2)}{\partial \beta_0} &= \frac{r}{\beta_0+\beta_1\tau_1} - \frac{1}{\alpha_1}\sum_{i=1}^{r}\frac{t_i^{\alpha_1}}{\tau_1^{\alpha_1-1}} + \sum_{i=r+1}^{r+m}\frac{1}{\beta_0+\beta_1 t_i} - \frac{(m+k)\tau_1}{\alpha_1} - \sum_{i=r+1}^{r+m}(t_i-\tau_1)\\ &\quad + \frac{k}{\beta_0+\beta_1\tau_2} - \frac{1}{\alpha_2}\sum_{i=r+m+1}^{r+m+k}\frac{t_i^{\alpha_2}}{\tau_2^{\alpha_2-1}} - k(\tau_2-\tau_1) + \frac{k\tau_2}{\alpha_2}\\ &\quad - \frac{sT^{\alpha_2}}{\alpha_2\tau_2^{\alpha_2-1}} = 0, \qquad (13)\end{aligned}$$

$$\begin{aligned}\frac{\partial l(\beta_0,\beta_1,\alpha_1,\alpha_2)}{\partial \beta_1} &= \frac{r\tau_1}{\beta_0+\beta_1\tau_1} - \frac{1}{\alpha_1}\sum_{i=1}^{r}\frac{t_i^{\alpha_1}}{\tau_1^{\alpha_1-2}} + \sum_{i=r+1}^{r+m}\frac{t_i}{\beta_0+\beta_1 t_i} - \frac{(m+k)\tau_1^2}{\alpha_1} + \frac{k\tau_2^2}{\alpha_2}\\ &\quad - \frac{1}{2}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2) + \frac{k\tau_2}{\beta_0+\beta_1\tau_2} - \frac{1}{\alpha_2}\sum_{i=m+r+1}^{m+r+k}\frac{t_i^{\alpha_2}}{\tau_2^{\alpha_2-2}} - \frac{k}{2}(\tau_2^2-\tau_1^2)\\ &\quad - \frac{sT^{\alpha_2}}{\alpha_2\tau_2^{\alpha_2-2}} = 0, \qquad (14)\end{aligned}$$

$$\begin{aligned}\frac{\partial l(\beta_0,\beta_1,\alpha_1,\alpha_2)}{\partial \alpha_1} &= \sum_{i=1}^{r}(\ln t_i - \ln\tau_1) - \frac{\beta_0+\beta_1\tau_1}{\alpha_1}\sum_{i=1}^{r}\frac{t_i^{\alpha_1}}{\tau_1^{\alpha_1-1}}\left(\ln t_i - \ln\tau_1 - \frac{1}{\alpha_1}\right)\\ &\quad + (\beta_0+\beta_1\tau_1)\frac{(m+k)\tau_1}{\alpha_1^2} = 0. \qquad (15)\end{aligned}$$

$$\begin{aligned}\frac{\partial l(\beta_0,\beta_1,\alpha_1,\alpha_2)}{\partial \alpha_2} &= \sum_{i=r+m+1}^{r+m+k}(\ln t_i - \ln\tau_2) - \frac{\beta_0+\beta_1\tau_2}{\alpha_2}\sum_{i=r+m+1}^{r+m+k}\frac{t_i^{\alpha_2}}{\tau_2^{\alpha_2-1}}\left(\ln t_i - \ln\tau_2 - \frac{1}{\alpha_2}\right)\\ &\quad - k(\beta_0+\beta_1\tau_2)\frac{\tau_2}{\alpha_2^2} - \frac{(\beta_0+\beta_1\tau_2)sT^{\alpha_2}}{\alpha_2\tau_2^{\alpha_2-1}}\left(\ln T - \ln\tau_2 - \frac{1}{\alpha_2}\right) = 0. \qquad (16)\end{aligned}$$

We need to solve four equations (13), (14), (15) and (16) simultaneously to compute MLEs of β_0, β_1, α_1 and α_2. Once the MLEs of β_0, β_1, α_1 and α_2 are obtained, the MLEs of λ_1 and λ_2 can be easily obtained using (4).

From the form of the normal equations, it is clear that we do not have any explicit solution for the equations (13), (14), (15), and (16). We need to use iterative procedure like Newton-Raphson algorithm to solve the four non-linear equations simultaneously or using multidimensional optimization procedure to maximize the log-likelihood function $l(\beta_0,\beta_1,\alpha_1,\alpha_2)$ directly. It may be mentioned finding the global optimum in any multidimensional optimization problem is an important issue. Most of the times it is quite difficult to prove the uniqueness of the global optimum. In this case also due to complicated nature of the function, we could not prove theoretically the uniqueness of the global optimum. It has been verified numerically during the data analysis, and it has been explained in details in Section 6.

Note that for any iterative procedure, it is very important to have good starting values. Choosing proper initial values can be quite challenging. Without the proper initial guesses, the algorithm may not even converge or it may converge to a local rather than a global optimum. In the next section, we derive the least squares estimators which may be used as initial guesses of any iterative algorithm or they may be used as alternative estimators of the underlying parameters.

3.2 Least Squares Estimators

In this section we provide an alternative method to estimate β_0 and β_1 when α_1 and α_2 are assumed to be known. Using the method of least squares, we are able to obtain explicit estimators of β_0 and β_1 when α_1 and α_2 are known. We will be able to use these estimators as initial values of the iterative process to compute the MLEs as follows. Suppose $\widetilde{\beta_0}(\alpha_1, \alpha_2)$ and $\widetilde{\beta}_1(\alpha_1, \alpha_2)$ are the least squares estimators (LSEs) of β_0 and β_1, respectively. Then from the function $l(\widetilde{\beta}_0(\alpha_1, \alpha_2), \widetilde{\beta}_1(\alpha_1, \alpha_2))$, we can get an idea about the initial values of α_1 and α_2.

