Comparison of Different Estimators of P[$Y < X$] for a Scaled Burr Type X Distribution

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Abstract

In this paper we consider the estimation of $P[Y < X]$, when Y and X are two independent scaled Burr Type X distribution having the same scale parameters. The maximum likelihood estimator and its asymptotic distribution is used to construct an asymptotic confidence interval of $P[Y < X]$. Assuming that the common scale parameter is known, the maximum likelihood estimator, uniformly minimum variance unbiased estimator and approximate Bayes estimators of $P[Y < X]$ are discussed. Different methods and the corresponding confidence intervals are compared using Monte Carlo simulations. One data set has been analyzed for illustrative purposes.

Key Words and Phrases: Stress-Strength model; maximum likelihood estimator; Bayes Estimator; Bootstrap Confidence intervals; Credible intervals; Asymptotic distributions.

Short Running Title: Estimation of $P[Y < X]$.

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1 Introduction

Burr [4] introduced twelve different forms of cumulative distribution functions for modelling lifetime data or survival data. Out of those twelve distributions, Burr Type X and Burr Type XII have received the maximum attention. Several authors considered different aspects of these two distributions, see for example Ahmad, Fakhry and Jaheen [1], Jaheen [12] [13], Raqab [22], Rodriguez [24], Sartawi and Abu-Salih [25], Surles and Padgett [26], [27] [28] and Wongo [32]. For an excellent review of these two distributions, the readers are referred to Johnson, Kotz and Balakrsihnan [14].

Recently Surles and Padgett [27] introduced the scaled Burr Type X distribution and named correctly as the generalized Rayleigh distribution. The scaled Burr Type X or generalized Rayleigh distribution (GRD) has the following distribution function for $X > 0$;

$$F(x;\alpha,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha}; \quad \text{for} \quad \alpha > 0, \lambda > 0. \tag{1}$$

Therefore, the GRD has the density function for $x > 0$ as;

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{\alpha-1}; \quad \text{for} \quad \alpha > 0, \lambda > 0. \tag{2}$$

Here α and λ are the shape and scale parameters respectively. Note that the GRD is a particular member of the exponentiated Weibull distribution, originally proposed by Mudholkar and Srivastava [19]. From now on, GRD with the shape parameter α and scale parameter λ will be denoted by $GR(\alpha,\lambda)$.

The main aim of this paper is to focus on the inference of $P(Y < X)$, where $Y \sim GR(\alpha,\lambda)$ and $X \sim GR(\beta,\lambda)$ and they are independently distributed. Here, the notation $\sim$ means 'follows' or 'has the distribution'. The estimation of R is a very common problem in the Statistical literature. The maximum likelihood estimator (MLE) of R was proposed by Surles and Padgett [26]. It is observed that the MLE can be obtained by solving a non-linear equation. We propose to use a simple iterative scheme to find the MLE of R. We also obtain the asymptotic distribution of the MLE to compute the confidence interval of R. Bias corrected accelerated parametric bootstrap confidence intervals are also proposed.

We also consider the case when the scale parameters are known. In this case the MLE of R has been obtained by Ahmad, Fakhry and Jaheen [1]. We obtain the exact distribution of the MLE of R and compute the exact confidence interval of R. We also propose Bayes and approximate Bayes estimates of R assuming different loss functions. Different methods are compared using Monte Carlo simulations and one data set has been analyzed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we discuss the MLE of R. The asymptotic distribution of the MLE of R and different confidence intervals of R are presented in Section 3. In Section 4, we consider the case when λ is known. Monte Carlo simulation results are presented in Section 5. We present the data analysis and its respective results in Section 6 and finally we conclude the paper in Section 7.

2 Maximum Likelihood Estimator of R

Let $Y \sim GR(\alpha, \lambda)$, $X \sim GR(\beta, \lambda)$ and they are independent. Therefore,

$$\begin{aligned} R &= P[Y < X] = 4\int_0^\infty \int_0^x \alpha\beta\lambda^4 xye^{-\lambda^2(x^2+y^2)} \times \left(1-e^{-(\lambda x)^2}\right)^{\beta-1}\left(1-e^{-(\lambda y)^2}\right)^{\alpha-1} dydx \\ &= \frac{\beta}{\alpha+\beta} \end{aligned} \tag{3}$$

Now to compute the MLE of R, first we obtain the MLEs of α and β. Let $X_1, \ldots, X_n$ be a random sample from $GR(\beta, \lambda)$ and $Y_1, \ldots, Y_m$ be a random sample from $GR(\alpha, \lambda)$. The extended likelihood function is;

$$\begin{aligned} L(\alpha,\beta,\lambda) &= C + m\ln\alpha + n\ln\beta + 2(m+n)\ln\lambda + \left(\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{m}\ln y_i\right) \\ &- \lambda^2\left(\sum_{j=1}^{n} x_j^2 + \sum_{i=1}^{m} y_i^2\right) + (\alpha-1)\sum_{i=1}^{m}\ln\left(1-e^{-(\lambda y_i)^2}\right) \end{aligned}$$

$$+ \quad (\beta - 1) \sum_{j=1}^{n} \ln \left(1 - e^{-(\lambda x_j)^2}\right), \tag{4}$$

where C is constant. Therefore, to obtain the MLE's of α β and λ, we can maximize (4) directly with respect to α, β and λ or equivalently, we can solve the following equations:

$$\frac{\partial L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} \ln \left(1 - e^{-(\lambda y_i)^2}\right) = 0, \tag{5}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln \left(1 - e^{-(\lambda x_i)^2}\right) = 0, \tag{6}$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= \frac{2(m+n)}{\lambda} - 2\lambda \left(\sum_{j=1}^{n} x_j^2 + \sum_{i=1}^{m} y_i^2 \right) + 2\lambda(\alpha - 1) \sum_{i=1}^{m} \frac{y_i^2 e^{-(\lambda y_i)^2}}{1 - e^{-(\lambda y_i)^2}} \\ &+ \quad 2\lambda(\beta - 1) \sum_{j=1}^{n} \frac{x_j^2 e^{-(\lambda x_j)^2}}{1 - e^{-(\lambda x_j)^2}} = 0, \end{aligned} \tag{7}$$

From (5) and (6), we obtain the MLE of α and β as functions of λ, say $\hat{\alpha}$ and $\hat{\beta}$, as

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^{m} \ln \left(1 - e^{-(\lambda y_i)^2}\right)}. \tag{8}$$

$$\hat{\beta} = -\frac{n}{\sum_{j=1}^{n} \ln \left(1 - e^{-(\lambda x_i)^2}\right)}. \tag{9}$$

If the scale parameter is known, the MLE of α and β can be obtained from (8) and (9). If all the parameters are unknown, we can first estimate the scale parameter by maximizing the profile likelihood function $L(\hat{\alpha}(\lambda), \hat{\beta}(\lambda), \lambda)$, with respect to λ or by solving the following non-linear equation;

$$\begin{aligned} g(\lambda) &= \frac{m+n}{\lambda} - \lambda \left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2 \right) - \frac{m\lambda}{\sum_{i=1}^{m} \ln(1 - e^{-(\lambda y_i)^2})} \sum_{i=1}^{m} \frac{y_i^2 e^{-(\lambda y_i)^2}}{1 - e^{-(\lambda y_i)^2}} \\ &- \frac{n\lambda}{\sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_j)^2})} \sum_{j=1}^{n} \frac{x_j^2 e^{-(\lambda x_j)^2}}{1 - e^{-(\lambda x_j)^2}} \\ &- \lambda \sum_{i=1}^{m} \frac{y_i^2 e^{-(\lambda y_i)^2}}{1 - e^{-(\lambda y_i)^2}} - \lambda \sum_{j=1}^{n} \frac{x_j^2 e^{-(\lambda x_j)^2}}{1 - e^{-(\lambda x_j)^2}} = 0. \end{aligned}$$

