Exponentiated Exponential Distribution

Abstract

Two-parameter exponentiated exponential distribution is a right skewed unimodal distribution. The density function and hazard function of the exponentiated exponential distribution are quite similar to the density function and hazard function of the gamma or Weibull distribution. It can be used quite effectively to analyze lifetime data in place of gamma or Weibull distribution. The genesis of the model, several properties and different estimation procedures are discussed here.

Key Words and Phrases: Hazard function; Maximum likelihood estimators; Fisher information matrix; Order statistics; Skewed distribution.

1 Genesis and Definition

One of the cumulative distribution function which was used during the first half of the nineteenth century by Gompertz [2] and Verhulst [14, 15, 16] to compare known human mortality tables and to represent population growth is as follows;

$$G(t) = \left(1 - \rho e^{-t\lambda}\right)^{\alpha}; \quad \text{for} \quad t > \frac{1}{\lambda} \ln \rho, \tag{1}$$

where ρ, α and λ are all positive real numbers (see [1] also). The exponentiated exponential distribution, also known as the generalized exponential distribution, is defined as a particular case of Gompertz-Verhulst distribution function (1) when $\rho = 1$. Therefore, X is a two-parameter exponentiated exponential random variable if it has the following distribution function;

$$F(x; \alpha, \lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha}; \quad x > 0, \tag{2}$$

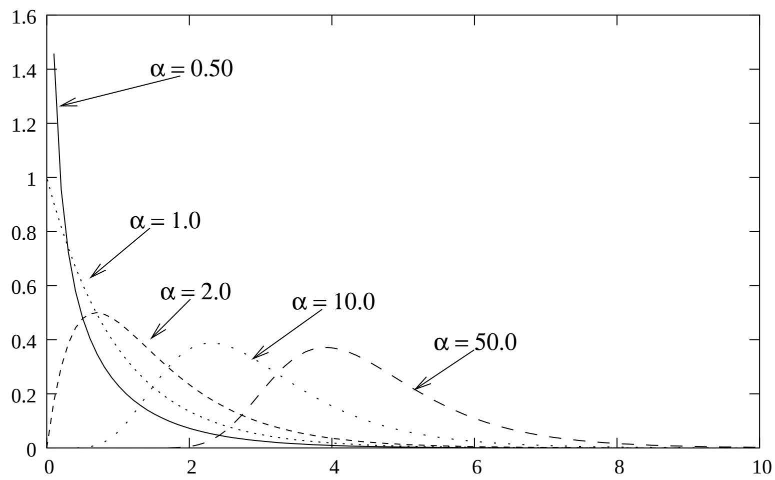

Figure 1: Different density functions of the exponentiated exponential distributions

for α, $\lambda > 0$. Here α and λ are the shape and scale parameters respectively. Note that the exponentiated exponential distribution is a particular member of the exponentiated Weibull distribution [11]. It is observed in [4] that the two-parameter exponentiated exponential distribution can be used quite effectively in analyzing several lifetime data, particularly in place of two-parameter gamma or two-parameter Weibull distribution. If the shape parameter is one, then all the three distributions coincide with the one parameter exponential distribution. Therefore, all the three distributions are extensions/ generalizations of the exponential distribution in different ways.

2 Properties

If the random variable X has the distribution function (2), then the corresponding density function is

$$f(x;\alpha,\lambda) = \alpha\lambda\left(1 - e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x}; \qquad x > 0. \tag{3}$$

Some of the typical exponentiated exponential density functions for different values of α and for $\lambda = 1$ are provided in Figure 1. It is clear from the Figure 1 that the density function of the exponentiated exponential distribution can take different shapes. It is unimodal for

$\alpha > 1$ and it is reversed 'J' shaped for $\alpha \leq 1$. The density function of the exponentiated exponential distribution is log-convex if $\alpha \leq 1$ and log-concave if $\alpha \geq 1$. Moreover, it has an increasing or decreasing hazard function if $\alpha > 1$ or $\alpha < 1$ respectively and for $\alpha = 1$, the hazard function is constant.