We note that the cumulative hazard function of the proposed CRM model is a linear function of β_0 and β_1 for fixed α_1 and α_2. Hence it is natural to obtain estimates of β_0 and β_1 by minimizing the least squares distance between the empirical cumulative hazard function and the fitted cumulative hazard function with respect to β_0 and β_1. Hence $\widetilde{\beta_0}(\alpha_1, \alpha_2)$ and

$\widetilde{\beta}_1(\alpha_1, \alpha_2)$ can be obtained by minimizing

$$g_{\alpha_1,\alpha_2}(\beta_0, \beta_1) = \sum_{i=1}^{r+m+k} \left(H(t_i) - \widehat{H}(t_i)\right)^2 + s(H(T) - \widehat{H}(T))^2 \qquad (17)$$

with respect to β_0 and β_1, respectively. Here $H(t_i)$ and $\widehat{H}(t_i)$ are the cumulative hazard function and estimated cumulative hazard function respectively. We use standard non-parametric estimators given by

$$\widehat{H}(t_i) = -\ln(\widehat{S}(t_i)) = \ln n - \ln(n - i + 1) = d_i \;\; \text{(say)}, \quad \text{and} \qquad (18)$$

$$\widehat{H}(T) = -\ln(\widehat{S}(T)) = \ln n - \ln s = d \quad \text{(say)}. \qquad (19)$$

Rewriting (4) as

$$\lambda_1 = c_1(\alpha_1)\beta_0 + c_2(\alpha_1)\beta_1 \quad \text{and} \qquad (20)$$

$$\lambda_2 = c_3(\alpha_2)\beta_0 + c_4(\alpha_2)\beta_2, \qquad (21)$$

where

$$c_1(\alpha_1) = \frac{1}{\alpha_1 \tau_1^{\alpha_1 - 1}}, \quad c_2(\alpha_1) = \frac{1}{\alpha_1 \tau_1^{\alpha_1 - 2}}, \quad c_3(\alpha_2) = \frac{1}{\alpha_2 \tau_2^{\alpha_2 - 1}}, \;\; \text{and} \;\; c_4(\alpha_2) = \frac{1}{\alpha_2 \tau_2^{\alpha_2 - 2}}.$$

We can write (17) as

$$\begin{aligned} g_{\alpha_1,\alpha_2}(\beta_0, \beta_1) = & \sum_{i=1}^{r} (d_i - (c_1(\alpha_1)\beta_0 + c_2(\alpha_1)\beta_1) t_i^{\alpha_1})^2 + \\ & \sum_{i=r+1}^{r+m} \left(d_i - (c_1(\alpha_1)\beta_0 + c_2(\alpha_1)\beta_1)\tau_1^{\alpha_1} - \beta_0(t_i - \tau_1) - \beta_1(t_i^2 - \tau_1^2)/2\right)^2 + \\ & \sum_{i=r+m+1}^{n} \Big(d_i - (c_1(\alpha)\beta_0 + c_2(\alpha)\beta_1)\tau_1^{\alpha} - \beta_0(\tau_2 - \tau_1) - \beta_1(\tau_2^2 - \tau_1^2)/2 - \\ & \quad (c_3(\alpha_2)\beta_0 + c_4(\alpha_2)\beta_1)(t_i^{\alpha} - \tau_2^{\alpha})\Big)^2 . \end{aligned}$$

Let $d_{r+m+k+1} = \cdots = d_n = d$, and $t_{r+m+k+1} = \cdots = t_n = T$, then the least squares estimators of β_0 and β_1 for a given α_1 and α_2, can be obtained respectively, as

$$\widetilde{\beta}_0(\alpha_1, \alpha_2) = \frac{D_1 C_2 - C_1 D_2}{C_0 D_1 - C_1 D_0} \quad \text{and} \quad \widetilde{\beta}_1(\alpha_1, \alpha_2) = \frac{-D_0 C_2 + C_0 D_2}{C_0 D_1 - C_1 D_0}. \qquad (22)$$

The explicit expressions of C_0, C_1, C_2, D_0, D_1 and D_2 are provided in Appendix A.

4 Monte Carlo Simulations

We have performed some simulation experiments to compare the performances of the MLEs and the LSEs in terms of their biases and the MSEs. We have considered different sample sizes; $n = 25$, 50, 75 and 100, different shape parameters; $\alpha_1 = 1.5$, 2.0, different τ_1; 10, 15 and different δ; 10, 15 values. For each combination of the sample size and parameter values, first we generated the sample from the given model as described in Section 2. Now based on the given sample we computed the MLEs and the LSEs of the parameters. We have used RAN2 for random deviate generator and AMOEBA for solving non-linear optimization problem of Press et al. (1992) in all these cases. Then we replicated the process 1000 times and obtained the average biases and the square root of the MSEs (SMSEs) of all the parameter estimators for both MLEs and LSEs. The results are reported in Tables 1 - 4. All the programs are written in FORTRAN.

From these simulation results some of the points are quite clear. It is observed in all these cases as the sample size increases, the biases and the SMSEs decrease. It indicates the consistency property of both the estimators. It is also observed that the mean squared errors of the MLEs are smaller than those of the LSEs. In some cases particularly for small sample sizes the biases of the LSEs are slightly smaller than the biases of the MLEs. Moreover, it is also observed that for fixed τ_2 as τ_1 decreases the SMSEs and biases of $\widehat{\alpha}_1$ and $\widehat{\lambda}_2$ increase and for $\widehat{\alpha}_2$ and $\widehat{\lambda}_2$ they decrease.

Table 1: The average MLEs, LSEs and the associated square root of the mean squared errors (in parenthesis), when $\tau_1 = 15$, $\tau_2 = 25$, $T = 100$, $\alpha_1 = 2.0$, $\alpha_2 = 1.0$, $\lambda_1 = 1/200$, $\lambda_2 = 1/50$

n	Methods	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
25	MLE	2.1989	1.3056	0.0067	0.0261
		(0.3878)	(0.5551)	(0.0052)	(0.0482)
	LSE	2.2697	1.2582	0.0064	0.0777
		(0.6058)	(0.7449)	(0.0080)	(0.1255)
50	MLE	2.1817	1.3209	0.0060	0.0174
		(0.3064)	(0.5126)	(0.0038)	(0.0376)
	LSE	2.2161	1.1974	0.0052	0.0722
		(0.4380)	(0.6246)	(0.0049)	(0.1187)
75	MLE	2.1732	1.2791	0.0048	0.0180
		(0.2506)	(0.4203)	(0.0031)	(0.0392)
	LSE	2.1817	1.0177	0.0049	0.0529
		(0.3409)	(0.5116)	(0.0040)	(0.0997)
100	MLE	2.1686	1.2365	0.0049	0.0196
		(0.2103)	(0.3495)	(0.0025)	(0.0227)
	LSE	2.1639	0.9973	0.0047	0.0437
		(0.2915)	(0.4409)	(0.0032)	(0.0687)

