Interval Estimation of the Unknown Exponential Parameter Based on Time Truncated Data

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Abstract

In this paper we consider the statistical inference of the unknown parameter of an exponential distribution based on the time truncated data. The time truncated data occurs quite often in the reliability analysis for type-I or hybrid censoring cases. All the results available today are based on the conditional argument that at least one failure occurs during the experiment. In this paper we provide some inferential results based on the unconditional argument. We extend the results for some two-parameter distributions also.

Key Words and Phrases: Exponential distribution; type-I censoring; hybrid censoring; conjugate prior; confidence interval; credible interval.

AMS Subject Classifications: 62F10, 62F03, 62H12.

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1 Introduction

Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter λ, then it has the following probability density function (PDF)

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{1}$$

Here $\lambda \in (0, \infty)$ is the natural parameter space. Since for $\lambda = 0$, $f(x;\lambda)$ as defined in (1) is not a proper PDF, we are not including the point 0 in the parameter space. Suppose n items are tested and their ordered failure times are denoted by $x_{1:n} < x_{2:n} < \ldots < x_{n:n}$. If the experiment is stopped at a prefixed time T then it results in a simple type-I censoring case. Let there be D failures in $[0, T]$. Then $x_{1:n} < x_{2:n} < \ldots < x_{D:n} \leq T$ and $T < x_{D+1:n} < \ldots x_{n:n}$, although $x_{D+1:n}, \ldots, x_{n:n}$ are not observed. Here D is a random variable that can take the values $0, 1, \ldots, n$. The main aim of this note is to draw inference on λ, based on D observations.

This is an old problem and Bartholomew (1963) was the first to consider it. He considered the following form of the exponential PDF;

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{2}$$

He considered $[0, \infty)$ as the parameter space of θ. In this case the maximum likelihood estimator (MLE) of θ does not exist when $D = 0$. Hence all the inferences related to θ are based on the conditional argument that $D \geq 1$. Later on a series of papers starting with Chen and Bhattacharyya (1988) and then continuing with Gupta and Kundu (1998) and Childs et al. (2003) considered type-I hybrid censoring case for the model (2), and obtained the exact inference on θ based on the conditional argument that $D \geq 1$.

In case of type-I censoring, $P(D = 0) = exp(-n\lambda T)$ and this probability can be quite high for small value of $n\lambda T$. The natural question here is whether it is possible to draw any inference on λ, when $D = 0$. The second aim of the study is to determine if there exists any significant difference between the conditional and unconditional inference. For example, in this paper it has been shown that it is possible to construct an exact $100(1-\alpha)\%$ confidence interval of λ even when $D = 0$. Following the approach of Chen and Bhattacharyya (1998), an exact $100(1-\alpha)\%$ confidence interval of λ also can be obtained based on the conditional MLE of λ, conditioning on the event that $D \geq 1$. The question is whether the lengths of the confidence intervals based on the two different approaches are significantly different or not. We perform some simulation experiments to compare the confidence intervals based on the two different methods. We further obtain the Bayes estimate and the associated credible interval of the unknown parameter based on the non-informative prior. Finally the results are extended for the two-parameter exponential, Weibull and generalized exponential distributions also.

Rest of the paper is organized as follows. In Section 2, we provide two different constructions of confidence intervals and the Bayesian credible intervals. The simulation results for confidence and credible intervals of the parameter λ are provided in Section 3. In Section 4, we extend the results for the two parameter exponential, Weibull and generalized exponential distributions, and finally we conclude the paper in Section 5.

2 Construction of Confidence and Credible Intervals

In this section we proceed to construct the confidence and credible intervals of the parameter of interest λ, based on a new estimator of λ and the posterior distribution of λ, respectively.

2.1 Confidence interval (CI)

Based on the observations $x_{1:n} < \ldots < x_{D:n}$ from the model (1), the likelihood function becomes

$$L(\lambda|x_{1:n},\ldots,x_{D:n}) = \begin{cases} e^{-n\lambda T}, & \text{if } D = 0, \\ \frac{n!}{(n-D)!}\lambda^D e^{-\lambda\left[\sum_{i=1}^{D} x_{i:n}+(n-D)T\right]}, & \text{if } D > 0. \end{cases} \tag{3}$$

Note that the MLE of λ does not exist when $D = 0$. It exists only if $D > 0$ and it is given by

$$\widehat{\lambda}_{MLE} = \frac{D}{\sum_{i=1}^{D} x_{i:n} + (n-D)T}.$$

Now based on $(D, \sum_{i=1}^{D} x_{i:n})$, the joint sufficient statistic for λ, we define a new estimator of λ for all $D \geq 0$ as follows

$$\widehat{\lambda} = \frac{D}{\sum_{i=1}^{D} x_{i:n} + (n-D)T}. \tag{4}$$

Note that, $\widehat{\lambda}$ is equal to $\widehat{\lambda}_{MLE}$ only when $D > 0$. Now we provide the exact distribution of $\widehat{\lambda}$, which will be useful in constructing an exact confidence interval of λ.

