On Bivariate Birnbaum-Saunders Distribution

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Abstract

Univariate Birnbaum-Saunders distribution has been used quite effectively to analyze positively skewed lifetime data. It has received considerable amount of attention in the last few years. In this paper, we study the bivariate Birnbaum-Saunders distribution from a reliability and dependence point of view. It is observed that the bivariate Birnbaum-Saunders distribution can be obtained as a Gaussian copula. It helps in deriving several dependency properties and also to compute several dependency measures of the bivariate Birnbaum-Saunders distribution. Further, we consider the estimation of the unknown parameters based on copula, and study their performances using Monte Carlo simulations. One data set has been analyzed for illustrative purposes. Finally we extend some of the results for multivariate Birnbaum-Saunders distribution also.

Key Words and Phrases: Birnbaum-Saunders distribution; copula; hazard gradient; total positivity; maximum likelihood estimators.

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1 Introduction

Birnbaum and Saunders (1969) proposed a two-parameter lifetime distribution for fatigue failure caused by cyclic loading. It was also assumed that the failure is due to development and growth of a dominant crack. Desmond (1985) later strengthened the physical justification for the use of this distribution by relaxing the assumption made by Birnbaum and Saunders (1969). Desmond (1986) also established the relationship between a Birnbaum-Saunders distribution and an inverse Gaussian distribution. Since the introduction of the model, considerable amount of work has been done related to different developments of the model, and for some recent work readers are referred to Chang and Tang (1993, 1994), Dupuis and Mills (1998), Rieck (1995, 1999), Ng et al. (2003, 2006), Zhang et al. (2016). An excellent exposition on Birnbaum-Saunders distribution can be found in Johnson et al. (1995).

The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable T, can be written as

$$F_T(t;\alpha,\beta) = \Phi(a(t;\alpha,\beta)); \quad t > 0, \tag{1}$$

where $\Phi(\cdot)$ is the CDF of a standard normal distribution function, $\alpha > 0$, $\beta > 0$, and

$$a(t;\alpha,\beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right). \tag{2}$$

The corresponding probability density function (PDF) becomes;

$$f_T(t;\alpha,\beta) = \phi(a(t;\alpha,\beta))A(t;\alpha,\beta), \tag{3}$$

where

$$A(t;\alpha,\beta) = \frac{d}{dt}a(t;\alpha,\beta) = \frac{1}{2\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right\} = \frac{t+\beta}{2\alpha\sqrt{\beta}t^{3/2}}. \tag{4}$$

Here the parameters α and β are the shape and scale parameters, respectively. From now on a Birnbaum-Saunders distribution with parameters α and β will be denoted by BS(α,β).

It is known that the PDF of a Birnbaum-Saunders distribution is unimodal, and the hazard function is also unimodal, see for example Gupta and Akman (1995) or Kundu et al. (2008). It can be easily seen that if we make the transformation

$$X = a(T; \alpha, \beta), \tag{5}$$

where $a(\cdot; \cdot, \cdot)$ is same as defined in (2), then X is normally distributed with mean zero and variance one.

Recently, Kundu et al. (2010) introduced a bivariate Birnbaum-Saunders (BVBS) distribution with five parameters by using the same monotone transformation as the univariate Birnbaum-Saunders distribution. They discussed different properties of the bivariate Birnbaum-Saunders distribution, and provided several inferential issues. The main aim of this paper is to re-visit bivariate Birnbaum-Saunders distribution and discuss several new properties. It is further observed that the bivariate Birnbaum- Saunders distribution can be obtained from a Gaussian copula with Birnbaum-Saunders marginals. We explore different dependency properties based on the copula structure and also strengthen some of the existing results. We obtain different dependency measures also. We provide a new two-step estimation procedure based on the copula structure and derive the asymptotic properties of these estimators. Extensive simulations are performed to see the effectiveness of the proposed method, and it is observed that the performances of the proposed estimators are quite comparable with the corresponding maximum likelihood estimators. Further, we provide the analysis of a real data set for illustrative purposes. Finally we extend some of the results to the multivariate case also.

Rest of the paper is organized as follows. In Section 2, we provide some new properties of a bivariate Gaussian copula. Different properties of a BVBS distribution are derived in Section 3. Two-step estimation procedure based on copula are provided in Section 4. Simulation results and data analysis are presented in Section 5. Multivariate Birnbaum-

Saunders distribution has been considered in Section 6, and finally we conclude the paper in Section 7.

2 Bivariate Gaussian Copula

To every bivariate distribution function $F_{X_1,X_2}(\cdot,\cdot)$, with continuous marginals, $F_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$, corresponds a unique function $C : [0,1] \times [0,1] \to [0,1]$, called a copula such that

$$F_{X_1,X_2}(x_1,x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)); \quad (x_1,x_2) \in (-\infty,\infty) \times (-\infty,\infty). \tag{6}$$

Conversely, it is possible to construct a bivariate distribution function having the desired marginal distribution functions and a chosen dependence structure, *i.e.* copula. We have the following relation

$$C(u,v) = F_{X_1,X_2}(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)). \tag{7}$$

The bivariate Gaussian copula is defined as follows;

$$C_G(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \phi_2(x,y;\rho)dxdy = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho). \tag{8}$$

Here $\phi(\cdot)$, $\Phi(\cdot)$ and $\Phi_2(\cdot)$ are same as defined before.