3 Moments and Order Statistics

The different moments of the exponentiated exponential distribution can be obtained using the moment generating function. If the random variable X has the density function (3) with $\lambda = 1$, then the moment generating function, $M(t)$, of X for $|t| < 1$ is

$$M(t) = Ee^{tX} = \frac{\Gamma(\alpha+1)\Gamma(1-t)}{\Gamma(\alpha-t+1)}, \tag{4}$$

one is referred to [3] for the exact derivations. Therefore, for $|t| < 1$, the cumulant generating function of X is

$$\kappa(t) = \ln(M(t)) = \ln(\Gamma(\alpha+1)) + \ln(\Gamma(1-t)) - \ln(\Gamma(\alpha-t+1)). \tag{5}$$

Using (5) first four cumulants can be obtained as

$$\kappa_1 = \psi(\alpha+1) - \psi(1), \quad \kappa_2 = \psi'(1) - \psi'(\alpha+1) \tag{6}$$

$$\kappa_3 = \psi''(\alpha+1) - \psi''(1), \quad \kappa_4 = \psi'''(1) - \psi'''(\alpha+1). \tag{7}$$

Here $\psi(.)$ and its derivatives are digamma and polygamma functions. Using (6), mean and variance of Xcan be obtained as

$$E(X) = \psi(\alpha+1) - \psi(1), \quad V(X) = \psi'(1) - \psi'(\alpha+1). \tag{8}$$

The mean is an increasing function of α. It has the the mode at $\ln\alpha$ for $\alpha > 1$ and it has the median at $-\ln(1-(0.5)^{\frac{1}{\alpha}})$. For different values of α, they satisfy the following relation;

$$\text{mean} > \text{median} > \text{mode}.$$

It indicates that the density function is skewed to the right. The variance is an increasing function of α and it increases to $\frac{\pi^2}{6}$, see [3].

Now we discuss the distributions of the order statistics of the exponentiated exponential distribution. Let $X_1, \ldots, X_n$ be independent and identically distributed exponentiated exponential random variables with shape parameter α and scale parameter 1. Further, let $X_{(1)} < \ldots < X_{(n)}$ denote the order statistics obtained from these n variables. The density function of the largest order statistics $X_{(n)}$ is

$$f_{X_{(n)}}(x) = n\alpha e^{-x}\left(1 - e^{-x}\right)^{n\alpha - 1}. \tag{9}$$

Therefore, $X_{(n)}$ also has the exponentiated exponential distribution with the shape parameter $n\alpha$ and scale parameter 1. The density function of the $r-th$ order statistic $X_{(r)}$, $1 \le r \le n$, for $x > 0$ is;

$$\begin{aligned} f_{X_{(r)}}(x) &= \frac{n!}{(n-r)!(r-1)!}\alpha e^{-x}(1-e^{-x})^{\alpha r - 1}(1-(1-e^{-x})^{\alpha})^{n-r} \\ &= \sum_{j=0}^{n-r}(-1)^j \frac{n! \;\; f(x;\alpha(r+j),1)}{(r-1)!j!(n-r-j)!(r+j)}. \end{aligned} \tag{10}$$

From (10) the corresponding distribution function or the survival function can be easily obtained. The moment generating function of $X_{(r)}$, for $|t| < 1$, is

$$\begin{aligned} M_{X_{(r)}}(t) &= Ee^{tX_{(r)}} = \int_0^\infty e^{tx} f_{X_{(r)}}(x)dx \\ &= \sum_{j=0}^{n-r}(-1)^j \frac{n!}{(r-1)!j!(n-r-j)!(r+j)} \times \frac{\Gamma(\alpha(r+j)+1)\Gamma(1-t)}{\Gamma(\alpha(r+j)-t+1)}. \end{aligned}$$

Therefore,

$$E(X_{(r)}) = \sum_{j=0}^{n-r}(-1)^j \frac{n!}{(r-1)!j!(n-r-j)!(r+j)}\left[\psi(\alpha(r+j)+1) - \psi(1)\right]$$

and

$$E(X_{(r)}^2) = \sum_{j=0}^{n-r}(-1)^j \frac{n!}{(r-1)!j!(n-r-j)!(r+j)} \times$$
$$\left[\psi(\alpha(r+j)+1) - \psi(1)\right] + \left[\psi'(1) - \psi'(\alpha(r+j)+1) + (\psi(\alpha(r+j)+1) - \psi(1))^2\right].$$

The higher order moments of $X_{(r)}$ can be easily obtained using (10).

