

# ANALYZING NON-STATIONARY SIGNALS USING GENERALIZED MULTIPLE FUNDAMENTAL FREQUENCY MODEL

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ABSTRACT. In this paper we propose a new generalized multiple frequency model to analyze non-stationary signals. The model under the assumptions of additive stationary errors can be used quite effectively to analyze different signals. We propose the usual least squares estimators to estimate the unknown parameters and it is shown that the estimators are strongly consistent. We obtain the asymptotic distributions also. The performance of the proposed model is compared with the multiple frequency model using Monte Carlo simulations. Finally several real data are analyzed using both the proposed model and the multiple frequency model.

## 1. INTRODUCTION

Parametric modeling of signals and estimating the parameters of different models are important problems in Statistical Signal Processing. Several models have been proposed in the past twenty five years to analyze several stationary signals. Statistical performance of different methods depend very much on the suitability of the selected model and also on the estimation procedure. Several models like Moving Average (MA), Autoregressive (AR) or Autoregressive Moving Average (ARMA) are being used extensively for analyzing stationary signals. In practice, many signals are non-stationary in nature and it is well known that analyzing non-stationary signals is quite difficult in general. Quasi-stationarity is often used in analyzing non-stationary signals. The main idea about quasi-stationarity is to assume stationarity over a short data segment (see for example Isaksson, Wennberg and Zetterberg; 1981, Kay; 1988 and McAulay and Quatieri; 1986, Dahlhaus; 1997, Ombao et al.; 2001) and the analysis of the signal is performed over this short data length. Therefore, by this method a compromise is needed between the model

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validity over the entire data length and the estimation of the unknown parameters. In this paper we introduce a new model to analyze non-stationary signals which is a generalization of the fundamental frequency model as well as the harmonic model with multiple fundamentals and provide an estimation procedure under the assumption of stationary noise random variables.

We consider the following model in stationary noise,

$$(1) \quad y(t) = \sum_{k=1}^M f_k(t; \boldsymbol{\theta}_k) + X(t), \quad t = 1, \dots, N.$$

We assume that there are  $M$  fundamental frequencies and the other frequencies appear in the model with a certain relationship associated with each fundamental frequency. Here  $f_k(t; \boldsymbol{\theta}_k)$  is the contribution of the  $k$ -th fundamental frequency and is a sum of  $q_k$  sinusoidal components of the following form:

$$(2) \quad f_k(t; \boldsymbol{\theta}_k) = \sum_{j=1}^{q_k} \rho_{k_j}^0 \cos \left\{ [\lambda_k^0 + (j-1)\omega_k^0]t - \phi_{k_j}^0 \right\},$$

where  $\lambda_k^0$  is the fundamental frequency and the other frequencies associated with  $\lambda_k^0$  are occurring at  $\lambda_k^0, \lambda_k^0 + \omega_k^0, \dots, \lambda_k^0 + (q_k - 1)\omega_k^0$ . Note that,  $\lambda_k^0 + \omega_k^0, \dots, \lambda_k^0 + (q_k - 1)\omega_k^0$ , need not be harmonics of  $\lambda_k^0$ . If  $\lambda_k^0 = \omega_k^0$ , then they are harmonics of  $\lambda_k^0$ . Corresponding to the frequency  $\lambda_k^0 + (j-1)\omega_k^0$ ,  $\rho_{k_j}^0$  and  $\phi_{k_j}^0$  represent the amplitude and phase components respectively and they are also unknown.

We make the following assumptions on the model parameters and noise random variables  $X(t)$ .

**Assumption 1.**

$$(3) \quad \rho_{k_j}^0 > 0, \quad \phi_{k_j}^0 \in (-\pi, \pi), \quad \lambda_k^0, \omega_k^0 \in (0, \pi), \quad j = 1, \dots, q_k, \quad k = 1, \dots, M.$$

**Assumption 2.**  $\lambda_k^0$  and  $\omega_k^0$ ,  $k = 1, \dots, M$  are such that

$$\lambda_k^0 + (i_1 - 1)\omega_k^0 \neq \lambda_l^0 + (i_2 - 1)\omega_l^0$$

for  $i_1 = 1, \dots, q_k$ ;  $i_2 = 1, \dots, q_l$  and  $k \neq l = 1, \dots, M$ .

**Assumption 3.** The number of fundamental frequencies,  $M$  and the number of components  $q_k$  associated with the  $k^{\text{th}}$  fundamental frequency,  $k = 1, \dots, M$  are known.