The bivariate Gaussian copula density can be obtained as

$$c_G(u,v;\rho) = \frac{\partial^2}{\partial u \partial v} C_G(u,v;\rho) = \frac{\phi_2(\Phi^{-1}(u), \Phi^{-1}(v);\rho)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \tag{9}$$

$$= \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v) - \rho^2(\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2)}{2(1-\rho^2)}\right). \tag{10}$$

It may be noted that the bivariate Gaussian copula cannot be obtained in explicit form, it has to computed numerically. To compute bivariate Gaussian copula one needs to compute $\Phi^{-1}(\cdot)$, and several very well known approximation exist which can be used quite effectively for computational purposes.

We recall that a non-negative function g defined in $\mathbb{R}^2$ is total positivity of order two, abbreviated by TP_2 if for all $x_1 < x_2$, $y_1 < y_2$, with $x, y \in \mathbb{R}$

$$g(x_1, y_1)g(x_2, y_2) \geq g(x_2, y_1)g(x_1, y_2). \tag{11}$$

If the equality (11) is reversed, it is called reverse rule of order two (RR_2).

The following result will be useful for future development, and it may have some independent interest also.

Result 1: The Gaussian copula density is (a) TP_2 for $0 < \rho < 1$, (b) RR_2 if $-1 < \rho < 0$.

Proof: To prove (a), we need to show that for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$, if $0 < a_1 < a_2 < 1$, and $0 < b_1 < b_2 < 1$, then for $0 < \rho < 1$,

$$c_G(a_1, b_1; \rho)c_G(a_2, b_2; \rho) \geq c_G(a_2, b_1; \rho)c_G(a_1, b_2; \rho). \tag{12}$$

Proving (12) is equivalent to proving that for all $-\infty < x_1 < x_2 < \infty$, $-\infty < y_1 < y_2 < \infty$

$$\phi_2(a_1, b_1; \rho)\phi_2(a_2, b_2; \rho) \geq \phi_2(a_2, b_1; \rho)\phi_2(a_1, b_2; \rho). \tag{13}$$

Now considering all possible six cases, namely (i) $x_1 < x_2 < y_1 < y_2$, (ii) $x_1 < y_1 < x_2 < y_2$, (iii) $x_1 < y_1 < y_2 < x_2$, (iv)$y_1 < x_1 < x_2 < y_2$, (v) $y_1 < x_1 < y_2 < x_2$, (vi) $y_1 < y_2 < x_1 < x_2$ (13) can be easily verified for $0 < \rho < 1$. Similarly, (b) can be easily obtained along the same line. ∎

3 Different Properties of a BVBS Distribution

The random vector $(T_1, T_2)^T$ is said to have a BVBS distribution, if the joint CDF of $(T_1, T_2)^T$ is of the form;

$$F_{T_1,T_2}(t_1, t_2) = P(T_1 \leq t_1, T_2 \leq t_2) = \Phi_2(a(t_1; \alpha_1, \beta_1), a(t_2; \alpha_2, \beta_2); \rho); \quad t_1 > 0, t_2 > 0, \tag{14}$$

where $\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0,\ -1 < \rho < 1$ and $\Phi_2(u, v; \rho)$ is the CDF of a standard normal vector $(Z_1, Z_2)^T$, with correlation coefficient ρ. The corresponding joint PDF is

$$f_{T_1,T_2}(t_1, t_2) = \phi_2(a(t_1; \alpha_1, \beta_1), a(t_2; \alpha_2, \beta_2); \rho) \times \prod_{i=1}^{2} A(t_i; \alpha_i, \beta_i), \tag{15}$$

where $\phi_2(u, v; \rho)$ denotes the standard bivariate normal density function;

$$\phi_2(u, v; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(u^2 + v^2 - 2\rho uv) \right\}. \tag{16}$$

Here after, we shall denote the joint PDF (15) by $\text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. The density surface of $\text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ is unimodal and can take different shapes depending on the values of α_1, α_2 and ρ, see for example Figure 1 of Kundu et al. (2010). For this reason, the BVBS distribution can be used quite effectively to analyze positively skewed bivariate data.

If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then $T_i \sim \text{BS}(\alpha_i, \beta_i)$, for $i = 1, 2$. T_1 and T_2 are positively (negatively) correlated if $\rho > (<)0$, and for $\rho = 0$, they are independent. Moreover, it is observed that if $(T_1, T_2)^T$ has a BVBS distribution, then $(T_1^{-1}, T_2^{-2})^T$, $(T_1^{-1}, T_2)^T$ and $(T_1, T_2^{-1})^T$ all have BVBS distributions. The generation from a BVBS is quite simple, hence the simulation experiments can be performed quite easily.

The following result can be used for testing purposes, or it may have some independent interest also.

Result 2: If $(T_1, T_2) \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for all values of α_1, α_2 and ρ, the stress-strength parameter $R = P(T_1 < T_2) = 1/2$ when $\beta_1 = \beta_2$.