Consequently, $\hat{\lambda}$ can be obtained by solving the non-linear equation

$$h(\lambda) = \lambda^2, \tag{10}$$

where

$$\begin{aligned} h(\lambda) &= (m+n)\left[\lambda\left(\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{m} y_j^2\right) + \frac{m\lambda}{\sum_{i=1}^{m}\ln(1-e^{-(\lambda y_i)^2})}\sum_{i=1}^{m}\frac{y_i^2 e^{-(\lambda y_i)^2}}{1-e^{-(\lambda y_i)^2}}\right. \\ &+ \frac{n\lambda}{\sum_{j=1}^{n}\ln(1-e^{-(\lambda x_j)^2})}\sum_{j=1}^{n}\frac{x_j^2 e^{-(\lambda x_j)^2}}{1-e^{-(\lambda x_j)^2}} \\ &+ \left.\lambda\sum_{i=1}^{m}\frac{y_i^2 e^{-(\lambda y_i)^2}}{1-e^{-(\lambda y_i)^2}} + \lambda\sum_{j=1}^{n}\frac{x_j^2 e^{-(\lambda x_j)^2}}{1-e^{-(\lambda x_j)^2}}\right]^{-1}. \end{aligned}$$

Since $\hat{\lambda}$ is a fixed-point like solution of the non-linear equation (10), therefore, it can be obtained by using a simple iterative scheme as follows;

$$h(\lambda_{(j)}) = \lambda^2_{(j+1)}, \tag{11}$$

where $\lambda_{(j)}$ is the $j-th$ iterate of $\hat{\lambda}$. Once we obtain $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from (8) and (9) respectively. Therefore, the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha}+\hat{\beta}}. \tag{12}$$

3 Asymptotic Distribution and Different Confidence Intervals

Surles and Padgett [27] derived the Fisher information matrix of (α, β, λ). It can be easily seen that the GR family satisfies all the regularity conditions, therefore, we have the following asymptotic results:

Theorem 1: As $m \to \infty$ and $n \to \infty$ and $\frac{m}{n} \to p$, then

$$\left[\sqrt{m}(\hat{\alpha}-\alpha), \sqrt{n}(\hat{\beta}-\beta), \sqrt{m}(\hat{\lambda}-\lambda)\right] \to N_3(\mathbf{0}, \mathbf{A}^{-1}(\alpha,\beta,\lambda)).$$

Here

$$\mathbf{A}(\alpha,\beta,\lambda)=\begin{bmatrix} a_{11} & 0 & a_{13}\\ 0 & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},$$

where

$$a_{11}=\frac{1}{\alpha^2},\quad a_{13}=a_{31}=\frac{c}{m},\quad a_{22}=\frac{1}{\beta^2},\quad a_{23}=a_{32}=\frac{d\sqrt{p}}{m},\quad a_{33}=-g(\alpha)-\frac{1}{p}g(\beta),$$

and

$$\begin{aligned} c &= \frac{2m}{\lambda}\left[\frac{1}{\alpha-1}(\psi(\alpha)-\psi(1))-\frac{1}{\alpha}\right] \quad \text{if} \quad \alpha\neq 1\\ &= \frac{2m}{\lambda}\sum_{i=0}^{\infty}\frac{1}{(i+2)^2} \quad \text{if} \quad \alpha=1,\\ d &= \frac{2n}{\lambda}\left[\frac{1}{\beta-1}(\psi(\beta)-\psi(1))-\frac{1}{\beta}\right] \quad \text{if} \quad \beta\neq 1\\ &= \frac{2n}{\lambda}\sum_{i=0}^{\infty}\frac{1}{(i+2)^2} \quad \text{if} \quad \beta=1,\end{aligned}$$

$$\begin{aligned} g(\mu) &= -\frac{2}{\lambda^2}\Bigg[1+(\psi(\mu+1)-\psi(1)-\frac{1}{\mu})+2((\psi(2)-\psi(\mu+1))^2+(\psi'(2)-\psi'(\mu+1)))\\ &\quad +\frac{2}{\mu-2}((\psi(3)-\psi(\mu+1))^2+(\psi'(3)-\psi'(\mu+1)))\Bigg] \quad \text{if} \quad \mu\neq 1 \text{ or } 2\\ &= -\frac{2}{\lambda^2}\left[1+\psi(2)-\psi(1)\right] \quad \text{if} \quad \mu=1\\ &= -\frac{2}{\lambda^2}\left[\frac{3}{2}+\psi(3)-\psi(2)+2((\psi(2)-\psi(3))^2+(\psi'(2)-\psi'(3)))+4\sum_{i=0}^{\infty}\frac{2}{(3+i)^3}\right]\\ &\quad \text{if} \quad \mu=2.\end{aligned}$$

Proof of Theorem 1: The proof follows by expanding the derivative of the log-likelihood function using Taylor series and using the Central Limit Theorem.

Using Theorem 1, we have the following result regarding $\hat{R}$.

Theorem 2: As $m\to\infty$ and $n\to\infty$ so that $\frac{m}{n}\to p$, then

$$\sqrt{m}\left(\hat{R}-R\right)\to N(0,B), \tag{13}$$

where

$$B = \frac{1}{u(\alpha+\beta)^4}\left[\beta^2(a_{22}a_{33}-a_{23}^2) - 2\alpha\beta\sqrt{p}a_{23}a_{31} + \alpha^2 p(a_{11}a_{33}-a_{13}^2)\right]$$

and

$$u = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}.$$

Proof of Theorem 2: It follows using Theorem 1.

Remark 1: Note that the normalizing constants $\sqrt{m}$ and $\sqrt{n}$ in Theorem 1, can be interchanged.

Remark 2: The asymptotic confidence interval of R can be obtained using Theorem 2. For constructing the asymptotic confidence interval of R, the variance B needs to be estimated. The empirical Fisher information matrix and the MLE estimates of α, β and λ may be used to estimate B. For example,

$$\hat{a}_{11} = \frac{1}{\hat{\alpha}^2}, \quad \hat{a}_{22} = \frac{1}{\hat{\beta}^2}, \quad \hat{a}_{13} = \hat{a}_{31} = -\frac{2\hat{\lambda}}{m}\sum_{i=1}^{m}\frac{y_i^2 e^{-\hat{\lambda}y_i^2}}{(1-e^{-(\hat{\lambda}y_i)^2})}, \quad \hat{a}_{23} = \hat{a}_{32} = -\frac{2\hat{\lambda}\sqrt{p}}{m}\sum_{j=1}^{n}\frac{x_j^2 e^{-(\hat{\lambda}x_j)^2}}{(1-e^{-(\hat{\lambda}x_j)^2})}$$

$$\hat{a}_{33} = \frac{2}{\lambda^2}\left(1+\frac{1}{p}\right) + 2\left[\frac{1}{m}\sum_{j=1}^{n}x_j^2 + \frac{1}{m}\sum_{i=1}^{n}y_i^2\right] +$$
$$4(\hat{\alpha}-1)\frac{\hat{\lambda}^2}{m}\sum_{i=1}^{m}\frac{y_i^4 e^{-(\hat{\lambda}y_i)^2}}{(1-e^{-(\hat{\lambda}y_i)^2})^2} + 4(\hat{\beta}-1)\frac{\hat{\lambda}^2}{m}\sum_{j=1}^{n}\frac{x_j^4 e^{-(\hat{\lambda}x_j)^2}}{(1-e^{-(\hat{\lambda}x_j)^2})^2} -$$
$$2(\hat{\alpha}-1)\frac{1}{m}\sum_{i=1}^{m}\frac{y_i^2 e^{-(\hat{\lambda}y_i)^2}}{1-e^{-(\hat{\lambda}y_i)^2}} - 2(\hat{\beta}-1)\frac{1}{m}\sum_{j=1}^{n}\frac{x_j^2 e^{-(\hat{\lambda}x_j)^2}}{1-e^{-(\hat{\lambda}x_j)^2}}$$

We also propose two parametric bootstrap confidence intervals mainly for small sample sizes.