**Assumption 4.**  $X(t)$  has the following representation:

$$(4) \quad X(t) = \sum_{k=-\infty}^{\infty} \alpha(k)e(t-k),$$

where  $\{e(t)\}$  is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance  $\sigma^2$ . The arbitrary constants  $\alpha(k)$ 's are such that

$$\sum_{k=-\infty}^{\infty} |\alpha(k)| < \infty.$$

Assumptions 1-4 are quite general. We need assumption 2 for identifiability. It only says that the effective frequencies are distinct. Note that  $\lambda_M + (q_M - 1)\omega_M < \pi$ . We should mention here that under assumption 4, the signal  $y(t)$ 's are non-stationary in mean, not in second or higher order structure. Thus, given an observed signal  $\{y(t); t = 1, \dots, N\}$ , the aim is to estimate the unknown parameters, namely  $\rho$ 's,  $\lambda$ 's,  $\omega$ 's and  $\phi$ 's under assumptions 1-4.

We are interested to study the model (1) under assumptions 1-4. Several authors have considered various forms of the model (1) with  $M = 1$  and without any restriction on the frequencies, namely,

$$(5) \quad y(t) = \sum_{k=1}^p a_k^0 \cos(\beta_k^0 t - \phi_k^0) + X(t), \quad t = 1, \dots, N,$$

where  $a_k^0$ 's are non-negative amplitudes,  $\beta_k^0$ 's are frequencies and  $\phi_k^0$ 's are phases and they are unknown. Note that the multiple frequency model (MFM) (5) is a more general model and can be used when no relationship exists among the frequencies. But, if the frequencies are related, then this additional information is helpful to reduce the number of non-linear parameters. In section 4, we shall see how the model (1) is used to analyze different real data.

The model (5) is a well-studied model and several authors considered the model with different assumptions on the noise random variables and proposed different estimation procedures. References may be made to works of Walker (1971), Hannan (1971, 1973), Rice and Rosenblatt (1988), Kundu (1997) and so on. Observe that the proposed model is a generalization of the following fundamental frequency model

$$(6) \quad y(t) = \sum_{j=1}^q \rho_j^0 \cos(j\lambda^0 t - \phi_j^0) + X(t), \quad t = 1, \dots, N.$$

When  $M = 1$ ,  $q_1 = q$  and  $\lambda_1^0 = \omega_1^0 = \lambda^0$ , model (1) coincides with model (6). The model (6) was considered by Hannan (1974), Baldwin and Thomson (1978), Quinn and Thomson (1991), Nandi and Kundu (2003) and Kundu and Nandi (2004) to analyze different real life data sets. Quinn and Thomson (1991) obtained the theoretical properties of an equivalent estimator of the generalized least squares estimator. Nandi and Kundu (2003)

discussed the theoretical properties of the least squares estimators for the model (6). The model (6) has only one non-linear parameter, and so if all the frequencies are harmonics of a particular frequency, the fundamental frequency model is the best one as maximum reduction of the number of parameters is possible. Recently, Irizarry (2000) considered a similar model to analyze several musical sound (harmonical) data. The model is expressed as follows:

$$(7) \quad y(t) = \sum_{j=1}^J \left\{ \sum_{k=1}^{K_j} (A_{j,k} \cos(k\theta_j t) + B_{j,k} \sin(k\theta_j t)) \right\} + X(t)$$

and is a harmonic model with multiple fundamental frequencies  $\theta_1, \dots, \theta_J$ . This model is exactly equal to the fundamental frequency model (6) with  $J = 1$ ,  $A_{1,k} = \phi_k \cos(\phi_k)$  and  $B_{1,k} = \rho_k \sin(\phi_k)$ . Irizarry (2000) proposed window-based estimators using the method of weighted least squares to estimate the unknown parameters and established the consistency and asymptotic normality properties of the estimators. The model (1) is a generalization of model (7). If  $\lambda_k^0 = \omega_k^0$  in model (1), it coincides with (7) (writing  $\rho_{k_j} \cos(\phi_{k_j}) = A_{j,k}$  and  $\rho_{k_j} \sin(\phi_{k_j}) = B_{j,k}$ ). Therefore, the proposed model is a generalization of the harmonic model (7) with multiple fundamentals (so also of the fundamental frequency model (6)) and a particular case of the frequency model (5) which has several applications in different field of science.