Proof: If $\beta_1 = \beta_2 = \beta$, then

$$R = P(T_1 < T_2) = P\left[\left(\sqrt{\frac{T_1}{\beta}} - \sqrt{\frac{\beta}{T_1}} \right) < \left(\sqrt{\frac{T_2}{\beta}} - \sqrt{\frac{\beta}{T_2}} \right) \right].$$

Since

$$\begin{pmatrix} \sqrt{\frac{T_1}{\beta}} - \sqrt{\frac{\beta}{T_1}} \\ \sqrt{\frac{T_2}{\beta}} - \sqrt{\frac{\beta}{T_2}} \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2\rho \\ \alpha_1\alpha_2\rho & \alpha_2^2 \end{pmatrix} \right),$$

therefore, the result immediately follows. ∎

Note that if $(T_1, T_2) \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then $F_{T_1,T_2}(\cdot, \cdot)$, the joint CDF of T_1 and T_2 as defined in (14), can be written as

$$F_{T_1,T_2}(t_1, t_2) = \Phi_2(a(t_1; \alpha_1, \beta_1), a(t_2; \alpha_2, \beta_2); \rho)) = C_G(F_{T_1}(t_1; \alpha_1, \beta_1), F_{T_2}(t_2; \alpha_2, \beta_2); \rho). \quad (17)$$

Here $C_G(\cdot)$ is the Gaussian copula as defined in (8). The corresponding joint PDF can be obtained as

$$f_{T_1,T_2}(t_1, t_2) = c_G(F_{T_1}(t_1; \alpha_1, \beta_1), F_{T_2}(t_2; \alpha_2, \beta_2); \rho) f_{T_1}(t_1; \alpha_1, \beta_1) f_{T_2}(t_2; \alpha_2, \beta_2). \quad (18)$$

Therefore, using the uniqueness property of copula corresponding to a bivariate distribution with continuous marginals, it is immediate that the BVBS distribution can be obtained from the bivariate Gaussian copula with Birnbaum-Saunders distributions as the marginals. Therefore, we immediately have the following results:

Result 3: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > (<) 0$ it is TP_2 or RR_2, for all values of α_1, β_1, α_2 and β_2.

Proof: The result mainly follows from the fact that TP_2 or RR_2 property is a copula property, see Nelsen (2006), and using Result 2. ∎

Note that Kundu et al. (2010) (Theorem 3.3) established similar results under the restriction $\alpha_1 = \alpha_2$, and $\beta_1 = \beta_2$. Hence, Result 3 is a stronger result than Theorem 3.3 of Kundu et al. (2010).

From Result 3, and using the results of Shaked (1977), the following results (4 & 5) can be easily obtained.

Result 4: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > (<) 0$, the conditional failure rate of T_1, given $T_2 = t_2$, is a decreasing (increasing) function in t_2.

RESULT 5: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > (<)0$, the conditional failure rate of T_1, given $T_2 > t_2$, is a decreasing (increasing) function in t_2.

RESULT 6: The local dependence of a BVBS distribution is positive (negative) if $\rho > (<)0$.

PROOF: It mainly follows from the result that if the joint PDF of (T_1, T_2) is TP_2 (RR_2) then $\gamma_f > (<)0$, see Holland and Wong (1987). ∎

Let us recall the following definition. Suppose X_1 and X_2 are two random variables. If $P(X_1 > x_1 | X_2 = x_2)$ is a non-decreasing function of x_2 for all x_1, then X_1 is called stochastically increasing in X_2. Similarly, if $P(X_2 > x_2 | X_1 = x_1)$ is a non-decreasing function of x_1 for all x_2, then X_2 is called stochastically increasing in X_1.

Now we have the following result.

RESULT 7: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > 0$ (a) T_1 is stochastically increasing in T_2 (b) T_2 is stochastically increasing in T_1, for all values of α_1, β_1, α_2 and β_2.

PROOF: We will prove (a), (b) follows along the same line. Note that the results can be established if we can show that $C_G(u, v; \rho)$ is a concave function in u for fixed v when $\rho > 0$, see Nelsen (2006, page 197). It is equivalent to prove that $\frac{\partial}{\partial u} C_G(u, v; \rho)$ is a decreasing function in u (v). Using Meyer (2009), we have

$$\frac{\partial}{\partial u} C(u, v; \rho) = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right). \tag{19}$$

Clearly, for $\rho > 0$, the right hand side is a decreasing function in u for fixed v and the result follows. ∎

The following result easily follows using the copula property.

RESULT 8: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for for all values of α_1, β_1, α_2, β_2, then

(a) Blomqvist's beta, (b) Kendall's tau and (c) Spearman's rho become

$$\beta = \frac{2}{\pi}\arcsin(\rho), \qquad \tau = \frac{2}{\pi}\arcsin(\rho), \qquad \rho_S = \frac{6}{\pi}\arcsin\left(\frac{\rho}{2}\right), \tag{20}$$

respectively.

Johnson and Kotz (1975) defined the joint bivariate hazard rate as follows:

$$h(x_1, x_2) = \left(-\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}\right) \ln S_{X_1,X_2}(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)). \tag{21}$$

Here S_{X_1,X_2} denotes the joint survival function of X_1 and X_2. It is well known that the bivariate hazard gradient function $h(\cdot,\cdot)$ uniquely determines the joint distribution function of X_1 and X_2. We have the following result for a BVBS distribution.