Theorem 1 *The distribution of $\widehat{\lambda}$ for $x \geq 0$, can be written as,*

$$P(\widehat{\lambda} \leq x) \quad = \quad \begin{cases} e^{-n\lambda T}, & \text{if } x = 0, \\ e^{-n\lambda T} + \sum_{d=1}^{n}\sum_{k=0}^{d} C_{k,d}\Gamma(d, A_d(x, T_{k,d})), & \text{if } x > 0, \end{cases} \tag{5}$$

where

$$C_{k,d} = (-1)^k \binom{n}{d}\binom{d}{k} e^{-\lambda T(n-d+k)}, \quad T_{k,d} = (n-d+k)T/d,$$

$$A_k(x,a) = \begin{cases} \lambda k\left(\frac{1}{x} - a\right), & \text{if } x < \frac{1}{a}, \\ 0, & \text{if } x \geq \frac{1}{a}, \end{cases}$$

and $\Gamma(a,z) = \dfrac{1}{\Gamma(a)}\displaystyle\int_z^{\infty} t^{a-1}e^{-t}dt$, is the incomplete gamma function.

Proof: For $x \geq 0$, $P(\widehat{\lambda} \leq x)$ can be written as,

$$\begin{aligned} P(\widehat{\lambda} \leq x) &= P(\widehat{\lambda} \leq x, D = 0) + P(\widehat{\lambda} \leq x, D > 0) \\ &= P(\widehat{\lambda} \leq x | D = 0) P(D = 0) + P(\widehat{\lambda} \leq x | D > 0) P(D > 0) \\ &= \begin{cases} e^{-n\lambda T}, & \text{if } x = 0, \\ e^{-n\lambda T} + \sum_{d=1}^{n} \sum_{k=0}^{d} C_{k,d} \Gamma(d, A_d(x, T_{k,d})), & \text{if } x > 0. \end{cases} \end{aligned} \tag{6}$$

Note that the expression of $P(\widehat{\lambda} \leq x | D > 0)$ can be obtained by using the moment generating function approach. Refer to Corollary 2.2 of Childs et al. (2003) or equation (8) of Gupta and Kundu (1998) for it. It is possible to obtain (5) in terms of the chi-square integral as in Bartholomew (1963). The distribution of $\widehat{\lambda}$ is a mixture of discrete and continuous distributions. The following corollary comes from Theorem 1.

Corollary 1

$$P(\widehat{\lambda} = 0) = e^{-n\lambda T}$$

and for $x > 0$, the PDF of $\widehat{\lambda}$ is given by

$$f_{\widehat{\lambda}}(x) = \frac{1}{x^2} \sum_{d=1}^{n} \sum_{k=0}^{d} C_{k,d} \;\; g\left(\frac{1}{x} - T_{k,d}; \lambda d, d\right), \qquad x \geq \frac{1}{nT}, \tag{7}$$

where

$$g(x; \alpha, p) = \begin{cases} \frac{\alpha^p}{\Gamma(p)} x^{p-1} e^{-\alpha x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Now we consider the construction of an exact $100(1-\alpha)\%$ confidence interval of λ. We need the following lemma for further development.

Lemma 1 *For a fixed $b \geq 0$, $P_\lambda(\widehat{\lambda} \leq b)$ is a monotonically decreasing function of λ.*

Proof: The proof can be obtained along the same line as the proof of the three monotonic lemmas by Balakrishnan and Iliopoulos (2009).

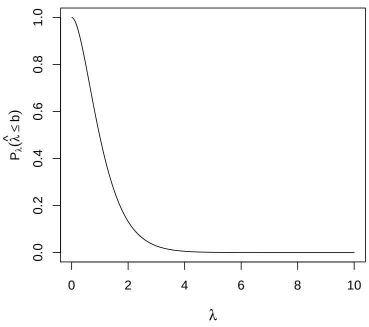


Figure 1: The plot of $P_\lambda(\widehat{\lambda} \le b)$ as a function of λ, when b is fixed.

A graphical plot (Figure 1) of $P_\lambda(\widehat{\lambda} \le b)$ as a function of λ for a fixed value of $b \ge 0$ is provided for a visual illustration. Here we have taken $n = 5$, $T = 0.5$ and $b = 1$. We now provide an exact $100(1-\alpha)\%$ confidence interval of λ for different values of D.