Table 2: The average MLEs, LSEs and the associated square root of the mean squared errors (in parenthesis), when $\tau_1 = 15$, $\tau_2 = 25$, $T = 100$, $\alpha_1 = 1.5$, $\alpha_2 = 1.0$, $\lambda_1 = 1/200$, $\lambda_2 = 1/50$

n	Methods	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
25	MLE	1.7234	1.1461	0.0113	0.0115
		(0.4524)	(0.3283)	(0.0117)	(0.0321)
	LSE	1.8341	0.8436	0.0100	0.0776
		(0.6763)	(0.4231)	(0.0133)	(0.0922)
50	MLE	1.6828	1.1017	0.0098	0.0140
		(0.3287)	(0.1996)	(0.0075)	(0.0163)
	LSE	1.7813	0.8625	0.0089	0.0654
		(0.5734)	(0.2372)	(0.0095)	(0.0722)
75	MLE	1.6516	1.0655	0.0085	0.0159
		(0.2408)	(0.1661)	(0.0057)	(0.0156)
	LSE	1.7685	0.8790	0.0078	0.0538
		(0.4969)	(0.1659)	(0.0072)	(0.0523)
100	MLE	1.6362	1.0587	0.0063	0.0178
		(0.2012)	(0.1476)	(0.0047)	(0.0087)
	LSE	1.7327	0.8999	0.0076	0.0343
		(0.4239)	(0.1217)	(0.0060)	(0.0211)

Table 3: The average MLEs, LSEs and the associated square root of the mean squared errors (in parenthesis), when $\tau_1 = 10$, $\tau_2 = 25$, $T = 100$, $\alpha_1 = 2.0$, $\alpha_2 = 1.0$, $\lambda_1 = 1/200$, $\lambda_2 = 1/50$

n	Methods	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
25	MLE	2.2425	1.2896	0.0067	0.0191
		(0.5096)	(0.4984)	(0.0072)	(0.0410)
	LSE	2.4538	1.1434	0.0064	0.0801
		(0.8174)	(0.6610)	(0.0087)	(0.0816)
50	MLE	2.2394	1.1860	0.0055	0.0164
		(0.3901)	(0.3335)	(0.0048)	(0.0326)
	LSE	2.4227	0.8729	0.0050	0.0721
		(0.6630)	(0.4722)	(0.0058)	(0.0688)
75	MLE	2.2303	1.1578	0.0053	0.0178
		(0.3247)	(0.2452)	(0.0038)	(0.0191)
	LSE	2.3784	0.8965	0.0048	0.0611
		(0.5720)	(0.3221)	(0.0047)	(0.0488)
100	MLE	2.2087	1.1426	0.0048	0.0189
		(0.2679)	(0.1952)	(0.0031)	(0.0139)
	LSE	2.3187	0.9165	0.0048	0.0289
		(0.5365)	(0.2554)	(0.0041)	(0.0189)

Table 4: The average MLEs, LSEs and the associated square root of the mean squared errors (in parenthesis), when $\tau_1 = 10$, $\tau_2 = 25$, $T = 100$, $\alpha_1 = 1.5$, $\alpha_2 = 1.0$, $\lambda_1 = 1/200$, $\lambda_2 = 1/50$

n	Methods	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
25	MLE	1.7203	1.1145	0.0127	0.0189
		(0.4922)	(0.3048)	(0.0123)	(0.0260)
	LSE	2.0169	0.7609	0.0102	0.0954
		(0.8695)	(0.3657)	(0.0148)	(0.0997)
50	MLE	1.7057	1.0629	0.0106	0.0165
		(0.3810)	(0.1758)	(0.0076)	(0.0171)
	LSE	1.9164	0.8727	0.0069	0.0716
		(0.8392)	(0.1972)	(0.0094)	(0.0721)
75	MLE	1.6685	1.0330	0.0100	0.0169
		(0.2789)	(0.1392)	(0.0058)	(0.0144)
	LSE	1.8136	0.8825	0.0055	0.0652
		(0.7539)	(0.1420)	(0.0078)	(0.0523)
100	MLE	1.6617	1.0249	0.0100	0.0175
		(0.2587)	(0.1246)	(0.0051)	(0.0091)
	LSE	1.7436	0.8998	0.0052	0.0432
		(0.6527)	(0.1001)	(0.0049)	(0.0312)

5 Bayes Estimators

In this section we discuss Bayes estimation of the unknown parameters and the construction of the associated credible intervals. We make the following prior assumptions on α_1, λ_1, α_2 and λ_2:

$$\pi(\alpha_1) \sim \text{GA}(a_1, b_1), \;\; \pi(\lambda_1) \sim \text{GA}(c_1, d_1), \;\; \pi(\alpha_2) \sim \text{GA}(a_2, b_2), \;\; \text{and } \pi(\lambda_2) \sim \text{GA}(c_2, d_2), \tag{23}$$

respectively, and they are independently distributed. Therefore, the joint posterior density function of α_1, λ_1, α_2 and λ_2 given the observation ($\mathcal{D}$) can be obtained as

$$\begin{aligned}&\pi(\alpha_1, \lambda_1, \alpha_2, \lambda_2|\mathcal{D}) \\ &= \frac{L(\alpha_1, \lambda_1, \alpha_2, \lambda_2) \times \pi(\alpha_1) \times \pi(\lambda_1) \times \pi(\alpha_2) \times \pi(\lambda_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_1, \lambda_1, \alpha_2, \lambda_2) \times \pi(\alpha_1) \times \pi(\lambda_1) \times \pi(\alpha_2) \times \pi(\lambda_2) d\alpha_1 d\lambda_1 d\alpha_2 d\lambda_2}.\end{aligned}$$

Here $L(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ can be obtained from (11) by replacing β_0 and β_1 as in (5). Hence, the Bayes estimator under the squared error loss function, of any function of α_1, λ_1, α_2 and λ_2, say $g(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ can be obtained as

$$\begin{aligned}&\widehat{g}_B(\alpha_1, \lambda_1, \alpha_2, \lambda_2) \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g(\alpha_1, \lambda_1, \alpha_2, \lambda_2) \pi(\alpha_1, \lambda_1, \alpha_2, \lambda_2|\mathcal{D}) d\alpha_1 d\lambda_1 d\alpha_2 d\lambda_2.\end{aligned}$$

Clearly, the Bayes estimators of α_1, λ_1, α_2 and λ_2 cannot be obtained in closed form. Hence, we propose to use Gibbs sampling technique to compute the Bayes estimates of the unknown parameters, and also to construct the associated credible intervals.