The following two confidence intervals mainly for small sample sizes, which might be computationally very demanding for large sample, are proposed, namely; (i) percentile bootstrap method (we call it from now on as Boot-p) based on the original idea of Efron [7] and (ii) bootstrap-t method (we refer it as Boot-t from now on) based on the idea of Hall [10].

Boot-p Methods:

Step 1: First estimate α, β and λ from the given sample $\{y_1, \ldots, y_m\}$ and $\{x_1, \ldots x_n\}$, using the maximum likelihood method.

Step 2: Using any random mechanism, generate a random sample of size m, from GR($\hat{\alpha}$, $\hat{\lambda}$), say $\{y_1^*, \ldots, y_m^*\}$. Similarly, generate a random sample of size n, say $\{x_1^*, \ldots, x_n^*\}$ from GR($\hat{\beta}$, $\hat{\lambda}$). Based on $\{y_1^*, \ldots, y_m^*\}$ and $\{x_1^*, \ldots, x_n^*\}$ compute the bootstrap estimate of R, say $\hat{R}^*$.

Step 3: Repeat step 2, NBOOT times.

Step 4: Let $G^*(x) = P(\hat{R}^* \leq x)$, be the cumulative distribution function of $\hat{R}^*$. Define $\hat{R}_{Boot-p}(x) = G^{*^{-1}}(x)$ for a given x. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-p}(\frac{\gamma}{2}), \hat{R}_{Boot-p}(1-\frac{\gamma}{2})\right). \tag{14}$$

Boot-t Method

Step 1: Same as Boot-p

Step 2: Same as Boot-p

Step 3: Compute the following statistic:

$$T^* = \frac{\sqrt{m}(\hat{R}^* - \hat{R})}{\sqrt{V(\hat{R}^*)}}.$$

Note that, $V(\hat{R}^*) = \frac{B^*}{m}$ and it can be computed as mentioned in Remark 2.

Step 4: Repeat steps 2 and 3, NBOOT times.

Step 5: From the NBOOT T* values obtained, determine the upper and lower bound of the $100(1-\gamma)\%$ confidence interval of R as follows: Let $H(x) = P(T^* \leq x)$ be the cumulative distribution function of T^*. For a given x, define

$$\hat{R}_{Boot-t} = \hat{R} + m^{-\frac{1}{2}}\sqrt{V(\hat{R})}H^{-1}(x).$$

Here also, $V(\hat{R}) = \frac{B}{m}$ can be computed using Remark 2. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-t}(\frac{\gamma}{2}), \hat{R}_{Boot-t}(1-\frac{\gamma}{2})\right).$$

4 Estimation of R if λ is Known

4.1 Classical Methods

In this section we assume that the scale parameters are known. Therefore, even if they are not same, we can always transform the data and make the scale parameters to be one in both cases. It is already observed by Surles and Padgett [26] that the MLE of R is

$$\hat{R}_{MLE} = \frac{n\sum_{i=1}^{m}\ln(1-e^{-Y_i^2})}{n\sum_{i=1}^{m}\ln(1-e^{-Y_i^2}) + m\sum_{j=1}^{n}\ln(1-e^{-X_j^2})}. \tag{15}$$

The exact distribution of $\hat{R}_{MLE}$ can be obtained and based on the distribution of $\hat{R}_{MLE}$, $100(1-\gamma)\%$ confidence interval of R can be obtained as

$$\left[\frac{1}{1+F_{2m,2n;1-\frac{\alpha}{2}} \times \left(\frac{1}{\hat{R}}-1\right)}, \frac{1}{1+F_{2m,2n;\frac{\alpha}{2}} \times \left(\frac{1}{\hat{R}}-1\right)}\right], \tag{16}$$

where, $F_{2m,2n;\frac{\gamma}{2}}$ and $F_{2m,2n;1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}^{th}$ percentile points of a F distribution with $2m$ and $2n$ degrees of freedom.

Using the idea of Tong [30], [31], Surles and Padgett [26] also obtained the UMVUE of R and it can be written as;

$$\begin{aligned}\hat{R}_{UMVUE} &= 1-\sum_{i=0}^{n-1}(-1)^i\frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}\left(\frac{T_2}{T_1}\right)^i, \quad \text{if} \quad T_2 \le T_1 \\ &= \sum_{i=0}^{m-1}(-1)^i\frac{(m-1)!(n-1)!}{(m-i-1)!(n+i-1)!}\left(\frac{T_1}{T_2}\right)^i, \quad \text{if} \quad T_1 \le T_2,\end{aligned}$$

where $T_1 = -\sum_{i=1}^{n} \ln(1 - e^{-X_i^2})$ and $T_2 = -\sum_{i=1}^{m} \ln(1 - e^{-Y_i^2})$.

4.2 Bayesian Approach

In this subsection, we consider Bayesian estimate of R, under the assumptions that the shape parameters α and β are random variables for both the populations. It is assumed that α and β have independent gamma priors with the PDF's;

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1 - 1} e^{-b_1 \alpha}; \quad \alpha > 0, \tag{17}$$

$$\pi(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)} \beta^{a_2 - 1} e^{-b_2 \beta}; \quad \beta > 0, \tag{18}$$

respectively. Here $a_1, b_1, a_2, b_2 > 0$. Therefore, α and β follow Gamma(a_1, b_1) and Gamma(a_2, b_2) respectively. It is observed by Ahmad, Fakhry and Jaheen [1] and Surles and Padgett [27] that the posterior PDF R, for $0 < r < 1$ is

$$f_R(r) = C \frac{r^{a_2+n-1}(1-r)^{a_1+m-1}}{((b_1+T_1)(1-r) + (b_2+T_2)r)^{m+n+a_1+a_2}}, \quad \text{for} \quad 0 < r < 1, \tag{19}$$

where

$$T_1 = -\sum_{i=1}^{m} \ln(1 - e^{-Y_i^2}), \quad \text{and} \quad T_2 = -\sum_{j=1}^{n} \ln(1 - e^{-X_j^2})$$

and

$$C = \frac{\Gamma(n + m + a_1 + a_2)}{\Gamma(a_1 + m)\Gamma(a_2 + n)} (b_1 + T_1)^{a_1+m} (b_2 + T_2)^{a_2+n}.$$

Now let us study the shape of the density function of R. Note that $\frac{d}{dr} \ln f_R(r) = 0$ has only two roots. Moreover, $\lim_{r \to 0+} \frac{d}{dr} \ln f_R(r) > 0$ and $\lim_{r \to 1-} \frac{d}{dr} \ln f_R(r) < 0$. Therefore, the density $f_R(r)$ has a unique mode in $[0, 1]$ and the mode can be obtained as the unique root which lies between 0 and 1 of the following quadratic equation:

$$2r^2(B_1 - B_2) + r(2B_2 - 2B_1 + A_1 B_2 + A_2 B_1) - A_2 B_1 = 0, \tag{20}$$

where $B_1 = b_1 + T_1$, $B_2 = b_2 + T_2$, $A_1 = a_1 + m - 1$, $A_2 = a_2 + n - 1$. The posterior mean or the median of R, can not be obtained in a closed form. They can be obtained

Table 1: Biases, MSEs, Confidence Lengths and Coverage Percentages of the different methods.