The presence of this kind of periodicity is a convenient approximation, but many real-life phenomena can be described quite effectively, using models (6), (7) and similarly by using model (1). We shall see later on in this paper that incidentally several short duration speech data can be successfully modeled using (1). Baldwin and Thompson (1978) and Quinn and Thompson (1991) used the model (6) to describe the visual observation of S. Carinae, a variable star in the Southern Hemisphere sky. Greenhouse, Kass and Tsay (1987) proposed the use of higher-order harmonic terms of one or more fundamentals and ARMA processes for the errors (so model (6) and (7)) for fitting biological rhythms and illustrated it by analyzing human core body temperature data. The harmonic regression model has also been used to assess the static properties of human circadian systems; see Brown and Czeisler (1992) and Brown and Luithardt (1999). To analyze the periodic changes in the functional activity of specific groups of neurons in the human SCN (suprachiasmatic nucleus), the annual cycles of peptidergic activity could be described by a multiple harmonic regression model with ARMA errors (Hofman; 2001). Musical sound waves produced by musical instruments can be analyzed using above mentioned models (Rodet: 1997). Irizarry (2000) studied a segment of sound produced by a pipe organ playing two consecutive notes using model (7). Sircar and Syali (1996) proposed an amplitude modulated model with i.i.d. error by exploiting some special features of some short-length

voiced speech signals and analyzed “aaa” and “uuu” sound data. Nandi, Kundu and Iyer (2004) also studied this amplitude modulated model with stationary error and analyzed the same data sets. The analysis of these two data sets are also included in this paper using model (1).

We propose the usual least squares estimators (LSEs) to estimate the unknown parameters of the model (1) and obtain the theoretical properties of the estimators. We note that the model (1) is highly non-linear in its parameters. Therefore, all the theoretical results of the least squares estimators are asymptotic and it is not possible to obtain the finite sample behavior theoretically. It also does not satisfy the sufficient conditions given in Jennrich (1969) or Wu (1981) for the LSEs to be consistent. So the consistency and asymptotic normality properties are not automatically followed from standard results already available in the literature. We need to prove them in some different way. We observe that the least squares estimators are consistent and they are asymptotically normally distributed under the assumption that the error process is a linear process. The asymptotic distribution provides us to approximate the variances of the estimates for finite samples and to construct the error bounds of all the estimators. We compare models (1) and (5) using numerical examples based on simulated as well as real life data sets.

The rest of the paper is organized as follows. In section 2, we define the LSEs of the unknown parameters and derive the asymptotic properties of the LSEs. Simulation results are presented in section 3 to see the small sample performance. Different real data sets are analyzed in section 4 and finally we conclude the paper in section 5. The proofs are given in the appendix.

## 2. ASYMPTOTIC PROPERTIES

In this section we define the usual least squares estimators and obtain their theoretical properties. We denote the parameter vector  $\Psi$  as  $\Psi = (\theta_1, \dots, \theta_M)$  for the model (1), where  $\theta_k = (\rho_{k_1}, \dots, \rho_{k_{q_k}}, \phi_{k_1}, \dots, \phi_{k_{q_k}}, \lambda_k, \omega_k)$ ,  $k = 1, \dots, M$ .  $\Psi^0$  denotes the true parameter value. Here  $M$  refers to the total number of fundamental frequencies present and  $q_k$ , the number of frequencies associated with the  $k^{th}$  fundamental frequency. Under assumption 3, the other parameters of this model are identifiable provided they satisfy assumption 2.

Least squares method consists of choosing  $\hat{\Psi}$  by minimizing the criterion

$$(8) \quad Q_N(\Psi) = \sum_{t=1}^N \left( y(t) - \sum_{k=1}^M \sum_{j=1}^{q_k} \rho_{k_j} \cos \{ (\lambda_k + (j-1)\omega_k)t - \phi_{k_j} \} \right)^2.$$

Note that obtaining the LSEs involves a  $2 \sum_{k=1}^M q_k + 2M$  dimensional minimization search. When  $M$  and  $q_k$ ,  $k = 1, \dots, M$  are large the least squares estimators may be very expensive. But  $\rho_{k_j}$ s and  $\phi_{k_j}$ s can be expressed as functions of the frequencies  $\lambda_k$ s and  $\omega_k$ s, so using the separable regression technique of Richards (1961), it involves a  $2M$  dimensional search.

We now present results describing the asymptotic properties of the least squares estimators for the parameter  $\Psi^0$  of the model defined in (1). We prove all the results in the appendix.