We have used the following Gibbs sampling procedure. Based on the priors (23) we first obtain the full conditionals, namely

$$\pi(\alpha_1|\lambda_1, \lambda_2, \alpha_2), \quad \pi(\lambda_1|\alpha_1, \alpha_2, \lambda_2), \quad \pi(\alpha_2|\alpha_1, \lambda_1, \lambda_2), \quad \pi(\lambda_2|\alpha_1, \lambda_1, \alpha_2). \tag{24}$$

The explicit expressions of all the full conditionals defined in (24) are provided in the Appendix C. We use the ratio of uniforms method, see for example Gentle (1998), to generate samples from the full conditional distributions. We use the following algorithm to compute the Bayes estimates and the associated credible intervals. Start with an initial values of α_1, λ_1, α_2 and λ_2, say $\alpha_1^{(0)}$, $\lambda_1^{(0)}$, $\alpha_2^{(0)}$ and $\lambda_2^{(0)}$. We take the MLEs as the initial values. Suppose at the k-th stage the values of α_1, λ_1, α_2 and λ_2 are $\alpha_1^{(k)}$, $\lambda_1^{(k)}$, $\alpha_2^{(k)}$ and $\lambda_2^{(k)}$, respectively, then $\alpha_1^{(k+1)}$, $\lambda_1^{(k+1)}$, $\alpha_2^{(k+1)}$ and $\lambda_2^{(k+1)}$ can be obtained as follows.

ALGORITHM:

1. Generate $\alpha_1^{(k+1)}$ from $\pi(\alpha_1|\lambda_1^{(k)}, \alpha_2^{(k)}, \lambda_2^{(k)})$.
2. Generate $\lambda_1^{(k+1)}$ from $\pi(\lambda_1|\alpha_1^{(k+1)}, \alpha_2^{(k)}, \lambda_2^{(k)})$.
3. Generate $\alpha_2^{(k+1)}$ from $\pi(\alpha_2|\alpha_1^{(k+1)}, \lambda_1^{(k+1)}, \lambda_2^{(k)})$.
4. Generate $\lambda_2^{(k+1)}$ from $\pi(\lambda_2|\alpha_1^{(k+1)}, \lambda_1^{(k+1)}, \alpha_2^{(k+1)})$.

Continue the process M times and obtain $\{(\alpha_1^{(k)}, \lambda_1^{(k)}, \alpha_2^{(k)}, \lambda_2^{(k)}), k = 1, \ldots, M\}$. Remove the first B samples for burn-in, and use the rest of the samples to compute the Bayes estimates and the associated highest posterior density (HPD) credible intervals.

In the Bayesian computation, the posterior summaries and theoretical quantities associated with the individual parameters often include the posterior mean, variance and some selected percentiles. From the percentile points, the HPD credible intervals can be easily obtained. We order the generated samples, and from the ordered samples, the HPD credible intervals can be obtained in the standard manner.

6 Decompression Sickness Example

In this section, we will consider data from an experiment conducted at Brooks Air Force Base to study the effects of risk factors on decompression sickness (DCS), see for example

Pilmanis et al. (2004). Altitude DCS occurs as a result of a change in environmental pressure that causes nitrogen dissolved in the body to come out of solution rapidly. The resulting nitrogen bubbles forming in different parts of the body give rise to a variety of symptoms including joint pain, headaches, blurred vision, and numbness. As part of the DCS study, subjects were placed in a hypobaric chamber that simulated exposure to altitude. The subjects were monitored continuously during the exposure duration: if the subject reported any of the symptoms of DCS, the experiment was terminated immediately. In the particular scenario we are considering, subjects were exposed to an initial altitude of 18,000 feet. After 4 hours at this altitude, the chamber was pressurized to simulate an altitude of 35,000 ft. The experiment continued for another 3 hours. The experiment involved a total of $n = 40$ subjects and we have $\tau_1 = 240$ minutes and $T = 420$ minutes. Seven subjects reported DCS symptoms prior to the change in altitude (failures in the first interval). A total of 31 subjects reported symptoms with 9 censored observations.

Clearly, the lag period δ, and consequently, τ_2 are unknown. We fit the proposed step-stress model to the DCS data assuming a fixed value for τ_2 and then estimating the unknown model parameters. The process is repeated for several values of τ_2 starting at 240 minutes and then increasing at increments of 5 minutes. For each choice of τ_2, the likelihood value is computed. Using this discrete optimization process, we choose the value of τ_2 for which the likelihood is maximized. We observed that for $\tau_2 = 350$, 355 and 360, the log-likelihood and the associated MLEs are almost identical. Therefore, we choose $\tau_2 = 350$ minutes, i.e. $\widehat{\delta} = 110$. This tells us that there is a lag of 110 minutes (almost 2 hours) before the effects of the increased stress are observed. This lag can be explained by the underlying physical processes that govern the formation and growth of bubbles at altitude.

The LSEs of α_1, α_2, λ_1 and λ_2 are 3.8144, 3.0003, 0.1701 and 0.1101, respectively. We take these as the initial estimators and obtain the MLEs using AMOEBA for solving non-linear optimization of Press et al. (1992). The MLEs of the unknown parameters, the associated standard errors (within parentheses) and the maximized log-likelihood (MLL) value are as follows: $\widehat{\alpha}_1$ = 3.8753 (0.3221), $\widehat{\alpha}_2$ = 3.0868 (0.2654), $\widehat{\lambda}_1$ = 0.1661 (0.0198), $\widehat{\lambda}_2$ = 0.1375 (0.0161), and MLL = -53.1870. Since it is difficult to prove theoretically that the above values maximize the log-likelihood function, we have also used the grid search method to maximize the log-likelihood function, in four dimensions with the ranges of α_1, α_2, λ_1 and λ_2 as (0,10), (0,10), (0,1.0), (0.1.0), respectively, with grid size 0.0001 each. We have reached the same solution, and it indicates that in this case the global maximum is attained. To assess the fit of the model, we computed the Kolmogorov-Smirnov (KS) distance between the observed and the fitted distribution. The KS statistic is 0.0713 with an associated p value being 0.5524, indicating that the Weibull step-stress model provides a good fit to the data.

From the values of the shape parameter, it is clear that the exponential model is not appropriate for this example. We may want to test whether the shape parameters of the two Weibull distributions are equal, i.e.

$$H_0 : \alpha_1 = \alpha_2 = \alpha \quad \text{vs.} \quad H_1 : \alpha_1 \neq \alpha_2.$$

The normal equations under the null hypothesis are provided in Appendix B. Based on the likelihood ratio test, we can test the hypothesis stated above. Under H_0, the MLEs of α, λ_1 λ_2 become 4.4448, 0.0514, 0.0100, respectively. The associated log-likelihood value becomes -58.1103. Based on the likelihood ratio test, the corresponding $p < 0.001$, that implies the strong rejection of the null hypothesis.