Case-I: Construction of CI when $D = 0$:

Since $P_\lambda(D = 0) = exp(-n\lambda T)$ and it is a decreasing function of λ, a one sided $100(1-\alpha)\%$ confidence interval of λ can be obtained as $A = \{\lambda : P_\lambda(D = 0) \ge 1 - \alpha\}$. Hence,

$$A = [(0, -\log(1-\alpha)/(nT)].$$

Case-II: Construction of CI when $D > 0$:

A symmetric $100(1-\alpha)\%$ confidence interval, (λ_L, λ_U), of λ can be obtained by solving the following two non-linear equations

$$1 - \frac{\alpha}{2} = P_{\lambda_L}(\widehat{\lambda} \le \widehat{\lambda}_{obs}) \quad \text{and} \quad \frac{\alpha}{2} = P_{\lambda_U}(\widehat{\lambda} \le \widehat{\lambda}_{obs}). \tag{9}$$

These non-linear equations are to be solved using standard non-linear solver *viz.* Newton Rapshon, bisection method etc.

2.2 Credible interval (CRI)

In this subsection we will discuss constructing a $100(1-\alpha)\%$ credible interval of λ based on the conjugate prior on λ. It is assumed that λ has a natural gamma prior with the shape and scale parameters as $a > 0$ and $b > 0$, respectively with the following PDF

$$\pi(\lambda) = \begin{cases} \frac{b^a}{\Gamma a}\lambda^{a-1}e^{-\lambda b}, & \text{if } \lambda > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Therefore the posterior density function of λ becomes

$$\pi(\lambda|data) \propto \lambda^{a+D-1}e^{-\lambda\left[b+\sum_{i=1}^{D} x_{i:n}+(n-D)T\right]}, \quad \lambda > 0. \tag{11}$$

Hence the Bayes estimate of λ under the squared error loss function becomes

$$\widehat{\lambda}_{Bayes} = \frac{a+D}{b+\sum_{i=1}^{D} x_{i:n} + (n-D)T}. \tag{12}$$

The associated $100(1-\alpha)\%$ symmetric credible interval of λ can be obtained as $(\lambda_{LB}, \lambda_{UB})$, where λ_{LB} and λ_{UB} can be obtained as the solutions of

$$\Gamma(a+D, \lambda_{LB}(b+S)) = \left(1-\frac{\alpha}{2}\right)\Gamma(a+D) \quad \text{and} \quad \Gamma(a+D, \lambda_{UB}(b+S)) = \Gamma(a+D)\,\frac{\alpha}{2}, \tag{13}$$

respectively. Here $\Gamma(a+D, x)$ is the incomplete gamma function and $S = \sum_{i=1}^{D} x_{i:n} + (n-D)T$. Observe that, when $a = b = 0$, the Bayes estimate of λ matches with $\widehat{\lambda}$, although it is an improper prior. Therefore, the comparison of the confidence intervals based on (9) and the credible interval based on (13) makes sense, although their interpretations are different. When $D = 0$, the posterior density function of λ becomes improper for $a = b = 0$. Due to this reason we propose to use a proper prior with $a = b \approx 0$ as suggested by Congdon (2014).

3 Numerical Comparisons

In this section we present some simulation results to compare the performances of the two confidence intervals and the corresponding credible intervals. We compare the performances in terms of the average lengths and the coverage percentages. Different values of n, λ and T are taken. We consider $n = 5$, 10, 15, 20, $\lambda = 0.5$, 1.0, 2.0 and $T = 1$, 2. For Bayesian inference we have taken $a = b = 0.001$. All the results are based on 5,000 replications. The results are reported in Tables 1 to 3.

The following points have been revealed from these simulation experiments. In all these cases it is observed that biases and mean squared errors (MSEs) decrease as sample size increases, T increases or λ increases as expected. Now comparing the confidence intervals and credible intervals in Tables 1 to 3 it is clear that for small sample sizes, small λ and small T values, the confidence intervals based on unconditional distribution performs better that the credible intervals based on non-informative priors in terms of the coverage percentages. The coverage percentages of the confidence intervals based on unconditional distribution are very close to the nominal value (95%) in all cases. Although, as expected for large sample sizes they are almost equal. The confidence intervals based on the conditional distribution do not perform very well when the sample size is very small and λ is also small, although when the sample size is not very small it performs well. Performances of the two confidence intervals match as expected when sample size is large.

4 Some Related Models

In this section we consider some of the related two-parameter models, and construct an exact $100(1-\alpha)\%$ confidence set of the two parameters, when n items are tested and there is no observed failure during $[0, T]$.