**Theorem 2.1.** *Under assumption 1, 3 and 4, the least squares estimator  $\hat{\Psi}$  of  $\Psi^0$  is a strongly consistent estimator of  $\Psi^0$ .*

Now for asymptotic distribution, let us define a diagonal matrix  $\mathbf{V}$  as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_M \end{bmatrix},$$

where for each  $k = 1, \dots, M$ ,  $\mathbf{D}_k$  is a diagonal matrix of the following form:

$$\mathbf{D}_k = \begin{bmatrix} \frac{1}{N^{1/2}} \mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N^{1/2}} \mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{N^{3/2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{N^{3/2}} \end{bmatrix}.$$

Here  $\mathbf{I}_{q_k}$  denotes the identity matrix of order  $q_k$ . Let  $R = 2 \sum_{k=1}^M q_k + 2M$ . The following theorem states the asymptotic distribution of the LSE  $\hat{\Psi}$  of  $\Psi^0$ .

**Theorem 2.2.** *Under the same assumption as Theorem 2.1 and Assumption 2,*

$$(\hat{\Psi} - \Psi^0)\mathbf{V}^{-1} \rightarrow \mathcal{N}_R(\mathbf{0}, 2\sigma^2\boldsymbol{\Sigma}^{-1}\mathbf{G}\boldsymbol{\Sigma}^{-1}),$$

as  $N \rightarrow \infty$ . The matrices  $\boldsymbol{\Sigma}$  and  $\mathbf{G}$  are as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Sigma}_M \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_M \end{bmatrix},$$

where

$$\boldsymbol{\Sigma}_k = \begin{pmatrix} \mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{J}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{J}_k^T\mathbf{P}_k & \frac{1}{3}\mathbf{J}_k^T\mathbf{P}_k\mathbf{J}_k & \frac{1}{3}\mathbf{J}_k^T\mathbf{P}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}_k^T\mathbf{P}_k & \frac{1}{3}\mathbf{L}_k^T\mathbf{P}_k\mathbf{J}_k & \frac{1}{3}\mathbf{L}_k^T\mathbf{P}_k\mathbf{L}_k \end{pmatrix}, \quad k = 1, \dots, M$$

and

$$\mathbf{G}_k = \begin{pmatrix} \mathbf{C}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_k\mathbf{C}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{J}_k^T\mathbf{P}_k\mathbf{C}_k & \frac{1}{3}\mathbf{J}_k^T\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & \frac{1}{3}\mathbf{J}_k^T\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}_k^T\mathbf{P}_k\mathbf{C}_k & \frac{1}{3}\mathbf{L}_k^T\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & \frac{1}{3}\mathbf{L}_k^T\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \end{pmatrix}, \quad k = 1, \dots, M.$$

Here

$$\begin{aligned} \mathbf{P}_k &= \text{diag}\{\rho_{k1}^{0^2}, \dots, \rho_{kq_k}^{0^2}\}, & \mathbf{J}_k &= (1, 1, \dots, 1)_{q_k \times 1}^T, \\ \mathbf{L}_k &= (0, 1, \dots, q_k - 1)_{q_k \times 1}^T, & \mathbf{C}_k &= \text{diag}\{c_k(1), \dots, c_k(q_k)\}, \quad k = 1, \dots, M \end{aligned}$$

and

$$\begin{aligned} c_k(j) &= \left( \sum_{l=-\infty}^{\infty} \alpha(l) \cos\{(\lambda_k^0 + (j-1)\omega_k^0)l\} \right)^2 + \left( \sum_{l=-\infty}^{\infty} \alpha(l) \sin\{(\lambda_k^0 + (j-1)\omega_k^0)l\} \right)^2 \\ &= \left| \sum_{l=-\infty}^{\infty} \alpha(l) e^{-i\{(\lambda_k^0 + (j-1)\omega_k^0)l\}} \right|^2, \quad j = 1, \dots, q_k; \quad k = 1, \dots, M. \end{aligned}$$

**Remark 2.1.** *The asymptotic distribution of the LSEs indicates that  $\hat{\theta}_j$  and  $\hat{\theta}_k$  are asymptotically independent if  $j \neq k$ , that is, the estimators of the unknown parameters corresponding to different fundamental frequencies are independent.*

**Remark 2.2.** *As  $\Sigma$  and  $\mathbf{G}$  are block-diagonal matrices,  $\Sigma^{-1}\mathbf{G}\Sigma^{-1}$  is also a block-diagonal matrix with diagonal blocks as  $\Sigma_{\mathbf{k}}^{-1}\mathbf{G}_{\mathbf{k}}\Sigma_{\mathbf{k}}^{-1}$ ,  $k = 1, \dots, M$ . The off-diagonal blocks are  $\mathbf{0}$  matrices. This implies that for each  $k = 1, \dots, M$ ,*

$$(\hat{\theta}_k - \theta_k^0)\mathbf{D}_{\mathbf{k}}^{-1} \rightarrow \mathcal{N}_{2q_k+2}(\mathbf{0}, 2\sigma^2\Sigma_{\mathbf{k}}^{-1}\mathbf{G}_{\mathbf{k}}\Sigma_{\mathbf{k}}^{-1}),$$

as  $N \rightarrow \infty$ .