Result 9: If $(T_1, T_2)^T \sim \text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for all values of α_1, β_1, α_2, β_2, for fixed t_2 (t_1), $h_1(t_1, t_2)$ $(h_2(t_1, t_2))$ is an unimodal function of t_1 (t_2) if $\rho > 0$.

Proof: The proof can be obtained using the hazard rate property of the normal distribution, and the details are avoided. ∎

4 Two-Stage Estimators Based on Copula

In this section we propose to estimate the unknown parameters using copula structure based on two stage estimation method and derive their asymptotic properties. Suppose $\{(t_{1i}, t_{2i}); i = \ldots, n\}$ is a bivariate sample of size n from a $\text{BS}_2(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. The log-likelihood function becomes;

$$\begin{aligned} l(\theta) = \sum_{i=1}^{n} \ln f_{T_1,T_2}(t_{1i}, t_{2i}; \theta) &= \sum_{i=1}^{n} \ln\ c_G(F_{T_1}(t_{1i}; \alpha_1, \beta_1), F_{T_2}(t_{2i}; \alpha_2, \beta_2); \rho) \\ &\quad + \sum_{i=1}^{n} \ln\ f_{T_1}(t_{1i}; \alpha_1, \beta_1) + \sum_{i=1}^{n} \ln\ f_{T_2}(t_{2i}; \alpha_2, \beta_2) \end{aligned} \tag{22}$$

Clearly, the maximization of (22) with respect to five unknown parameters will not produce explicit solutions. Kundu et al. (2010) proposed to use profile log-likelihood method to compute the MLEs of the unknown parameters.

Two step estimation procedure can be incorporated as follows. First compute the estimates of α_1 and β_1, say $\widetilde{\alpha}_1$ and $\widetilde{\beta}_1$, respectively, by maximizing

$$g_1(\alpha_1, \beta_1) = \sum_{i=1}^{n} \ln \ f_{T_1}(t_{1i}; \alpha_1, \beta_1) \tag{23}$$

with respect to α_1 and β_1. Similarly, compute the estimates of α_2 and β_2, say $\widetilde{\alpha}_2$ and $\widetilde{\beta}_2$, respectively, by maximizing

$$g_2(\alpha_2, \beta_2) = \sum_{i=1}^{n} \ln \ f_{T_2}(t_{2i}; \alpha_2, \beta_2) \tag{24}$$

with respect to α_2 and β_2.

Once we obtain $\widetilde{\alpha}_1$, $\widetilde{\beta}_1$, $\widetilde{\alpha}_2$ and $\widetilde{\beta}_2$, compute the estimate of ρ, say $\widetilde{\rho}$ by maximizing

$$g(\rho) = \sum_{i=1}^{n} \ln \ c_G(u_{1i}, u_{2i}; \rho) \tag{25}$$

with respect to ρ, where

$$u_{1i} = F_{T_1}(t_{1i}; \widetilde{\alpha}_1, \widetilde{\beta}_1) = \Phi(a(t_{1i}; \widetilde{\alpha}_1, \widetilde{\beta}_1)) \text{ and } u_{2i} = F_{T_2}(t_{2i}; \widetilde{\alpha}_2, \widetilde{\beta}_2) = \Phi(a(t_{2i}; \widetilde{\alpha}_2, \widetilde{\beta}_2)).$$

It is immediate that $(\widetilde{\alpha}_1, \widetilde{\beta}_1)$ and $(\widetilde{\alpha}_2, \widetilde{\beta}_2)$, are the MLEs of the corresponding parameters based on the marginals $\{t_{1i}; i = 1, \ldots, n\}$ and $\{t_{2i}; i = 1, \ldots, n\}$, respectively. They can be obtained as follows. Suppose

$$s_1 = \sum_{i=1}^{n} t_{1i}, \quad r_1 = \left[\frac{1}{n}\sum_{i=1}^{n} t_{1i}^{-1}\right]^{-1}, \quad s_2 = \sum_{i=1}^{n} t_{2i}, \quad r_2 = \left[\frac{1}{n}\sum_{i=1}^{n} t_{2i}^{-1}\right]^{-1},$$

and

$$K_1(x) = \left[\frac{1}{n}\sum_{i=1}^{n} (x + t_{1i})^{-1}\right]^{-1}, \quad K_2(x) = \left[\frac{1}{n}\sum_{i=1}^{n} (x + t_{2i})^{-1}\right]^{-1}.$$

Then obtain $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ from the unique positive roots of

$$\beta^2 - \beta(2r_1 + K_1(\beta)) + r_1(s_1 + K_1(\beta)) = 0 \tag{26}$$

$$\beta^2 - \beta(2r_2 + K_2(\beta)) + r_2(s_2 + K_2(\beta)) = 0, \tag{27}$$

see Birnbaum and Saunders (1969) or Ng et al. (2003), and obtain

$$\widetilde{\alpha}_1 = \left[\frac{s_1}{\widetilde{\beta}_1} + \frac{\widetilde{\beta}_1}{r_1} - 2\right]^{1/2} \quad \text{and} \quad \widetilde{\alpha}_2 = \left[\frac{s_2}{\widetilde{\beta}_2} + \frac{\widetilde{\beta}_2}{r_2} - 2\right]^{1/2}. \tag{28}$$