S.S.	2.00	2.50	3.00	3.50	4.00
(10,10)	- 0.0042(0.0128)	-0.0037(0.0119)	-0.0030(0.0109)	-0.0024(0.0098)	-0.0018(0.0089)
	0.4228(92%)	0.4136(92%)	0.4025(92%)	0.3910(93%)	0.3796(93%)
	0.3736(96%)	0.3484(96%)	0.3239(96%)	0.3017(96%)	0.2818(95%)
	0.4657(94%)	0.4421(95%)	0.4151(96%)	0.3881(95%)	0.3627(95%)
	0.3979(93%)	0.3842(93%)	0.3678(93%)	0.3508(93%)	0.3341(93%)
(15,15)	-0.0030(0.0084)	-0.0027(0.0078)	-0.0023(0.0072)	-0.0019(0.0065)	-0.0016(0.0059)
	0.3473(93%)	0.3393(93%)	0.3297(94%)	0.3199(94%)	0.3101(94%)
	0.3518(95%)	0.3288(95%)	0.3064(95%)	0.2855(94%)	0.2665(95%)
	0.4024(96%)	0.3708(94%)	0.3411(93%)	0.3143(93%)	0.2904(92%)
	0.3330(94%)	0.3208(93%)	0.3062(93%)	0.2913(93%)	0.2768(93%)
(20,20)	-0.0015(0.0062)	-0.0011(0.0057)	-0.0008(0.0052)	-0.0004(0.0047)	-0.0001 (0.0042)
	0.3020(94%)	0.2946(94%)	0.2859(95%)	0.2769(95%)	0.2683(95%)
	0.2960(96%)	0.2794(95%)	0.2620(95%)	0.2455(95%)	0.2307(95%)
	0.3407(94%)	0.3240(94%)	0.3059(93%)	0.2882(93%)	0.2717(92%)
	0.2920(94%)	0.2808(94%)	0.2676(95%)	0.2541(94%)	0.2411(94%)
(25,25)	-0.0018(0.0050)	-0.0015(0.0047)	-0.0012(0.0043)	-0.0009(0.0039)	-0.0007(0.0035)
	0.2708(94%)	0.2640(95%)	0.2561(95%)	0.2481(95%)	0.2402(96%)
	0.2923(95%)	0.2780(96%)	0.2627(96%)	0.2479(96%)	0.2341(96%)
	0.2875(95%)	0.2705(95%)	0.2532(95%)	0.2368(95%)	0.2219(95%)
	0.2632(94%)	0.2529(95%)	0.2408(95%)	0.2285(94%)	0.2166(94%)
(30,30)	-0.0032(0.0038)	-0.0028(0.0035)	-0.0024(0.0032)	-0.0020(0.0029)	-0.0017(0.0026)
	0.2480(95%)	0.2419(95%)	0.2347(95%)	0.2273(96%)	0.2201(96%)
	0.2291(93%)	0.2222(93%)	0.2136(94%)	0.2044(95%)	0.1948(95%)
	0.2180(94%)	0.2086(94%)	0.1984(94%)	0.1883(94%)	0.1788(94%)
	0.2420(95%)	0.2326(95%)	0.2214(95%)	0.2100(95%)	0.1990(95%)

The first rows represent the average biases and the corresponding MSEs are reported within brackets. Second, third, fourth rows represent the average lengths and the corresponding coverage percentages of the asymptotic, boot-p and boot-t confidence intervals. The fifth rows represent the average lengths and the corresponding coverage percentages bases on the formula (16), simply putting $\lambda = \hat{\lambda}$.

Table 2: Biases and MSEs of the different estimators.

S.S.	2.00	2.50	3.00	3.50	4.00
(10,10)	-0.0049(0.0109)	-0.0069(0.0099)	-0.0081(0.0088)	-0.0088(0.0079)	-0.0092(0.0070)
	-0.0037(0.0118)	-0.0035(0.0104)	-0.0032(0.0092)	-0.0030(0.0081)	-0.0028(0.0071)
	-0.0011(0.0135)	0.0011(0.0124)	0.0027(0.0111)	0.0038(0.0099)	0.0046(0.0088)
	-0.0101(0.0104)	-0.0145(0.0099)	-0.0174(0.0093)	-0.0193(0.0087)	-0.0205(0.0080)
	-0.0097(0.0105)	-0.0138(0.0100)	-0.0166(0.0094)	-0.0184(0.0087)	-0.0195(0.0080)
(15,15)	-0.0037(0.0074)	-0.0050(0.0067)	-0.0058(0.0059)	-0.0062(0.0052)	-0.0065(0.0046)
	-0.0029(0.0078)	-0.0027(0.0070)	-0.0025(0.0061)	-0.0024(0.0054)	-0.0022(0.0048)
	-0.0010(0.0087)	0.0006(0.0081)	0.0017(0.0073)	0.0024(0.0065)	0.0030(0.0058)
	-0.0073(0.0074)	-0.0102(0.0070)	-0.0121(0.0065)	-0.0134(0.0060)	-0.0141(0.0055)
	-0.0071(0.0074)	-0.0099(0.0070)	-0.0118(0.0065)	-0.0130(0.0060)	-0.0137(0.0055)
(20,20)	-0.0028(0.0055)	-0.0037(0.0049)	-0.0043(0.0043)	-0.0046(0.0038)	-0.0048(0.0034)
	-0.0015(0.0057)	-0.0014(0.0050)	-0.0013(0.0044)	-0.0012(0.0038)	-0.0011(0.0033)
	0.0000(0.0063)	0.0012(0.0057)	0.0020(0.0052)	0.0026(0.0046)	0.0030(0.0041)
	-0.0049(0.0056)	-0.0071(0.0052)	-0.0086(0.0047)	-0.0095(0.0043)	-0.0101(0.0039)
	-0.0048(0.0056)	-0.0069(0.0052)	-0.0083(0.0047)	-0.0092(0.0043)	-0.0098(0.0039)
(25,25)	-0.0012(0.0043)	-0.0020(0.0038)	-0.0025(0.0034)	-0.0028(0.0030)	-0.0030(0.0026)
	-0.0020(0.0047)	-0.0019(0.0041)	-0.0018(0.0036)	-0.0017(0.0032)	-0.0016(0.0028)
	-0.0008(0.0051)	0.0001(0.0047)	0.0008(0.0042)	0.0013(0.0038)	0.0016(0.0034)
	-0.0047(0.0047)	-0.0065(0.0043)	-0.0077(0.0040)	-0.0084(0.0036)	-0.0088(0.0033)
	-0.0047(0.0047)	-0.0064(0.0043)	-0.0075(0.0040)	-0.0082(0.0036)	-0.0087(0.0033)
(30,30)	-0.0025(0.0036)	-0.0031(0.0032)	-0.0034(0.0028)	-0.0036(0.0025)	-0.0036(0.0022)
	-0.0032(0.0035)	-0.0029(0.0031)	-0.0027(0.0027)	-0.0025(0.0024)	-0.0023(0.0021)
	-0.0023(0.0039)	-0.0013(0.0036)	-0.0006(0.0032)	-0.0001(0.0028)	0.0003(0.0025)
	-0.0056(0.0036)	-0.0069(0.0033)	-0.0078(0.0030)	-0.0082(0.0028)	-0.0085(0.0025)
	-0.0055(0.0036)	-0.0068(0.0033)	-0.0077(0.0030)	-0.0081(0.0028)	-0.0084(0.0025)