**Remark 2.3.** *It can be seen from the definition of the matrix  $\mathbf{D}_{\mathbf{k}}$  and Remark 2.2 that the normalization factor associated with  $\hat{\lambda}_k$  and  $\hat{\omega}_k$  is  $N^{3/2}$  whereas with  $\hat{\rho}_{k_j}$  and  $\hat{\phi}_{k_j}$ , it is  $N^{1/2}$ . This indicates that for a given sample size  $N$ , the frequencies can be estimated more accurately than the other parameters and the rate of convergence is much higher in case of estimators of  $\lambda_k^0$  and  $\omega_k^0$ .*

**Remark 2.4.** *Note that the matrices  $\Sigma_{\mathbf{k}}$  and  $\mathbf{G}_{\mathbf{k}}$  are of the form  $\Sigma_{\mathbf{k}} = \begin{pmatrix} \mathbf{I}_{q_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\mathbf{k}} \end{pmatrix}$*

*and  $\mathbf{G}_{\mathbf{k}} = \begin{pmatrix} \mathbf{C}_{\mathbf{k}} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbf{k}} \end{pmatrix}$ . So  $\Sigma_{\mathbf{k}}^{-1}\mathbf{G}_{\mathbf{k}}\Sigma_{\mathbf{k}}^{-1} = \begin{pmatrix} \mathbf{C}_{\mathbf{k}} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\mathbf{k}}^{-1}\mathbf{H}_{\mathbf{k}}\mathbf{F}_{\mathbf{k}}^{-1} \end{pmatrix}$  which implies that, the*

*amplitude estimators are independent of the corresponding phase and frequency parameter estimators.*

**Remark 2.5.** *From theorem 2.2, it can be seen that asymptotic distribution of the LSEs of the unknown parameters is independent of the true values of the phases.*

**Remark 2.6.** *For the linear processes i.e processes satisfying Assumption 4,  $\sigma^2 c_k(j)$  is exactly equal to the spectral density function of the process. Also it can be shown that*

$$\sigma^2 c_k(j) = E \left( \frac{1}{N} \left| \sum_{t=1}^N X(t) e^{-i(\lambda_k^0 + (j-1)\omega_k^0)t} \right|^2 \right), \quad j = 1, \dots, q_k; \quad k = 1, \dots, M,$$

which is the expected value of the periodogram function

$$(9) \quad I(\lambda) = \frac{1}{N} \left| \sum_{t=1}^N X(t) e^{-i\lambda t} \right|^2$$

of the error random variables  $X(t)$ . Thus, for simulation study,  $\sigma^2 c_k(j)$  can be estimated by local averaging of the periodogram function of the error process across the point estimate of the effective frequencies  $\hat{\lambda}_k, \hat{\lambda}_k + \hat{\omega}_k, \dots, \hat{\lambda}_k + (q_k - 1)\hat{\omega}_k$ . Another approach to estimate  $\sigma^2 c_k(j)$  is to model  $\{X(t)\}$  as an autoregressive process and then using the estimated autoregressive parameters,  $c_k(j)$ 's can be estimated. The later approach is suitable for analyzing real data sets but can not be implemented in experiments based on simulations.

### 3. SIMULATION RESULTS

In this section we present results of numerical experiments based on simulations. We compare the performance of the LSEs of the proposed model defined in (1) with the LSEs of the unknown parameters of the corresponding MFM as defined in (5). In case of MFM,  $\beta_1 = \lambda_1, \beta_2 = \lambda_1 + \omega_1, \dots, \beta_{q_1} = \lambda_1 + (q_1 - 1)\omega_1, \dots, \beta_{\sum_{k=1}^M q_k} = \lambda_{q_M} + (q_M - 1)\omega_{q_M}$ . We consider the following model for simulation studies with  $M = 2, q_1 = 3, q_2 = 3$ :