Finally obtain $\widetilde{\rho}$ by maximizing (25), which is equivalent to maximize

$$g(\rho) = \sum_{i=1}^{n} \ln\ \phi_1(z_{1i}, z_{2i}; \rho) \tag{29}$$

with respect to ρ, where for $i = 1, \ldots, n$,

$$z_{1i} = \Phi^{-1}(u_{1i}) = a(t_{1i}; \widetilde{\alpha}_1, \widetilde{\beta}_1) \quad \text{and} \quad z_{2i} = \Phi^{-1}(u_{2i}) = a(t_{2i}; \widetilde{\alpha}_2, \widetilde{\beta}_2).$$

Therefore,

$$\widetilde{\rho} = \frac{\sum_{i=1}^{n} z_{1i} z_{2i}}{\sqrt{\sum_{i=1}^{n} z_{1i}^2}\sqrt{\sum_{i=1}^{n} z_{2i}^2}}. \tag{30}$$

Now we provide the asymptotic distribution of $(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{\alpha}_2, \widetilde{\beta}_2, \widetilde{\rho})$. Using the results of Joe (2005), we obtain that

$$\sqrt{n}(\widetilde{\alpha}_1 - \alpha_1, \widetilde{\beta}_1 - \beta_1, \widetilde{\alpha}_2 - \alpha_2, \widetilde{\beta}_2 - \beta_2, \widetilde{\rho} - \rho) \stackrel{d}{\longrightarrow} \mathbf{V}. \tag{31}$$

Here

$$\mathbf{V} = \mathbf{A}^{-1}\mathbf{B}(\mathbf{A}^{-1})^T,$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{2}{\alpha_1^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{0.25+\alpha_1^{-2}+I(\alpha_1)}{\beta_1^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\alpha_2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{0.25+\alpha_2^{-2}+I(\alpha_2)}{\beta_2^2} & 0 \\ -\frac{\rho}{(1-\rho^2)\alpha_1} & 0 & -\frac{\rho}{(1-\rho^2)\alpha_2} & 0 & \frac{2(1+\rho^2)}{(1-\rho^2)^3} \end{bmatrix} \tag{32}$$

$$\mathbf{B} = \begin{bmatrix} \frac{2}{\alpha_1^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{0.25+\alpha_1^{-2}+I(\alpha_1)}{\beta_1^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\alpha_2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{0.25+\alpha_2^{-2}+I(\alpha_2)}{\beta_2^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\rho^2)}{(1-\rho^2)^3} \end{bmatrix} \tag{33}$$

$$I(\alpha) = 2\int_0^\infty \{[1+g(\alpha x)]^{-1} - 0.5\}^2 d\Phi(x)$$

and

$$g(y) = 1 + \frac{y^2}{2} + y\left(1 + \frac{y^2}{4}\right)^{1/2}.$$

It may be mentioned that the elements of the matrices **A** and **B** are obtained from the Fisher information matrices of the univariate and bivariate Birnbaum-Saunders distributions, see for example Ng et al. (2003) and Kundu et al. (2010). After simplification it can be seen that

$$\mathbf{V} = \begin{bmatrix} \frac{\alpha_1^2}{2} & 0 & 0 & 0 & \frac{\rho\alpha_1(1-\rho^2)}{2(1+\rho^2)} \\ 0 & \frac{\beta_1^2}{0.25+\alpha_1^{-2}+I(\alpha_1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2^2}{2} & 0 & \frac{\rho\alpha_2(1-\rho^2)}{2(1+\rho^2)} \\ 0 & 0 & 0 & \frac{\beta_2^2}{0.25+\alpha_2^{-2}+I(\alpha_2)} & 0 \\ \frac{\rho\alpha_1(1-\rho^2)}{2(1+\rho^2)} & 0 & \frac{\rho\alpha_2(1-\rho^2)}{2(1+\rho^2)} & 0 & \frac{(1-\rho^2)^2}{2(1+\rho^2)} \end{bmatrix}. \tag{34}$$

Note that because of the simple structure of the matrix V, the asymptotic confidence intervals of the unknown parameters based on pivotal quantities can be constructed very effectively. This will be illustrated with an example in the next section.

5 Simulation and Data Analysis

5.1 Simulation Results

In this section we provide some results, based on simulations, mainly to observe how the proposed two-step method performs in practice for different sample sizes and for different parameter values. Moreover, we would like to compare the two stage estimators with the corresponding MLEs. In our simulation experiments we have taken $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$. We took different sample sizes namely $n = 10$, 20, 50 and 100, and different ρ values, *i.e.* $\rho = 0.0$, 0.25, 0.50 and 0.95. We have chosen these parameter values and the sample sizes, because these results can be compared directly with the corresponding results based on the MLEs, as reported by Kundu et al. (2010).