4 Different Methods of Estimation

If $X_1, \ldots, X_n$ is a random sample from (3), then there are several methods of estimating the unknown parameters α and λ are available in the literature [5].

If $\hat{\alpha}_{MLE}$ and $\hat{\lambda}_{MLE}$ denote the maximum likelihood estimators of α and λ respectively, then $\hat{\lambda}_{MLE}$ can be obtained as the fixed point solution of

$$h(\lambda) = \lambda,$$

where

$$h(\lambda) = \left[\frac{\sum_{i=1}^n (x_i e^{-\lambda x_i}/(1-e^{-\lambda x_i}))}{\sum_{i=1}^n \ln(1-e^{-\lambda x_i})} + \frac{1}{n}\sum_{i=1}^n \frac{x_i}{1-e^{-\lambda x_i}} \right]^{-1}.$$

Once $\hat{\lambda}_{MLE}$ is obtained, $\hat{\alpha}_{MLE}$ can be obtained as

$$\hat{\alpha}_{MLE} = -\frac{n}{\sum_{i=1}^n \ln(1-e^{-x_i \hat{\lambda}_{MLE}})}.$$

It is shown in [5] that

$$\{\sqrt{n}(\hat{\alpha}_{MLE} - \alpha), \sqrt{n}(\hat{\lambda}_{MLE} - \lambda)\} \xrightarrow{d} N_2(\mathbf{0}, \boldsymbol{I}^{-1}(\alpha, \lambda)).$$

Here '$\xrightarrow{d}$' means the convergence in distribution. $\boldsymbol{I}(\alpha, \lambda)$ is the Fisher information matrix and the explicit expression is available in [17] or [5].

The method of moment estimators of α and λ can be obtained by equating the first two population moments with the corresponding sample moments. For example, if $\hat{\alpha}_{ME}$ and $\hat{\lambda}_{ME}$ denote the moment estimators of α and λ respectively, then $\hat{\alpha}_{ME}$ can be obtained by solving the following non-linear equation

$$\frac{S}{\bar{X}} = \frac{\sqrt{\psi'(1) - \psi'(\alpha+1)}}{\psi(\alpha+1) - \psi(1)},$$

where $\bar{X}$ and S are the sample mean and the sample standard deviation respectively. Once $\hat{\alpha}_{ME}$ is obtained, then

$$\hat{\lambda}_{ME} = \frac{\psi(\hat{\alpha}_{ME}+1) - \psi(1)}{\bar{X}}.$$

It can be shown that

$$\{\sqrt{n}(\hat{\alpha}_{ME} - \alpha), \sqrt{n}(\hat{\lambda}_{ME} - \lambda)\} \xrightarrow{d} N_2(\mathbf{0}, \mathbf{\Sigma}),$$

the exact expression of $\mathbf{\Sigma}$ is available in [5].

Some of the other estimators like the percentile estimators, estimators based on order statistics, least squares estimators, weighted least squares estimators and estimators based on L-moments are also available in [13] and [5].

5 Conclusions

The two-parameter exponentiated exponential distribution can be used in place of Weibull or gamma distribution to analyze lifetime data. It is observed in certain situations that the exponentiated exponential distribution provides better fit (in terms of the lower Kolmogorov-Smirnov distance or higher log-likelihood value) than the two-parameter gamma or Weibull distributions, see for details in [4]. Construction of confidences intervals and different testing problems have been discussed in [6]. Closeness/distance of the two-parameter exponentiated exponential distribution with the other two-parameter distributions can be found in [7, 8, 9, 10]. The three-parameter exponentiated exponential distribution has the support on the whole real line and it be used to analyze any skewed data set, see [3].

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