The first, second, third, fourth and fifth rows represent the biases and the corresponding MSEs by MLEs, UMVUES, approximate Bayes (with respect to 0-1 loss function), approximate Bayes (Lindley's approximation) and the Bayes estimators (with respect to the squared error loss function) are reported within brackets

Table 3: The average confidence, HPD lengths and coverage percentages.

S.S.	2.00	2.50	3.00	3.50	4.00
(10,10)	0.4005(95%)	0.3881(95%)	0.3732(95%)	0.3577(95%)	0.3425(95%)
	0.3982(94%)	0.3848(94%)	0.3687(94%)	0.3521(94%)	0.3359(94%)
	0.4148(95%)	0.3956(95%)	0.3757(95%)	0.3566(95%)	0.3387(95%)
(15,15)	0.3342(94%)	0.3227(94%)	0.3091(94%)	0.2951(94%)	0.2815(94%)
	0.3332(93%)	0.3210(93%)	0.3067(93%)	0.2920(93%)	0.2778(94%)
	0.3504(96%)	0.3342(96%)	0.3172(96%)	0.3005(96%)	0.2849(96%)
(20,20)	0.2927(95%)	0.2821(95%)	0.2696(95%)	0.2567(95%)	0.2443(95%)
	0.2921(95%)	0.2811(95%)	0.2680(95%)	0.2547(95%)	0.2419(95%)
	0.3071(95%)	0.2925(95%)	0.2770(95%)	0.2620(95%)	0.2479(95%)
(25,25)	0.2636(95%)	0.2538(95%)	0.2423(95%)	0.2305(95%)	0.2191(95%)
	0.2632(95%)	0.2531(95%)	0.2411(95%)	0.2290(95%)	0.2173(95%)
	0.2760(94%)	0.2626(94%)	0.2484(94%)	0.2346(94%)	0.2216(94%)
(30,30)	0.2424(96%)	0.2333(96%)	0.2225(96%)	0.2115(96%)	0.2009(96%)
	0.2421(96%)	0.2327(96%)	0.2217(96%)	0.2104(96%)	0.1996(96%)
	0.2541(95%)	0.2422(95%)	0.2294(95%)	0.2168(95%)	0.2049(95%)

The first and second rows represent the confidence intervals based on (16) using the estimate of R as MLE or UMVUE. The third rows represent the average HPD intervals and the corresponding coverage percentages based on MCMC.

numerically. Alternatively, using the idea of Gibbs sampling we can compute the posterior mean and median by simulation technique. The crucial point about Gibbs sampling is to generate samples from the posterior distribution. Since the posterior density function is log-concave and bounded, using the idea Berger and Sun [2] the samples to be generated from the posterior density function can be obtained. But we propose the following simple procedure in this case. It is easy to see that the posterior density functions of α and β are Gamma(a_1+m, b_1+T_1) and Gamma(a_2+n, b_2+T_2) respectively and they are independent. Therefore once we generate random samples from the posterior density functions of α and β, we obtain by a simple transformation, random samples from the posterior density function of R. Once we have a sample of size N from the posterior density function of R, we can compute the estimates of the posterior mean and median. Using the idea of Chen and Shao [5], we can compute the highest posterior density (HPD) interval.

Now, consider the following loss function:

$$L(a,b) = \begin{cases} 0 & \text{if} \quad |a-b| \le c \\ 1 & \text{if} \quad |a-b| > c. \end{cases} \tag{21}$$

It is known that the Bayes estimate with respect the above loss function (21) is the midpoint of the 'modal interval' of length $2c$ of the posterior distribution (see Ferguson [8], page 51, problem 5). Therefore, the posterior mode is an approximate Bayes estimator of R with respect to the loss function (21) when the constant c is small.

As we had mentioned before, the Bayes estimate of R under squared error loss can not be computed analytically. Alternatively, using the approximation of Lindley [17] and following the approach of Ahmad, Fakhry and Jaheen [1], it can be easily seen that the approximate Bayes estimate of R, say $\hat{R}_{BS}$, under squared error loss is

$$\hat{R}_{BS} = \tilde{R}\left[1 + \frac{\tilde{\alpha}\tilde{R}^2}{\tilde{\beta}^2(n+a_2-1)(m+b_1-1)} \times \left(\tilde{\alpha}(m+a_1-1) - \tilde{\beta}(n+a_2-1)\right)\right], \tag{22}$$

where

$$\tilde{\beta} = \frac{n + a_2 - 1}{b_2 + T_2}, \quad \tilde{\alpha} = \frac{m + a_1 - 1}{b_1 + T_1}, \quad \tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}}.$$

5 Numerical Experiments and Discussions

In this section we mainly perform some simulation experiments to observe the behavior of the different methods for different sample sizes and for different parameter values. All computations are performed at the Indian Institute of Technology Kanpur using Pentium IV processor. All the programs are written in FORTRAN-77 and we used the random deviate generator RAN2, described in Press *et al.* [21].

We consider both the cases separately to draw inference on R, namely when (i) λ is unknown and (ii) λ is known. We consider the following sample sizes; (m, n) = (10,10), (15,15), (20,20), (25,25), (30,30) and the following parameter values; $\alpha = 1.50$ and $\beta = 2.00$, 2.50, 3.00, 3.50 and 4.00. Without loss of generality we take $\lambda = 1$ and all the results are based on 1000 replications.

Case I: λ is unknown

From the sample, we compute the estimate of λ using the iterative algorithm (11). We started the iterative process with the initial estimate 1 and the iterative process stops when the difference between the two consecutive iterates are less than 10^{-6}. Once we estimate λ, we estimate α and β using (8) and (9) respectively. Finally we obtain the MLE of R using (12). We report the average biases and mean squared errors (MSEs) over 1000 replications. We compute the 95% confidence intervals based on the asymptotic distribution of $\hat{R}$ and using Remark 2. We also compute the 95% confidence intervals based on Boot-p and Boot-t methods. For both Boot-p and Boot-t, we took 100 replications for sample size $(10, 10)$, 200 replications for sample sizes $(15, 15)$, $(20, 20)$ and $(25, 25)$ and 300 replications for sample size $(30, 30)$. We also compute approximate confidence interval of R using the formula (16) and replacing λ by $\hat{\lambda}$. All the results are reported in Table 1

Some of the points are quite clear from this experiment. Even for small sample sizes, the performance of the MLEs are quite satisfactory in terms of biases and MSEs. It is observed that when m, n increase then MSEs decrease. It verifies the consistency property of the MLE of R. Surprisingly, the confidence intervals based on the MLEs work quite well even when the sample size is very small, say (10,10). The performance of the bootstrap confidence intervals are quite good. Particularly, Boot-p intervals perform very well. It reaches the nominal level even when the sample size is very small. The approximate confidence intervals based on F-distribution also works very well even for small sample sizes. Among the different confidence intervals, Boot-p has the shortest confidence lengths.