$$(10) \quad y(t) = \sum_{j=1}^3 \rho_{1j}^0 \cos \left\{ (\lambda_1^0 + (j-1)\omega_1^0)t - \phi_{1j}^0 \right\} + \sum_{j=1}^3 \rho_{2j}^0 \cos \left\{ (\lambda_2^0 + (j-1)\omega_2^0)t - \phi_{2j}^0 \right\} + X(t),$$

with  $X(t) = 0.5 e(t-1) + e(t)$  and

$$\begin{aligned} \rho_{11}^0 &= .10, \rho_{12}^0 = 0.45, \rho_{13}^0 = 0.40, \phi_{11}^0 = 0.55, \phi_{12}^0 = 0.60, \phi_{13}^0 = 0.15, \\ \rho_{21}^0 &= 0.20, \rho_{22}^0 = 0.60, \rho_{23}^0 = 0.40, \phi_{21}^0 = 0.10, \phi_{22}^0 = 0.50, \phi_{23}^0 = 0.20, \\ \lambda_1^0 &= 0.439822978, \omega_1^0 = 0.157079635, \\ \lambda_2^0 &= 1.130973372, \omega_2^0 = 0.188495562. \end{aligned}$$

Here  $e(t)$ s are *i.i.d.* Gaussian random variables with mean zero and finite variance  $\sigma^2$ . We report the results for  $N = 200$  and for error variances  $\sigma^2 = 0.2$  and  $0.4$ . We generate a data set from the model (10) and compute LSEs of different parameters by minimizing the residual sum of squares given in (8) and 95% confidence intervals for each parameter using Theorem 2.2. For minimization we use routines ‘‘amoeba’’ and ‘‘amotry’’ (based on downhill simplex method) given in Press et al. (1992). As already mentioned in Section 2, for interval estimation we need to estimate  $\sigma^2 c_k(j)$ ,  $j = 1, \dots, q_k, k = 1, \dots, M$ . We use smoothed periodogram (averaging the periodogram function over a window  $(-L, L)$ )

across the point estimate of the frequency) of the estimated error process. We replicate the process 5000 times and report average estimates, mean squared errors (MSE), average confidence lengths and coverage percentages. The results are reported in Tables 1 and 2. For comparison, we have also reported asymptotic variances and expected confidence lengths computed using the true values of the parameters. We perform same experiments on model (10), but using MFM, instead of model (1). In this case total number of non-linear parameters is  $q_1 + q_2 = 6$ . The results for MFM are reported in Tables 3 and 4.

Some of the points are quite clear from Tables 1 and 2. It is observed that for all the parameter estimators as the variance increases, average biases and mean squared errors increase. It verifies the consistency property of the LSEs. The non-linear frequency estimators are more accurate than the amplitude and phase estimators as the theory suggests. The mean squared errors and the corresponding asymptotic variances of all the estimators are quite close to each other. The coverage percentages of the parameters associated with first fundamental frequency attain the nominal level for all the parameters except  $\phi_{1_1}$  when  $\sigma^2 = 0.4$ , but in case of the second one, the amplitude and phase estimators do not attain the nominal level and they are quite poor, whereas the performance of the frequency estimators are satisfactory in all the cases considered. Since the expected confidence lengths are quite close to the average confidence lengths for all the parameters, the estimation of  $\sigma^2 c_k(j)$  is quite reasonable and the asymptotic results can be used in making finite sample inference.

Now comparing the estimators obtained using the proposed model and MFM, we observe that the amplitude and phase parameter estimators are estimated more accurately in terms of biases and mean squared errors if MFM is used. On the other hand the non-linear frequency estimators are more accurate if model (1) is used. The average confidence lengths for amplitudes are larger in case of model (1), whereas for phase estimators, they are larger in case of MFM. In case of MFM, 95% coverage percentages cover all the time for all the phase and amplitude estimators except  $a_5$  and  $\phi_5$ , for the model considered. For the frequencies also, they do not attain the nominal level in general.

#### 4. DATA ANALYSIS

In this section we use the proposed model (1) for analyzing several real datasets. We would like to mention here that several short-duration voiced speech signal, namely “eee”, “aaa”, “aww”, “uuu” and “ahh”, can be analyzed using model (1) and we present the analysis in this section. But as already mentioned in introduction that there are many

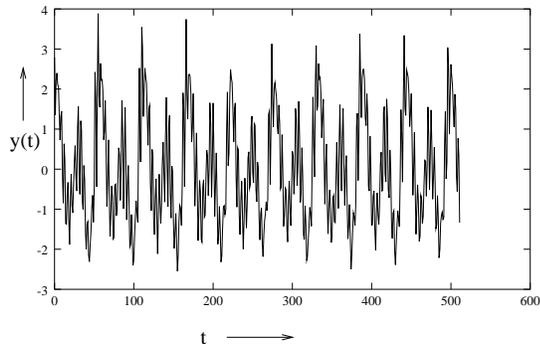


FIGURE 1. The plot of the observed “eee” vowel sound.