Figure 1 provides a plot of the predicted survival function and the underlying Kaplan-Meier estimator. The two curves are very close indicating that the model provides an excellent fit to the data. In Figure 2, we have provided a graph of the predicted hazard function. We note that the hazard function in the third interval, (350, ∞), is much steeper, indicating the increased risk over time at the higher altitude. We also see that the effect of the increased altitude is not instantaneous; as expected, there is a significant lag before the effects are observed.

For comparison purposes, we also fit the Khamis and Higgins (1988) model to this data set. The details about how to obtain the MLEs of the unknown parameters are available in Appendix D. In this case the MLEs of α_1, α_2, λ_1 and λ_2 are 2.7926, 1.4838, 0.0820 and 0.9242, respectively. The associated log-likelihood value is -58.2218 and based on both the AIC or BIC, we note that the CRM provides a better fit than the Khamis and Higgins model.

Now we would like to explore how the Bayesian method performs in this case for the proposed model. In this case we have assumed $\tau_2 = 350$. Since we do not have any prior information about the unknown parameters, we have assumed non-informative priors. As it has been suggested in the literature, see Congdon (2006), we have taken the hyper-parameters as follows: $a_1 = b_1 = a_2 = b_2 = c_1 = d_1 = c_2 = d_2 = 0.0001$. We have generated from the full conditionals as described in the previous section using the ratio of uniforms method. We generated 12000 samples and remove the first 2000 as burn-in period. Based on the generated samples we compute the posterior means and the posterior standard deviations (reported within parentheses) of all the parameters and they are as follows: $\widehat{\alpha}_1 = 3.4617$ (0.2817), $\widehat{\alpha}_2 = 3.1567$ (0.2345), $\widehat{\lambda}_1 = 0.1497$ (0.0173) and $\widehat{\lambda}_2 = 0.1116$ (0.0149). The KS statistic is 0.0826 and the associated p value is 0.4781. Therefore, in this case it is observed that based on the Kolmogorov-Smirnov distance criterion, the MLEs perform slightly better than the Bayes estimates.

7 Conclusions

The Weibull CRM is an extremely flexible model for step-stress experiments. While we have considered the case of a simple step-stress experiment, the model may be easily extended to the case of multiple stress levels assuming a functional relationship between the model parameters and the stress levels. The continuous hazard function is appropriate for many applications in survival analysis and reliability. The CRM also provides a more realistic representation of the effects of changes in stress on the hazard function, addressing a major limitation of existing models in the literature. The model also reduces to the model proposed by Khamis and Higgins (1998) in the limiting case, namely $\tau_2 \to \tau_1$.

The lag parameter provides the researcher with valuable insight into the effects of stress on the hazard function. If the experiment involved covariates, it is possible to incorporate their effects on the lag. Given the relative paucity of realistic models for step-stress experiments, we believe this research addresses a major gap in the current literature.

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Appendix A: Expressions of $C_0, C_1, C_2, D_0, D_1, D_2$

$$C_0 = c_1^2(\alpha_1)\sum_{i=1}^{r} t_i^{2\alpha_1} + \sum_{i=r+1}^{r+m}(c_1(\alpha_1)\tau_1^{\alpha_1} + t_i - \tau_1)^2 + \sum_{i=r+m+1}^{n}(c_1(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2 - \tau_1) + c_3(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))^2$$

$$C_1 = c_1(\alpha_1)c_2(\alpha_1)\sum_{i=1}^{r} t_i^{2\alpha_1} + \sum_{i=r+1}^{r+m}(c_1(\alpha_1)\tau_1^{\alpha_1} + (t_i - \tau_1))(c_2(\alpha_1)\tau_1^{\alpha_1} + (t_i^2 - \tau_1^2)/2) +$$

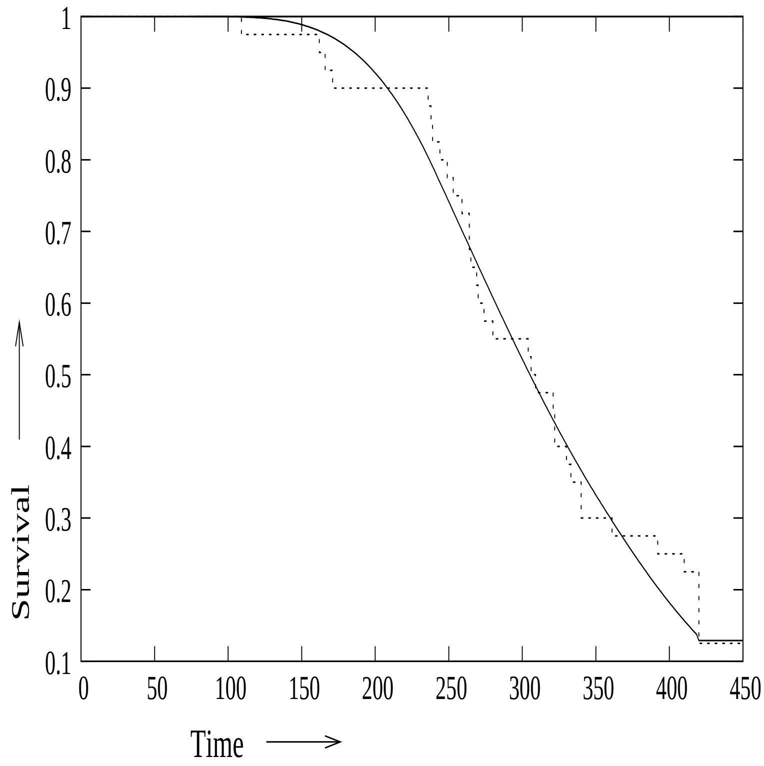

Figure 1: The solid line represents the predicted survival function and the dotted line represents the Kaplan-Meier survival function estimator of the DCS data.