Case II: λ is known

In this case we obtain the estimates of R by using the MLE and UMVUE. We do not have any prior information on R, and therefore, we prefer to use the non-informative prior namely, $a_1 = a_2 = b_1 = b_2 = 0$ to compute different Bayes estimates. Using the same prior distributions, we compute approximate Bayes estimates with respect to $0 - 1$ loss function (mode of the posterior distribution), approximate Bayes estimates using Ahmad, Fakhry and Jaheen [1]'s method and Bayes estimate with respect to squared error loss function using MCMC method. We report the average estimates and the MSEs based on 1000 replications. The results are reported in Table 2.

In this case, as expected for all the methods when m, n increase then the average biases and the MSEs decrease. It is observed that the MLEs and UMVUEs behave almost in a similar manner, both with respect to biases and MSEs. The approximate Bayes estimate obtained by Ahmad, Kakhry and Jaheen [1]s method and by the MCMC method behave very similarly. Interestingly, the approximate Bayes estimate obtained by using mode behaves quite differently from the other. It has significantly lower biases, in most of the cases, where as it has slightly higher MSEs than the rest.

We also compute confidence intervals and the corresponding coverage percentages by different methods. We compute the confidence intervals using (16), we also use (16) replacing $\hat{R}$ by $\tilde{R}$. We compute the HPD regions assuming non-informative priors. The results are reported in Table 3. In this case all the three confidence intervals behave very similarly in the sense of average confidence lengths and coverage percentages. Among the three, the confidence intervals based on (16) and using the UMVUE of R provide the shortest length.

Now we consider some numerical simulations for R very close to 1 (greatert than 0.95). Note that in the previous cases $0.5714 < R < 0.7213$. The performances of the different estimates can be quite different for $R > 0.95$ and particularly when the sample sizes are different. To study the properties of the different estimators for $R > 0.95$ and for different m and n we perform the following simulation experiments. We consider the two cases separately as before, namely (i) unknown λ, (ii) known λ. We take the following configurations of (m, n) = (10,10), (10,20), (10,30), (30,10), (30,30) and β = 30.00, 35.00, 40.00, 45.00, 50.00. As before, $\alpha = 1.5$ and $\lambda = 1.0$. Note that here $0.9523 < R < 0.9709$. Here also all the results are based on 1000 replications. For unknown λ, the results are reported in Table 4 and for known λ, the results are reported in Tables 5 and 6.

Comparing Tables 1 and 4 it is observed that although all the methods behave similarly for moderate R, the same is not true for large R. The confidence intervals based on the MLEs have higher coverage probabilities and also larger average confidence lengths. On the other hand the approximate confidence intervals based on F distribution have smaller coverage probabilities and also smaller average confidence lengths. It is observed that for large R, boot-p method works well in terms of the coverage probabilities when the scale parameter is unknown. Interestingly, comparing Tables 2, 5 and Tables 3, 6, it is observed that the performances of the estimators do not change when scale parameters are known.

6 Data Analysis

In this section we present a data analysis of the strength data reported by Badar and Priest [3]. The data represent the strength data measured in GPa, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of $1, 10, 20$, and 50mm. Impregnated tows of 1000 fibers were tested at gauge lengths of $20, 50, 150$ and 300 mm. It is already observed by Durham and Padgett [6] that Weibull model does not work well in this case. Surles and Padgett [26], [27] observed that generalized Rayleigh works quite well for these strength data. For illustrative purpose, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes $m = 69$ and $n = 63$, respectively. We are analyzing the data by subtracting 1.0 and 1.8 from the first and second data set respectively. The transformed data sets corresponds to 20 mm and 10 mm gauge lengths are assumed to follow $GR(\alpha, \lambda)$ and $GR(\beta, \lambda)$ respectively.

We use the iterative procedure (8) using the initial estimate of $\lambda = 1.0$. We used the stopping criterion as $|\lambda_{(j)} - \lambda_{(j+1)}| < 10^{-6}$. The iterative process stops in 14 steps and the final estimates are $\hat{\alpha} = 2.4421$, $\hat{\beta} = 1.4216$, and $\hat{\lambda} = 0.8598$. Before analyzing further, we checked the validity of the models. We plot the empirical survival functions and the fitted survival functions in Figures 1 and 2

We used the Kolmogorov-Smirnov (K-S) tests for each data sets to the fitted models. It is observed that for Data Sets I and II, the K-S distances are 0.09 and 0.12 with the corresponding p values are 0.6069 and 0.2845 respectively. It indicates that the GR model

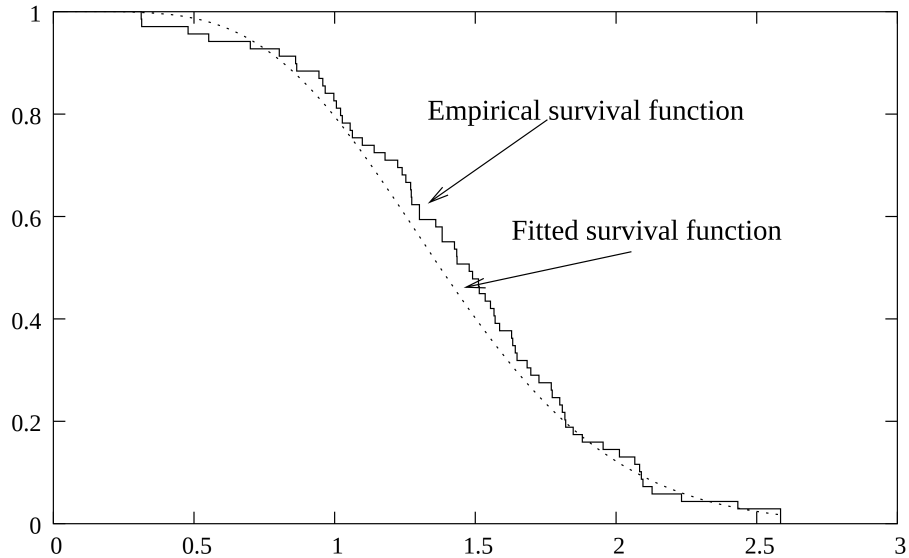

Figure 1: The empirical and fitted survival functions for the Data Set I.

provides reasonable fit to the transformed data sets.

Based on the estimates of α and β, the MLE of R, is $\hat{R} = 0.3679$ and the 95% confidence interval (asymptotic) is (0.2870, 0.4489), the Boot-p confidence interval is (0.2811,0.4428) and the corresponding Boot-t confidence interval is (0.2848, 0.4355). We also compute the 95% confidence interval based on the formula (16) and it is (0.2920, 0.4502). Note that the asymptotic confidence interval and the Boot-p confidence intervals are very similar. Based on the assumption that the common scale parameter is known, we obtain the UMVUE of λ and it is 0.3668.