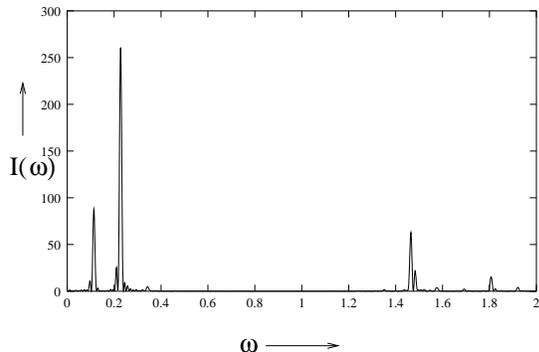


FIGURE 2. The plot of the periodogram function of “eee” sound.

other applications where this model can be satisfactorily used. The plots of the observed data sets and corresponding periodogram functions are provided in Figures 1-10. The data set “ahh” contains 340 signal values whereas each of all the other data sets contains 512 signal values, all sampled at 10 kHz frequency. We have estimated  $M$  and  $q_k$ ,  $k = 1, \dots, M$  from the periodogram plots. The periodogram is a powerful tool for locating the frequencies visually. If the observed data are periodic, the plot of the periodogram function exhibits large positive values (the squares of the amplitudes associated with the frequencies) at the true values of the underlying frequencies present in the data and at all the other points it is close to zero. Note that, we have considered the simple periodogram function, not the smoothed periodogram, which is commonly known as spectrogram in the time series literature. Also, we have calculated  $I(\lambda)$ , for each point of a grid (fine enough) of  $(0, \pi)$ . We have not calculated  $I(\lambda)$  only at the so-called Fourier frequencies  $\{\frac{2\pi j}{N}, j = 0, 1, \dots, N-1\}$ . So the number of peaks in the plot of the periodogram function gives an estimate of the number of effective frequencies present in the underlying model, or it roughly estimates the number of components  $p$  if one uses model (5). It may be quite subjective sometimes, depending on the error variance and magnitude of the amplitudes. The periodogram may show only the more dominant frequencies. In such cases, when the effects of these frequencies are removed from the observed series and the periodogram function of the residual series is plotted, then it may show some peaks corresponding to other frequencies. If the error variance is too high, periodogram plot may not exhibit a significant distinct peak at  $\lambda^*$ , even if this  $\lambda^*$  has a significant contribution to the data. Also, in case, two frequencies are “close enough” then periodogram may show only one peak. In such cases it is recommended to use of larger sample size, if it is possible and use of a finer grid may provide some more information about the presence of another frequency.

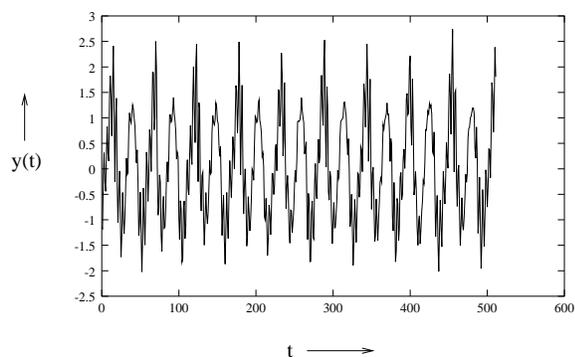


FIGURE 3. The plot of the observed “aaa” vowel sound.

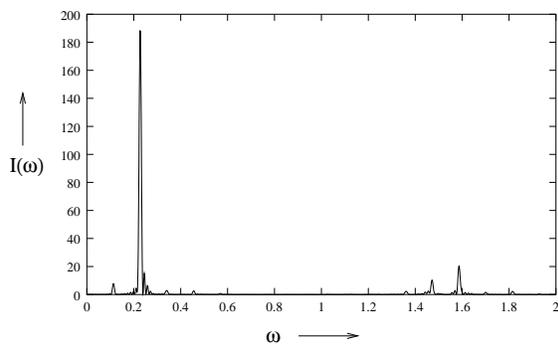


FIGURE 4. The plot of the periodogram function of “aaa” vowel sound.