To compute the MLEs of the marginals we have adopted the following procedure. We explain the procedure for α_1 and β_1, similar method has been adopted for α_2 and β_2 also. First we obtain the modified moment estimators of α_1 and β_1 as suggested by Ng et al. (2003) as follows:

$$\widetilde{\alpha}_{1M} = \left\{2\left[\left(\frac{s_1}{r_1}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\beta}_{1M} = (s_1 r_1)^{\frac{1}{2}}. \tag{35}$$

With this initial guess of β, we solve the non-linear equation (26). Note that (26) can be written as

$$\beta = \frac{r_1(s_1 + K_1(\beta))}{2r_1 + K_1(\beta) - \beta} = h(\beta) \quad \text{(say)}. \tag{36}$$

A simple iterative process like

$$\beta^{(i+1)} = h(\beta^{(i)}), \tag{37}$$

where $\beta^{(i)}$ and $\beta^{(i+1)}$ denote the estimator of β at the i-th and $(i+1)$-th stage respectively. In our simulation experiment it is observed that the above method works very well.

In each case we computed the average two stage estimators, and the associated mean squared errors (MSEs) over 1000 replications. The results so obtained are reported in Table 1. Some of the points are very clear from these experiments, as the sample size increases for all parameters, the biases and the MSEs decrease. It verifies the consistency properties of the two-stage estimators. The MSEs of the two stage estimators for $\alpha_1, \alpha_2, \beta_1, \beta_2$, do not depend on the correlation coefficient ρ, although MSEs of the estimators of ρ, decrease as it moves away from 0. Comparing the corresponding results for MLEs, as reported by Kundu et al. (2010), it is clear that two stage estimators behave almost as efficiently as the MLEs. Therefore, the two stage estimators may be used quite effectively in place of the MLEs to avoid solving higher dimensional optimization problem.

5.2 Data Analysis

In this section we provide the analysis of a real data set for illustrative purposes. We analyze the data arises from a study of affective facial expressions conducted in 22 subjects. The data was originally reported in Vasey and Thayer (1987), see also Davis (2002). In this study several pieces of music were played to each individual to elicit selected affective states. We report two stages, Stage 1: relaxing music condition, and Stage 2: It was designed to create positive effects. Each trial lasted for 90 seconds, and the response variable at each stage was the mean electromyographic (EMG) amplitudes (μV) from the left brow region. We are reporting the data in Table 2.

Basic data analysis indicate that the marginals are not symmetric and they are right skewed. We shall use the BVBS distribution to model this bivariate data set. From the observations we obtain $s_1 = 217.6364$, $r_1 = 260.7272$, $s_2 = 182.9919$, $r_2 = 212.8127$. Using these, the initial estimates of β_1 and β_2 can be obtained as $\widetilde{\beta}_{1M} = \sqrt{s_1 r_1} = 199.564$ and $\widetilde{\beta}_{2M} = \sqrt{s_2 r_2} = 235.555$. Finally we obtain the two stage estimators as

$$\widetilde{\alpha}_1 = 0.4256, \quad \widetilde{\beta}_1 = 199.6015, \quad \widetilde{\alpha}_2 = 0.4623, \quad \widetilde{\beta}_2 = 235.5729, \quad \widetilde{\rho} = 0.6925. \qquad (38)$$

The associated 95% confidence intervals of the parameters based on the asymptotic distribution are; $\widetilde{\alpha}_1$: (0.2998, 0.5513), $\widetilde{\beta}_1$: (164.9117, 234.2913), $\widetilde{\alpha}_2$: (0.3257, 5989), $\widetilde{\beta}_2$: (191.2843, 279.8615), $\widetilde{\rho}$: (0.5661, 0.8189).

The natural question that arises whether the BVBS distribution fits these bivariate data

n	ρ	$\widetilde{\alpha}_1$	$\widetilde{\alpha}_2$	$\widetilde{\beta}_1$	$\widetilde{\beta}_2$	$\widetilde{\rho}$
	0.95	0.9075 (0.0551)	0.9080 (0.0532)	2.0713 (0.3476)	2.0711 (0.3553)	0.9421 (0.0022)
10	0.50	0.9091 (0.0569)	0.9099 (0.0532)	2.0714 (0.3422)	2.0745 (0.3660)	0.4711 (0.0740)
	0.25	0.9104 (0.0572)	0.9108 (0.0539)	2.0717 (0.3427)	2.0763 (0.3680)	0.2282 (0.1024)
	0.00	0.9118 (0.0568)	0.9117 (0.0543)	2.0708 (0.3410)	2.0774 (0.3658)	-0.0094 (0.1119)
	0.95	0.9547 (0.0267)	0.9572 (0.0268)	2.0380 (0.1740)	2.0359 (0.1749)	0.9465 (0.0007)
20	0.50	0.9539 (0.0265)	0.9605 (0.0270)	2.0388 (0.1680)	2.0351 (0.1740)	0.4848 (0.0315)
	0.25	0.9544 (0.0263)	0.9615 (0.0271)	2.0385 (0.1656)	2.0358 (0.1745)	0.2380 (0.0463)
	0.00	0.9550 (0.0261)	0.9621 (0.0272)	2.0380 (0.1639)	2.0370 (0.1754)	-0.0056 (0.0515)
	0.95	0.9819 (0.0094)	0.9827 (0.0097)	2.0147 (0.0642)	2.0151 (0.0629)	0.9492 (0.0002)
50	0.75	0.9807 (0.0094)	0.9830 (0.0100)	2.0154 (0.0646)	2.0163 (0.0617)	0.4977 (0.0121)
	0.25	0.9802 (0.0095)	0.9828 (0.0101)	2.0159 (0.0644)	2.0167 (0.0616)	0.2499 (0.0185)
	0.00	0.9799 (0.0096)	0.9825 (0.0101)	2.0164 (0.0641)	2.0169 (0.0617)	0.0027 (0.0209)
	0.95	0.9889 (0.0048)	0.9896 (0.0050)	2.0106 (0.0330)	2.0125 (0.0328)	0.9495 (0.0001)
100	0.75	0.9885 (0.0048)	0.9900 (0.0051)	2.0094 (0.0328)	2.0132 (0.0324)	0.7483 (0.0020)
	0.50	0.9883 (0.0048)	0.9901 (0.0052)	2.0086 (0.0326)	2.0133 (0.0321)	0.4982 (0.0058)
	0.25	0.9881 (0.0048)	0.9902 (0.0052)	2.0081 (0.0324)	2.0132 (0.0321)	0.2491 (0.0089)
	0.00	0.9881 (0.0048)	0.9902 (0.0053)	2.0078 (0.0323)	2.0129 (0.0321)	0.0005 (0.0100)