Now, we obtain the Bayes estimates of R and the HPD region. Based on the non-informative prior, we obtain the posterior density function of R and it is plotted in Figure 3. From the figure, it is clear that the posterior density function is almost symmetric in nature. The Bayes estimate with respect to the squared error loss is 0.3687. An approximate Bayes estimate with respect to $0-1$ loss is 0.3661 and an approximate Bayes estimate using Ahmad, Fakhry and Jaheen [1]'s approximation is 0.3687. We obtain $(0.3096, 0.4448)$ as the 95% HPD region. Therefore, it is clear that all the Bayes estimates are quite similar

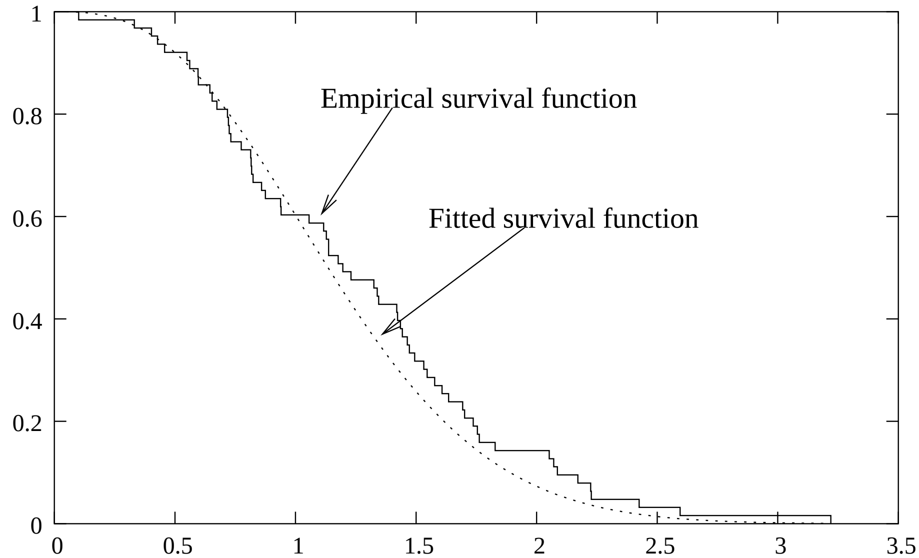

Figure 2: The empirical and fitted survival functions for Data Set II.

in nature. Moreover, the HPD region is slightly shorter in length, than the corresponding confidence intervals.

7 Conclusions

In this paper we compare different methods of estimating $R = P(Y < X)$ when Y and X both follow generalized Rayleigh distributions with different shape parameters but the same scale parameter. When the scale parameter is unknown, it is observed that the MLEs of the three unknown parameters can be obtained by solving one non-linear equation. We provide one simple iterative procedure to compute the MLEs of the unknown parameters and in turn to compute the MLE of R. We also obtain the asymptotic distribution of R and that was used to compute the asymptotic confidence intervals. It is observed that even when the sample size is quite small the asymptotic confidence intervals work quite well. We propose two bootstrap confidence intervals also and their performance are also quite satisfactory.

When the scale parameter is known we compare different estimators, namely MLE,

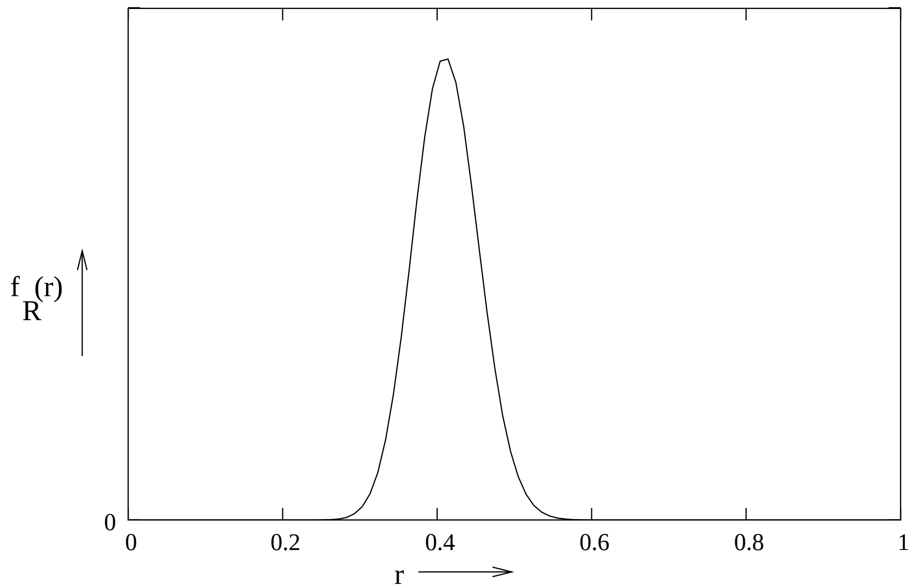

Figure 3: The posterior density function of r.

UMVUE with different Bayes estimators. It is observed that the Bayes estimators with non-informative priors behave quite similarly with the MLEs. We compute the HPD region of R also, using MCMC and interestingly, the HPD region and the confidence intervals obtained using the distribution of the MLE are quite comparable.

We should mention two points. Firstly; the asymptotic distribution of the MLE of R can be used for testing purposes also. Secondly; all the methods can be easily generalized for estimating $P(Y < cX)$ for some known c. In fact, the problem becomes quite difficult when the scale parameters are not equal. Recently Surles and Padgett [27] addressed this problem, but still satisfactory solutions are not available. More work is needed in that direction.

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Table 4: Biases, MSEs, Confidence Lengths and Coverage Percentages of the different methods.

S.S.	30.00	35.00	40.00	45.00	50.00
(10,10)	0.0018(0.0007)	0.0017(0.0005)	0.0014(0.0004)	0.0013(0.0003)	0.0011(0.0003)
	0.1682(99%)	0.1560(99%)	0.1473(99%)	0.1401(99%)	0.1317(99%)
	0.1019(95%)	0.0912(94%)	0.0819(94%)	0.0744(94%)	0.0683(94%)
	0.1738(97%)	0.1593(95%)	0.1482(95%)	0.1394(94%)	0.1319(94%)
	0.0841(85%)	0.0732(84%)	0.0649(83%)	0.0582(82%)	0.0528(81%)
(10,20)	0.0001(0.0005)	0.0001(0.0004)	0.0002(0.0003)	0.0002(0.0003)	0.0002(0.0002)
	0.1527(99%)	0.1468(99%)	0.1495(99%)	0.1439(99%)	0.1444(100%)
	0.0673(93%)	0.0592(93%)	0.0529(93%)	0.0477(93%)	0.0436(93%)
	0.1099(86%)	0.1129(89%)	0.0909(87%)	0.1019(87%)	0.0824(89%)
	0.0710(86%)	0.0618(85%)	0.0547(84%)	0.0491(83%)	0.0445(83%)
(10,30)	0.0014(0.0005)	0.0011(0.0003)	0.0010(0.0003)	0.0008(0.0002)	0.0007 (0.0002)
	0.1590(99%)	0.1671(100%)	0.1671(100%)	0.1711(100%)	0.1832(100%)
	0.0715(97%)	0.0628(97%)	0.0563(97%)	0.0518(98%)	0.0482(96%)
	0.0980(95%)	0.0912(95%)	0.0968(90%)	0.1050(91%)	0.0976(91%)
	0.0677(88%)	0.0588(86%)	0.0521(85%)	0.0467(85%)	0.0423(84%)
(30,10)	0.0025(0.0004)	0.0023(0.0003)	0.0021(0.0002)	0.0019(0.0002)	0.0018(0.0002)
	0.1170(98%)	0.1064(98%)	0.0979(98%)	0.0908(98%)	0.0848(98%)
	0.0697(93%)	0.0612(93%)	0.0547(93%)	0.0495(92%)	0.0453(92%)
	0.0914(91%)	0.0814(89%)	0.0735(89%)	0.0671(89%)	0.0617(88%)
	0.0705(89%)	0.0613(88%)	0.0543(87%)	0.0487(86%)	0.0442(86%)
(30,30)	0.0007(0.0002)	0.0007(0.0002)	0.0007(0.0001)	0.0006(0.0001)	0.0006(0.00001)
	0.0933(99%)	0.0857(99%)	0.0796(99%)	0.0745(99%)	0.0702(100%)
	0.0590(95%)	0.0530(95%)	0.0482(96%)	0.0443(96%)	0.0410(96%)
	0.0550(97%)	0.0491(97%)	0.0445(97%)	0.0408(97%)	0.0377(97%)
	0.0469(87%)	0.0407(86%)	0.0360(86%)	0.0323(85%)	0.0292(84%)