Model 1: A two-parameter exponential distribution with the location parameter μ and scale parameter λ having the following PDF

$$f(x; \lambda, \mu) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & \text{if } x \geq \mu, \\ 0, & \text{if } x < \mu. \end{cases} \tag{14}$$

Here $\lambda > 0$ and $-\infty < \mu < \infty$. In this case $P_{\{\lambda,\mu\}}(D = 0) = e^{-n\lambda(T-\mu)}$. Hence, a $100(1-\alpha)\%$ confidence set of (μ, λ) can be obtained as $A = \{(\mu, \lambda); \lambda > 0, -\infty < \mu < \infty, P_{\{\lambda,\mu\}}(D = 0) \geq (1-\alpha)\}$. Therefore,

$$A = \{(\mu, \lambda); \lambda > 0, -\infty < \mu < T, \lambda(T-\mu) \leq -\ln(1-\alpha)/n\}.$$

Model 2: Let us consider now a two-parameter Weibull distribution with the shape parameter β and scale parameter λ which has the following PDF

$$f(x; \lambda, \beta) = \begin{cases} \beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{15}$$

Here $\beta > 0$ and $\lambda > 0$. In this case $P_{\{\lambda,\beta\}}(D = 0) = e^{-n\lambda T^{\beta}}$. A $100(1-\alpha)\%$ confidence set of (λ, β), can be obtained as $A = \{(\lambda, \beta); \lambda > 0, \beta > 0, P_{\{\lambda,\beta\}}(D = 0) \geq (1-\alpha)\}$. Hence,

$$A = \{(\lambda, \beta); \lambda > 0, \beta > 0, \lambda T^{\beta} \leq -\ln(1-\alpha)/n\}.$$

Model 3: Similarly, if we consider two-parameter generalized exponential distribution which has the following PDF

$$f(x; \lambda, \beta) = \begin{cases} \beta\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\beta-1}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{16}$$

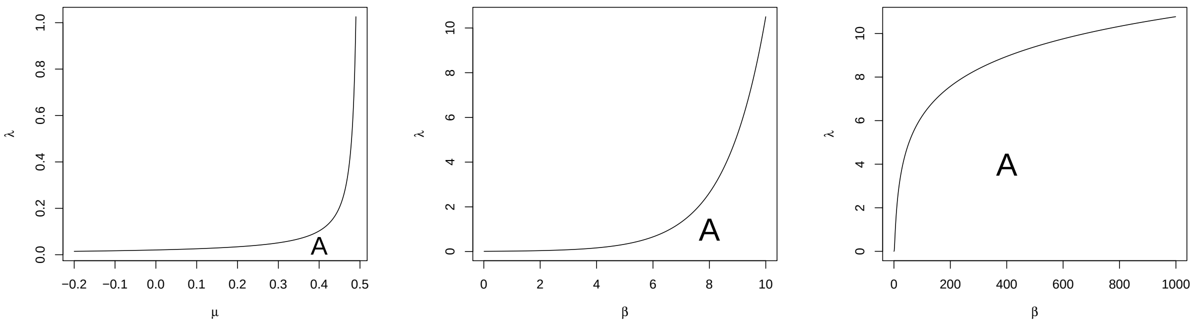
Here $\beta > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. Here, $P_{\{\lambda,\beta\}}(D = 0) = (1 - (1 - e^{-\lambda T})^{\beta})^{n}$. A $100(1-\alpha)\%$ confidence set of (λ, β), can be obtained as $A = \{(\lambda, \beta); \lambda > 0, \beta > 0, P_{\{\lambda,\beta\}}(D = 0) \geq (1-\alpha)\}$. Therefore,

$$A = \{(\lambda, \beta); \lambda > 0, \beta > 0, (1 - e^{-\lambda T})^{\beta} \leq 1 - (1-\alpha)^{1/n}\}.$$

Joint Confidence Set

Just for illustrative purposes, taking $n = 5$ and $T = 0.5$, we provide the joint $100(1-\alpha)\%$ confidence set as the shaded region of (i) (λ, μ) for two-parameter exponential distribution,

(ii) (β, λ) for two-parameter Weibull distribution and (iii) (β, λ) for two-parameter generalized exponential distribution, in Figure 2 when $D = 0$.



(a) Joint confidence set of (λ, μ) for the two-parameter exponential distribution.

(b) Joint confidence set of (β, λ) for the two-parameter Weibull distribution.

(c) Joint confidence set of (β, λ) for the two-parameter generalized exponential distribution.