Table 1: The average estimates and the corresponding mean squared errors (reported within brackets below) of the two stage estimators, when $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2.0$ and for different ρ's.

Subject	1	2	3	4	5	6	7	8	9	10	11
Stage 1	143	142	109	123	276	235	208	267	183	245	324
Stage 2	368	155	167	135	216	386	175	358	193	268	507
Subject	12	13	14	15	16	17	18	19	20	21	22
Stage 1	148	130	119	102	279	244	196	279	167	345	524
Stage 2	378	142	171	94	204	365	168	358	183	238	507

Table 2: Left brow EMG amplitudes from 22 subjects.

or not. To check that we have computed the Kolmogorov-Smirnov (KS) distances between the empirical marginals and the fitted marginals. It is observed that the KS distances between the empirical marginals and the fitted marginals and the associated p values (reported within parenthesis) for T_1 and T_2 are 0.1149 (0.9335) and 0.1523 (0.6876), respectively. These results suggest that the BVBS distribution is indeed an appropriate model for the EMG data set.

6 Multivariate Birnbaum-Saunders Distribution

In this section we provide the multivariate Birnbaum-Saunders distribution in terms of copula. A p-variate random vector $(T_1, \ldots, T_p)^T$ is said to have a multivariate Birnbaum-Saunders distribution if the joint CDF of $(T_1, \ldots, T_p)^T$ is of the following form;

$$F_{T_1,\ldots,T_p}(t_1, \ldots, t_p) = P(T_1 \le t_1, \ldots, T_p \le t_p) = \Phi_p(a(t_1; \alpha_1, \beta_1), \ldots, a_p(t_p; \alpha_p, \beta_p); \mathbf{\Sigma}), \quad (39)$$

here $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p > 0$, $\mathbf{\Sigma} > \mathbf{0}$ is a correlation matrix, and $\Phi_p(u_1, \ldots, u_p; \mathbf{\Sigma})$ is the CDF of a standard normal vector $(Z_1, \ldots, Z_P)^T$ with correlation matrix $\mathbf{\Sigma}$, see Kundu et al. (2013). The corresponding joint PDF is

$$f_{T_1,\ldots,T_p}(t_1, \ldots, t_p) = \phi_p(a(t_1; \alpha_1, \beta_1), \ldots, a_p(t_p; \alpha_p, \beta_p); \mathbf{\Sigma}) \times \prod_{i=1}^{p} A(t_i; \alpha_i, \beta_i), \quad (40)$$

where $\phi_p(u_1, \ldots, u_p; \mathbf{\Sigma})$ is the standard multivariate normal PDF with correlation matrix $\mathbf{\Sigma}$, *i.e.*,

$$\phi_p(u_1, \ldots, u_p; \mathbf{\Sigma}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\mathbf{u}^T \mathbf{\Sigma}^{-1} \mathbf{u}\right\}. \tag{41}$$

It is immediate that (39) can be written as

$$F_{T_1,\ldots,T_p} = C_G^p(F_{T_1}(t_1; \alpha_1, \beta_1), \ldots, F_{T_p}(t_p; \alpha_p, \beta_p); \mathbf{\Sigma}), \tag{42}$$

here $C_G^p(u_1, \ldots, u_p; \mathbf{\Sigma})$ is the p-variate Gaussian copula defined as follows;

$$C_G^p(u_1, \ldots, u_p; \mathbf{\Sigma}) = \Phi_p(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_p); \mathbf{\Sigma}). \tag{43}$$

Similarly, (40) can be written as

$$f_{T_1,\ldots,T_p}(t_1, \ldots, t_p) = c_G^p(F_{T_1}(t_1; \alpha_1, \beta_1), \ldots, F_{T_p}(t_p; \alpha_p, \beta_p); \mathbf{\Sigma}) \prod_{i=1}^{p} F_{T_i}(t_i; \alpha_i, \beta_i), \tag{44}$$

here $c_G^p(u_1, \ldots, u_p; \mathbf{\Sigma})$ is the p-variate Gaussian copula density and it is defined as

$$c_G^p(u_1, \ldots, u_p; \mathbf{\Sigma}) = \frac{\partial^p}{\partial u_1, \ldots \partial u_p} C_G^p(u_1, \ldots, u_p; \mathbf{\Sigma}) = \frac{\phi_p(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_p); \mathbf{\Sigma})}{\phi(\Phi^{-1}(u_1)) \ldots \phi(\Phi^{-1}(u_p))}. \tag{45}$$

It is possible to explore different dependence properties of the multivariate Birnbaum-Saunders distribution using the above copula structure. We just provide one important result for the multivariate dependence property of the multivariate Birnbaum-Saunders distribution.