The first rows represent the average biases and the corresponding MSEs are reported within brackets. Second, third, fourth rows represent the average lengths and the corresponding coverage percentages of the asymptotic, boot-p and boot-t confidence intervals. The fifth rows represent the average lengths and the corresponding coverage percentages bases on the formula (16), simply putting $\lambda = \hat{\lambda}$.

Table 5: Biases and MSEs of the different estimators.

S.S.	30.00	35.00	40.00	45.00	50.00
(10,10)	-0.0037(0.0004)	-0.0033(0.0003)	-0.0030(0.0002)	-0.0027(0.0002)	-0.0024(0.0002)
	-0.0005(0.0003)	-0.0005(0.0003)	-0.0004(0.0002)	-0.0004(0.0002)	-0.0003(0.0001)
	0.0027(0.0003)	0.0024(0.0002)	0.0021(0.0002)	0.0019(0.0001)	0.0017(0.0001)
	-0.0086(0.0005)	-0.0076(0.0004)	-0.0068(0.0003)	-0.0061(0.0003)	-0.0056(0.0002)
	-0.0084(0.0005)	-0.0075(0.0004)	-0.0067(0.0003)	-0.0060(0.0003)	-0.0055(0.0002)
(10,20)	-0.0040(0.0003)	-0.0035(0.0002)	-0.0031(0.0002)	-0.0028(0.0001)	-0.0026(0.0001)
	-0.0000(0.0003)	-0.0000(0.0002)	-0.0000(0.0002)	-0.0000(0.0001)	-0.0000(0.0001)
	0.0015(0.0003)	0.0014(0.0002)	0.0012(0.0002)	0.0011(0.0001)	0.0010(0.0001)
	-0.0058(0.0004)	-0.0051(0.0003)	-0.0046(0.0002)	-0.0041(0.0002)	-0.0038(0.0001)
	-0.0058(0.0004)	-0.0051(0.0003)	-0.0045(0.0002)	-0.0041(0.0002)	-0.0037(0.0001)
(10,30)	-0.0037(0.0003)	-0.0033(0.0002)	-0.0029(0.0002)	-0.0027(0.0001)	-0.0024(0.0001)
	-0.0005(0.0002)	-0.0005(0.0002)	-0.0004(0.0001)	-0.0004(0.0001)	-0.0003(0.0001)
	0.0005(0.0002)	0.0004(0.0001)	0.0004(0.0001)	0.0004(0.0001)	0.0003(0.0001)
	-0.0057(0.0003)	-0.0050(0.0003)	-0.0045(0.0002)	-0.0041(0.0002)	-0.0037(0.0001)
	-0.0057(0.0003)	-0.0050(0.0003)	-0.0045(0.0002)	-0.0040(0.0002)	-0.0037(0.0001)
(30,10)	-0.0012(0.0002)	-0.0011(0.0001)	-0.0010(0.0001)	-0.0009(0.0001)	-0.0008(0.0001)
	-0.0003(0.0002)	-0.0003(0.0001)	-0.0002(0.0001)	-0.0002(0.0001)	-0.0002(0.0001)
	-0.0030(0.0002)	0.0027(0.0001)	0.0024(0.0001)	0.0021(0.0001)	0.0019(0.0001)
	-0.0055(0.0003)	-0.0048(0.0002)	-0.0043(0.0002)	-0.0039(0.0001)	-0.0035(0.0001)
	-0.0054(0.0003)	-0.0048(0.0002)	-0.0043(0.0002)	-0.0038(0.0001)	-0.0035(0.0001)
(30,30)	-0.0013(0.0001)	-0.0011(0.0001)	-0.0010(0.0001)	-0.0009(0.0001)	-0.0008(0.00004)
	-0.0004(0.0001)	-0.0004(0.0001)	-0.0003(0.0001)	-0.0003(0.0001)	-0.0003(0.00004)
	0.0007(0.0001)	0.0006(0.0001)	0.0006(0.0001)	0.0005(0.00004)	0.0005(0.00003)
	-0.0029(0.0001)	-0.0026(0.0001)	-0.0023(0.0001)	-0.0021(0.0001)	-0.0019(0.00005)
	-0.0029(0.0001)	-0.0026(0.0001)	-0.0023(0.0001)	-0.0020(0.0001)	-0.0019(0.00004)

The first, second, third, fourth and fifth rows represent the biases and the corresponding MSEs by MLEs, UMVUES, approximate Bayes (with respect to 0-1 loss function), approximate Bayes (Lindley's approximation) and the Bayes estimators (with respect to the squared error loss function) are reported within brackets

Table 6: The average confidence, HPD lengths and coverage percentages.

S.S.	30.00	35.00	40.00	45.00	50.00
(10,10)	0.0969(95%)	0.0850(95%)	0.0758(95%)	0.0683(95%)	0.0622(95%)
	0.0895(95%)	0.0783(95%)	0.0696(95%)	0.0626(95%)	0.0569(95%)
	0.0888(95%)	0.0777(95%)	0.0691(95%)	0.0622(95%)	0.0566(95%)
(10,20)	0.0778(94%)	0.0681(94%)	0.0605(94%)	0.0544(94%)	0.0494(94%)
	0.0715(93%)	0.0624(93%)	0.0554(93%)	0.0497(93%)	0.0451(93%)
	0.0759(95%)	0.0663(95%)	0.0589(95%)	0.0530(95%)	0.0481(95%)
(10,30)	0.0727(95%)	0.0636(95%)	0.0565(95%)	0.0508(95%)	0.0462(95%)
	0.0668(94%)	0.0583(94%)	0.0516(94%)	0.0464(94%)	0.0421(94%)
	0.0691(94%)	0.0603(94%)	0.0535(94%)	0.0480(94%)	0.0436(94%)
(30,10)	0.0766(96%)	0.0670(96%)	0.0595(96%)	0.0536(96%)	0.0487(96%)
	0.0748(96%)	0.0654(96%)	0.0580(96%)	0.0522(96%)	0.0474(96%)
	0.0704(95%)	0.0615(95%)	0.0545(95%)	0.0490(95%)	0.0445(95%)
(30,30)	0.0495(96%)	0.0431(96%)	0.0382(96%)	0.0342(96%)	0.0311(96%)
	0.0481(96%)	0.0419(96%)	0.0371(96%)	0.0333(96%)	0.0302(96%)
	0.0485(95%)	0.0421(95%)	0.0374(95%)	0.0335(95%)	0.0303(95%)

The first and second rows represent the confidence intervals based on (16) using the estimate of R as MLE or UMVUE. The third rows represent the average HPD intervals and the corresponding coverage percentages based on MCMC.