$$\sum_{i=r+m+1}^{n} (c_1(\alpha_1)\tau_1^{\alpha_1} + c_3(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))(c_2(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2^2 - \tau_1^2)/2 + c_4(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))$$

$$\begin{aligned}
C_2 &= c_1(\alpha_1)\sum_{i=1}^{r} d_i t_i^{\alpha_1} + \sum_{i=r+1}^{r+m} d_i(c_1(\alpha_1)\tau_1^{\alpha_1} + t_i - \tau_1) + \\
&\quad \sum_{i=r+m+1}^{n} d_i(c_1(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2 - \tau_1) + c_3(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2})) \\
D_0 &= c_1(\alpha_1)c_2(\alpha_1)\sum_{i=1}^{r} t_i^{2\alpha_1} + \sum_{i=r+1}^{r+m} (c_1(\alpha_1)\tau_1^{\alpha_1} + (t_i - \tau_1))(c_2(\alpha_1)\tau_1^{\alpha_1} + (t_i^2 - \tau_1^2)/2) + \\
&\quad \sum_{i=r+m+1}^{n} (c_1(\alpha_1)\tau_1^{\alpha_1} + c_3(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))(c_2(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2^2 - \tau_1^2)/2 + c_4(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2})) \\
D_1 &= c_2^2(\alpha_1)\sum_{i=1}^{r} t_i^{2\alpha_1} + \sum_{i=r+1}^{r+m} (c_2(\alpha_1)\tau_1^{\alpha_1} + (t_i^2 - \tau_1^2)/2)^2 + \\
&\quad \sum_{i=r+m+1}^{n} (c_2(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2^2 - \tau_1^2)/2 + c_4(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))^2
\end{aligned}$$

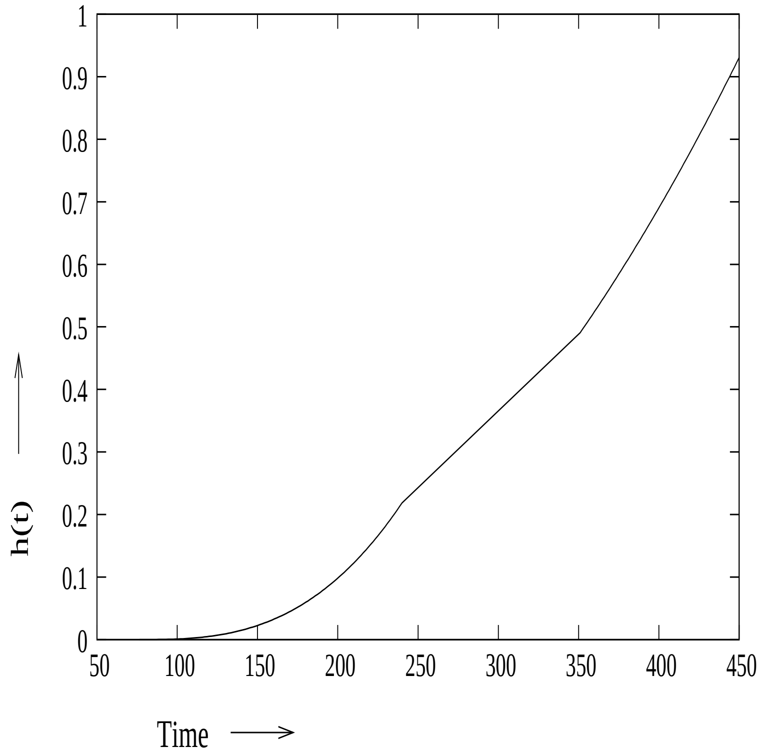

Figure 2: Hazard function of the predicted model of the DCS data.

$$
\begin{aligned}
D_2 &= c_2(\alpha_1)\sum_{i=1}^{r} d_i t_i^{\alpha_1} + \sum_{i=r+1}^{r+m} d_i(c_2(\alpha_1)\tau_1^{\alpha_1} + (t_i^2 - \tau_1^2)/2) + \\
&\quad \sum_{i=r+m+1}^{n} d_i(c_2(\alpha_1)\tau_1^{\alpha_1} + k(\tau_2^2 - \tau_1^2)/2 + c_4(\alpha_2)(t_i^{\alpha_2} - \tau_2^{\alpha_2}))
\end{aligned}
$$

Appendix B: Normal Equations

$$
\begin{aligned}
l(\beta_0, \beta_1, \alpha) &= r\ln(\beta_0 + \beta_1\tau_1) + (\alpha - 1)\sum_{i=1}^{r}(\ln t_i - \ln \tau_1) - (\beta_0 + \beta_1\tau_1) \times \frac{1}{\alpha}\sum_{i=1}^{r}\frac{t_i^{\alpha}}{\tau_1^{\alpha-1}} \\
&\quad + \sum_{i=r+1}^{r+m}\ln(\beta_0 + \beta_1 t_i) - m(\beta_0 + \beta_1\tau_1)\frac{\tau_1}{\alpha} - \beta_0\sum_{i=r+1}^{r+m}(t_i - \tau_1) - \frac{\beta_1}{2}\sum_{i=r+1}^{r+m}(t_i^2 - \tau_1^2) \\
&\quad + k\ln(\beta_0 + \beta_1\tau_2) + (\alpha - 1)\sum_{i=r+m+1}^{r+m+k}(\ln t_i - \ln \tau_2) - (\beta_0 + \beta_1\tau_2) \times \frac{1}{\alpha}\sum_{i=r+m+1}^{r+m+k}\frac{t_i^{\alpha}}{\tau_2^{\alpha-1}} \\
&\quad - k(\beta_0 + \beta_1\tau_1)\frac{\tau_1}{\alpha} - k\beta_0(\tau_2 - \tau_1) - \frac{k\beta_1}{2}(\tau_2^2 - \tau_1^2) + k(\beta_0 + \beta_1\tau_2)\frac{\tau_2}{\alpha}
\end{aligned}
$$

$$-s(\beta_0+\beta_1\tau_2)\frac{T^\alpha}{\alpha\tau_2^{\alpha-1}}. \tag{25}$$

The MLEs of the unknown parameters under H_{01}, can be obtained by maximizing (25) with respect to the unknown parameters. The normal equations can be written as follows:

$$\begin{aligned}
\frac{\partial l(\beta_0,\beta_1,\alpha)}{\partial\beta_0} &= \frac{r}{\beta_0+\beta_1\tau_1}-\frac{1}{\alpha}\sum_{i=1}^{r}\frac{t_i^\alpha}{\tau_1^{\alpha-1}}+\sum_{i=r+1}^{r+m}\frac{1}{\beta_0+\beta_1 t_i}-\frac{(m+k)\tau_1}{\alpha}-\sum_{i=r+1}^{r+m}(t_i-\tau_1)\\
&+\frac{k}{\beta_0+\beta_1\tau_2}-\frac{1}{\alpha}\sum_{i=n-k+1}^{n}\frac{t_i^\alpha}{\tau_2^{\alpha-1}}-k(\tau_2-t_r)+\frac{k\tau_2}{\alpha}-\frac{sT^\alpha}{\alpha\tau_2^{\alpha-1}}=0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l(\beta_0,\beta_1,\alpha)}{\partial\beta_1} &= \frac{r\tau_1}{\beta_0+\beta_1\tau_1}-\frac{1}{\alpha}\sum_{i=1}^{r}\frac{t_i^\alpha}{\tau_1^{\alpha-2}}+\sum_{i=r+1}^{r+m}\frac{t_i}{\beta_0+\beta_1 t_i}-\frac{m\tau_1^2}{\alpha}-\frac{1}{2}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2)\\
&+\frac{k\tau_2}{\beta_0+\beta_1\tau_2}-\frac{1}{\alpha}\sum_{i=r+m+1}^{r+m+k}\frac{t_i^\alpha}{\tau_2^{\alpha-2}}-\frac{k\tau_1^2}{\alpha}-\frac{k}{2}(\tau_2^2-\tau_1^2)+\frac{k\tau_2^2}{\alpha}\\
&-\frac{sT^\alpha}{\alpha\tau_2^{\alpha-2}}=0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l(\beta_0,\beta_1,\alpha)}{\partial\alpha} &= \sum_{i=1}^{r}(\ln t_i-\ln\tau_1)-\frac{\beta_0+\beta_1\tau_1}{\alpha}\sum_{i=1}^{r}\frac{t_i^\alpha}{\tau_1^{\alpha-1}}\left(\ln t_i-\ln\tau_1-\frac{1}{\alpha}\right)+(\beta_0+\beta_1\tau_1)\frac{m\tau_1}{\alpha^2}\\
&\sum_{i=r+m+1}^{r+m+k}(\ln t_i-\ln\tau_2)-\frac{\beta_0+\beta_1\tau_2}{\alpha}\sum_{i=r+m+1}^{r+m+k}\frac{t_i^\alpha}{\tau_2^{\alpha-1}}\left(\ln t_i-\ln\tau_2-\frac{1}{\alpha}\right)\\
&+(\beta_0+\beta_1\tau_1)\frac{k\tau_1}{\alpha^2}-k(\beta_0+\beta_1\tau_2)\frac{\tau_2}{\alpha^2}-\frac{(\beta_0+\beta_1\tau_2)sT^\alpha}{\alpha\tau_2^{\alpha-1}}\left(\ln T-\ln\tau_2-\frac{1}{\alpha}\right)=0.
\end{aligned}$$

We need to solve the above three equations simultaneously to compute the MLEs of β_0, β_1 and α under H_0. Once the MLEs of β_0, β_1 and α are obtained, the MLEs of λ_1 and λ_2 can be easily obtained as before.

Appendix C: Full Conditionals

$$\begin{aligned}
\pi(\alpha_1|\lambda_1,\alpha_2,\lambda_2) &\propto \alpha_1^{a_1+r-1}e^{-b_1\alpha_1}\prod_{i=1}^{r}t_i^{\alpha_1-1}e^{-\lambda_1\sum_{i=1}^{r}t_i^{\alpha_1}}\times\\
&\prod_{i=r+1}^{r+m}\left\{(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}-\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1})+(\alpha_2\lambda_2\tau_2^{\alpha_2-1}-\alpha_1\lambda_1\tau_1^{\alpha_1-1})t_i\right\}\times
\end{aligned}$$

$$e^{-m\lambda_1\tau_1^{\alpha_1-1}-(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1})\sum_{i=r+1}^{r+m}(t_i-\tau_1)/(\tau_2-\tau_1)} \times$$
$$e^{\alpha_1\lambda_1\tau_1^{\alpha_1-1}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2)/(2(\tau_2-\tau_1))-k\lambda_1\tau_1^{\alpha_1-1}-k\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}+k\alpha_1\lambda_1\tau_1^{\alpha_1-1}(\tau_2+\tau_1)}.$$

$$\begin{aligned}\pi(\lambda_1|\alpha_1,\alpha_2,\lambda_2) \propto\; & \lambda_1^{r+c_1-1}e^{-\lambda_1(\sum_{i=1}^{r}t_i^{\alpha_1}+d_1)} \times \\ & \prod_{i=r+1}^{m}\left\{(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}-\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1})+(\alpha_2\lambda_2\tau_2^{\alpha_2-1}-\alpha_1\lambda_1\tau_1^{\alpha_1-1})t_i\right\} \times \\ & e^{-m\lambda_1\tau_1^{\alpha_1-1}-(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1})\sum_{i=r+1}^{r+m}(t_i-\tau_1)/(\tau_2-\tau_1)} \times \\ & e^{\alpha_1\lambda_1\tau_1^{\alpha_1-1}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2)/(2(\tau_2-\tau_1))-k\lambda_1\tau_1^{\alpha_1-1}-k\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}+k\alpha_1\lambda_1\tau_1^{\alpha_1-1}(\tau_2+\tau_1)}.\end{aligned}$$

$$\begin{aligned}\pi(\alpha_2|\alpha_1,\lambda_1,\lambda_2) \propto\; & \prod_{i=r+1}^{m}\left\{(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}-\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1})+(\alpha_2\lambda_2\tau_2^{\alpha_2-1}-\alpha_1\lambda_1\tau_1^{\alpha_1-1})t_i\right\} \times \\ & e^{\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1}\sum_{i=r+1}^{r+m}(t_i-\tau_1)/(\tau_2-\tau_1)-\alpha_2\lambda_2\tau_2^{\alpha_2-1}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2)/(2(\tau_2-\tau_1))} \times \\ & \alpha_2^{k+a_2-1}e^{-b_2\alpha_2+k\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1}-k\alpha_2\lambda_2\tau_2^{\alpha_2-1}(\tau_2+\tau_1)/2+k\lambda_2\tau_2^{\alpha_2}-\lambda_2\sum_{i=r+m+1}^{r+m+k}t_i^{\alpha_2}-s\lambda_2T^{\alpha_2}} \times \\ & \prod_{i=r+m+1}^{r+m+k}t_i^{\alpha_2-1}.\end{aligned}$$