Figure 2: Joint confidence sets of the parameters for different distributions. Here '**A**' denotes the required set.

5 Conclusions

In this paper we have considered the time truncated exponential distribution and provide the exact confidence interval of the unknown scale parameter based on the unconditional argument. All the existing results are based on the conditional argument and it does not provide any statistical inference of the unknown parameter when there is no observation. In this paper we have provided the inference on the scale parameter even when there is no observation. Simulation experiments are performed and it is observed that the proposed method works quite well even when the sample size is very small. The joint confidence sets have been provided for two-parameter exponential, Weibull and generalized exponential distributions also for time truncated case when there is no observation during the time period of the experiment. The results can be extended to the competing risks model also, as considered in Kundu, Kannan and Balakrishnan (2004).

Table 1: CI of λ based on unconditional distribution of $\widehat{\lambda}$ when T=1, 2

		T=1				T=2			
	n	Bias	MSE	CI	CP	Bias	MSE	CI	CP
	5	0.074	0.207	1.577	97.36	0.091	0.166	1.253	94.42
	10	0.033	0.080	1.041	94.10	0.038	0.055	0.825	94.62
λ=0.5	15	0.020	0.049	0.833	94.86	0.023	0.033	0.659	94.58
	20	0.015	0.035	0.716	94.86	0.016	0.023	0.565	94.96
	5	0.182	0.662	2.505	94.42	0.226	0.619	2.240	94.50
	10	0.076	0.221	1.649	94.62	0.092	0.181	1.434	94.60
λ=1.0	15	0.046	0.132	1.318	94.58	0.056	0.104	1.139	94.56
	20	0.032	0.092	1.130	94.98	0.040	0.071	0.974	94.86
	5	0.452	2.477	4.481	94.50	0.510	2.428	4.351	94.66
	10	0.183	0.724	2.869	94.60	0.214	0.688	2.742	94.42
λ=2.0	15	0.112	0.416	2.278	94.56	0.131	0.390	2.162	94.92
	20	0.081	0.284	1.948	94.86	0.094	0.260	1.843	94.92

Table 2: CI of λ based on the conditional distribution of $\widehat{\lambda}$ when T=1, 2

		T=1				T=2			
	n	Bias	MSE	CI	CP	Bias	MSE	CI	CP
	5	0.067	0.185	1.398	69.28	0.088	0.156	1.241	91.28
	10	0.034	0.072	1.037	92.74	0.038	0.055	0.826	94.62
λ=0.5	15	0.021	0.048	0.837	94.96	0.023	0.033	0.659	94.58
	20	0.016	0.035	0.718	94.88	0.016	0.023	0.565	94.96
	5	0.176	0.623	2.482	91.28	0.226	0.618	2.245	94.48
	10	0.076	0.219	1.652	94.62	0.092	0.181	1.435	94.60
λ=1.0	15	0.046	0.132	1.318	94.58	0.056	0.104	1.139	94.56
	20	0.032	0.092	1.130	94.98	0.040	0.071	0.974	94.86
	5	0.452	2.471	4.490	94.48	0.510	2.428	4.352	94.66
	10	0.184	0.724	2.869	94.60	0.214	0.688	2.742	94.42
λ=2.0	15	0.112	0.416	2.278	94.56	0.131	0.390	2.162	94.92
	20	0.081	0.284	1.948	94.86	0.094	0.260	1.843	94.92

Table 3: CRI of λ based on non-informative priors when T=1, 2

		T=1				T=2			
	n	Bias	MSE	CRI	CP	Bias	MSE	CRI	CP
	5	0.074	0.207	1.390	89.36	0.091	0.166	1.202	90.42
	10	0.033	0.080	0.985	92.36	0.038	0.055	0.807	93.78
λ=0.5	15	0.020	0.049	0.804	94.94	0.023	0.033	0.649	93.76
	20	0.015	0.035	0.697	93.28	0.016	0.023	0.559	94.28
	5	0.182	0.662	2.404	90.42	0.226	0.619	2.212	93.54
	10	0.076	0.221	1.614	93.78	0.092	0.181	1.424	94.04
λ=1.0	15	0.046	0.132	1.299	93.76	0.056	0.104	1.134	94.04
	20	0.032	0.092	1.117	94.28	0.040	0.071	0.970	94.58
	5	0.452	2.477	4.422	93.54	0.510	2.428	4.343	94.46
	10	0.183	0.724	2.849	94.04	0.214	0.688	2.738	94.26
λ=2.0	15	0.112	0.416	2.267	94.06	0.131	0.390	2.161	94.76
	20	0.081	0.284	1.941	94.58	0.094	0.260	1.842	94.80

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