To establish the result we use the following standard notation. For any two real numbers a and b, let $a \vee b = \min\{a, b\}$ and $a \wedge b = \max\{a, b\}$. For $\boldsymbol{x} = (x_1, \cdots, x_p)^T$ and $\boldsymbol{y} = (y_1, \cdots, y_p)^T$, let $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, \cdots, x_p \vee y_p)^T$ and $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, \cdots, x_p \wedge y_p)^T$. Then, a function $g : \mathbb{R}^p \to \mathbb{R}^+$ is said to be MTP_2 (multivariate total positivity of order two), in the sense of Karlin and Rinott (1980), if $g(\boldsymbol{x})g(\boldsymbol{y}) \le g(\boldsymbol{x} \wedge \boldsymbol{y})g(\boldsymbol{x} \vee \boldsymbol{y})$. We then have the following result for the multivariate Gaussian copula.

Result 10: If all the off-diagonal elements of $\mathbf{\Sigma}^{-1}$ are less than or equal to zero, then the multivariate Gaussian copula density as defined in (45) has MTP_2 property.

Proof: To prove that (45) has the MTP_2 property, we need to show

$$c_G^p(\boldsymbol{u}_1;\boldsymbol{\Sigma})c_G^p(\boldsymbol{u}_2;\boldsymbol{\Sigma}) \le c_G^p(\boldsymbol{u}_1 \wedge \boldsymbol{u}_2;\boldsymbol{\Sigma})c_G^p(\boldsymbol{u}_1 \vee \boldsymbol{u}_2;\boldsymbol{\Sigma}), \tag{46}$$

where $\boldsymbol{u}_i = (u_{i1}, \ldots, u_{ip})^T$, for $i = 1{,}2$. Observe that to prove (46), it is sufficient to show that

$$\boldsymbol{x}_1^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}_1 + \boldsymbol{x}_2^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}_2 \ge (\boldsymbol{x}_1 \vee \boldsymbol{x}_2)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_1 \vee \boldsymbol{x}_2) + (\boldsymbol{x}_1 \wedge \boldsymbol{x}_2)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2), \tag{47}$$

where $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{ip})^T$, for $i = 1, 2$, and

$$x_{ij} = \Phi^{-1}(u_{ij}), \quad i = 1, 2, \quad j = 1, \cdots, p.$$

If the elements of $\boldsymbol{\Sigma}^{-1}$ are denoted by $((\sigma^{kj}))$, for $k, j = 1, \cdots, p$, then proving (47) is equivalent to showing

$$\sum_{\substack{k,j=1\\k\neq j}}^{p} (x_{1k}x_{1j}+x_{2k}x_{2j})\sigma^{kj} \ge \sum_{\substack{k,j=1\\k\neq j}}^{p} ((x_{1k}\wedge x_{2k})(x_{1j}\wedge x_{2j})+(x_{1k}\vee x_{2k})(x_{1j}\vee x_{2j}))\sigma^{kj}. \tag{48}$$

For all $k, j = 1, \cdots, p$,

$$x_{1k}x_{1j} + x_{2k}x_{2j} \le (x_{1k} \wedge x_{2k})(x_{1j} \wedge x_{2j}) + (x_{1k} \vee x_{2k})(x_{1j} \vee x_{2j}),$$

which can be easily shown by taking any ordering of $x_{1k}, x_{1j}, x_{2k}, x_{2j}$. Now, the result follows since $\sigma^{kj} \le 0$. ∎

Therefore, using the copula property of the MTP_2, we immediately have the following result;

Result 11: If $(T_1, \ldots, T_p)^T$ has multivariate Birnbaum-Saunders distribution with parameters $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$, and $\boldsymbol{\Sigma}$. If all the elements of $\boldsymbol{\Sigma}^{-1}$ are less than equal to zero, then $(T_1, \ldots, T_p)^T$ has the MTP_2 property.

7 Conclusions

In this paper we have considered the bivariate Birnbaum-Saunders distribution and studied its different reliability and dependence properties. It is observed that the bivariate Birnbaum-Saunders distribution can be obtained using a Gaussian copula and therefore, several copula results can be easily transformed to the present model. We have provided a more general TP_2 result using the copula property. Two stage estimators are proposed which can be obtained quite easily and their asymptotic distribution has been established. By extensive simulation studies it is observed that the two-stage estimators behave quite similarly as the MLEs, and therefore they can be used quite effectively in place of MLEs to avoid high dimensional optimization problem. Moreover, the bivariate Gaussian copula has been generalized to the multivariate case also, and multivariate total positivity of order two result has been established. It should be mentioned that several other properties and the inferential issues can be developed using the copula structure. More work is needed in that direction.

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