$$\begin{aligned}\pi(\lambda_2|\alpha_1,\lambda_1,\alpha_2) \propto\; & \lambda_2^{k+c_2-1}e^{-d_2\lambda_2} \times \\ & \prod_{i=r+1}^{m}\left\{(\alpha_1\lambda_1\tau_2\tau_1^{\alpha_1-1}-\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1})+(\alpha_2\lambda_2\tau_2^{\alpha_2-1}-\alpha_1\lambda_1\tau_1^{\alpha_1-1})t_i\right\} \times \\ & e^{\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1}\sum_{i=r+1}^{r+m}(t_i-\tau_1)/(\tau_2-\tau_1)-\alpha_2\lambda_2\tau_2^{\alpha_2-1}\sum_{i=r+1}^{r+m}(t_i^2-\tau_1^2)/(2(\tau_2-\tau_1))} \times \\ & e^{-k\alpha_2\lambda_2\tau_2^{\alpha_2-1}(\tau_2+\tau_1)/2+k\alpha_2\lambda_2\tau_1\tau_2^{\alpha_2-1}+k\lambda_2\tau_2^{\alpha_2}-\lambda_2\sum_{i=r+m+1}^{r+m+k}t_i^{\alpha_2}-s\lambda_2T^{\alpha_2}}.\end{aligned}$$

Appendix D: Computation of the MLEs for Khamis and Higgins Model

In case of Khamis and Higgins (1998) model, the hazard function takes the following form:

$$h(t) = \begin{cases} \alpha_1\lambda_1 t^{\alpha_1-1} & \text{if} \quad 0 < t \le \tau \\ \alpha_2\lambda_2 t^{\alpha_2-1} & \text{if} \quad t < \tau < \infty. \end{cases} \tag{26}$$

Hence the cumulative hazard function corresponds to (26) becomes

$$H(t) = \begin{cases} \lambda_1 t^{\alpha_1} & \text{if} \quad 0 < t \le \tau \\ \lambda_1 \tau^{\alpha_1} + \lambda_2 (t^{\alpha_2} - \tau^{\alpha_2}) & \text{if} \quad t < \tau < \infty. \end{cases} \tag{27}$$

The survival function $S(t)$ associated with (27) is given by

$$S(t) = e^{-H(t)} = \begin{cases} e^{-\lambda_1 t^{\alpha_1}} & \text{if} \quad 0 < t \le \tau \\ e^{-\lambda_1 \tau^{\alpha_1} - \lambda_2 (t^{\alpha_2} - \tau^{\alpha_2})} & \text{if} \quad t < \tau < \infty. \end{cases} \tag{28}$$

Hence, the associated PDF of (28) has the following form:

$$f(t) = -\frac{d}{dt} S(t) = \begin{cases} \alpha_1 \lambda_1 t^{\alpha_1 - 1} e^{-\lambda_1 t^{\alpha_1}} & \text{if} \quad 0 < t \le \tau \\ \alpha_2 \lambda_2 t^{\alpha_2 - 1} e^{-\lambda_2 t^{\alpha_2}} \times e^{-\lambda_1 \tau^{\alpha_1} + \lambda_2 \tau^{\alpha_2}} & \text{if} \quad t < \tau < \infty. \end{cases} \tag{29}$$

Therefore, based on the observations

$$t_1 < \ldots < t_r < \tau < t_{r+1} < \ldots < t_{r+k} < T,$$

the likelihood function becomes

$$\begin{aligned} L_{KH}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) &= \alpha_1^r \lambda_1^r \prod_{i=1}^{r} t_i^{\alpha_1 - 1} e^{-\lambda_1 \sum_{i=1}^{r} t_i^{\alpha_1}} \times \alpha_2^k \lambda_2^k e^{(n-r)(-\lambda_1 \tau^{\alpha_1} + \lambda_2 \tau^{\alpha_2})} \times \\ & \prod_{i=r+1}^{r+k} t_i^{\alpha_2 - 1} e^{-\lambda_2 \sum_{i=r+1}^{r+k} t_i^{\alpha_2}} \times e^{-(n-r-k)\lambda_2 T^{\alpha_2}}. \end{aligned} \tag{30}$$

The log-likelihood function corresponds to (30) can be written as

$$l_{KH}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = l_{KH1}(\alpha_1, \lambda_1) + l_{KH2}(\alpha_2, \lambda_2), \tag{31}$$

where

$$\begin{aligned} l_{KH1}(\alpha_1, \lambda_1) &= r \ln \alpha_1 + r \ln \lambda_1 + (\alpha_1 - 1) \sum_{i=1}^{r} \ln t_i - \lambda_1 \left(\sum_{i=1}^{r} t_i^{\alpha_1} + (n-r)\tau^{\alpha_1} \right) \\ l_{KH2}(\alpha_2, \lambda_2) &= k \ln \alpha_2 + k \ln \lambda_2 + (\alpha_2 - 1) \sum_{i=r+1}^{r+k} \ln t_i - \\ & \lambda_2 \left(\sum_{i=r+1}^{r+k} t_i^{\alpha_2} + (n-r-k)T^{\alpha_2} - (n-r)\tau^{\alpha_2} \right). \end{aligned}$$

Hence, the MLEs of $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ can be obtained by maximizing (31) with respect to the unknown parameters. They are equivalent in maximizing $l_{KH1}(\alpha_1, \lambda_1)$ with respect to α_1 and λ_1, and maximizing $l_{KH2}(\alpha_2, \lambda_2)$ with respect to α_2 and λ_2. Both the maximization can be performed by profile likelihood method.

The MLEs of α_1 and λ_1 based on Khamis and Higgins model can be obtained as follows. For a given α_1, the MLE of λ_1 for $r_1 > 0$, can be obtained as

$$\widehat{\lambda}_{1KH}(\alpha_1) = \frac{r}{\sum_{i=1}^{r} t_i^{\alpha_1} + (n-r)\tau^{\alpha_1}},$$

and the MLE of α_1 can be obtained by maximizing $l_{KH1}(\alpha_1, \widehat{\lambda}_{1KH}(\alpha_1))$ with respect to α_1. Similarly, for a given α_2, the MLE of λ_2, for $k > 0$, can be obtained as

$$\widehat{\lambda}_{2KH}(\alpha_1) = \frac{k}{\sum_{i=r+1}^{r+k} t_i^{\alpha_1} + (n-r-k)T^{\alpha_2} - (n-r)\tau^{\alpha_2}},$$

and the MLE of α_2 can be obtained by maximizing $l_{KH2}(\alpha_2, \widehat{\lambda}_{2KH}(\alpha_2))$ with respect to α_2.

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