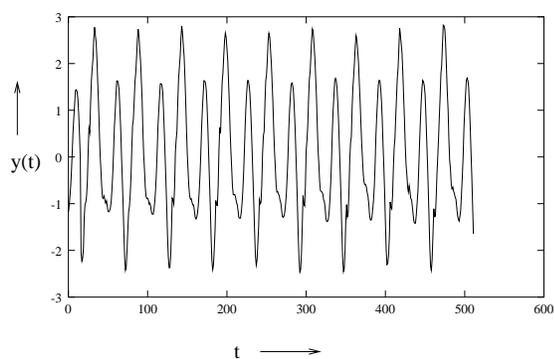


FIGURE 5. The plot of the observed “uuu” vowel sound.

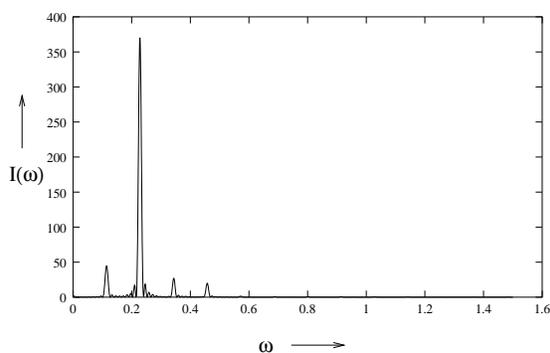


FIGURE 6. The plot of the periodogram function of “uuu” vowel sound.

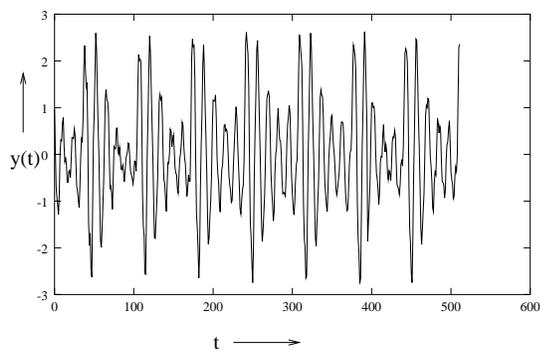


FIGURE 7. The plot of the observed “aww” sound.

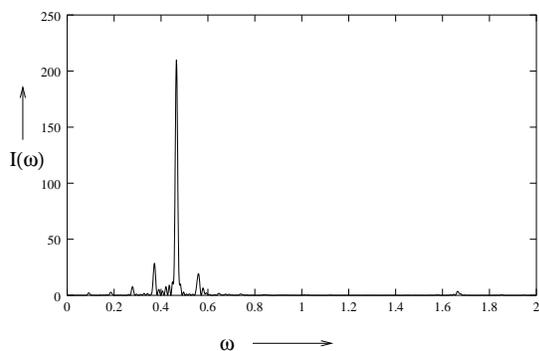


FIGURE 8. The plot of the periodogram function of “aww” sound.

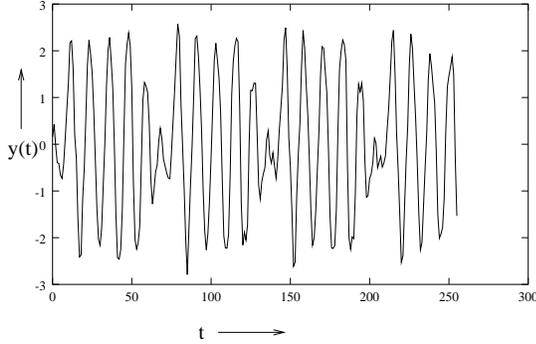


FIGURE 9. The plot of the observed “ahh” sound.

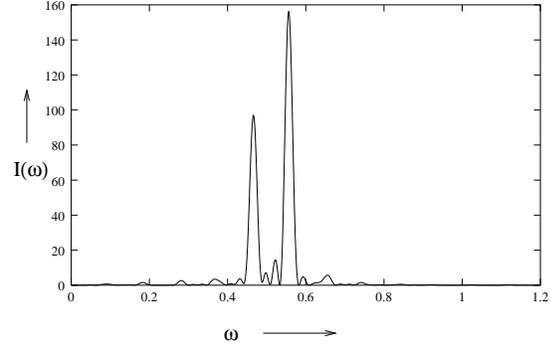


FIGURE 10. The plot of the periodogram function of “ahh” sound.

The initial estimates of the frequencies are obtained from the plot of the periodogram function. Using these initial estimates as starting values, the LSEs of the unknown parameters are obtained for all the data sets. Using Theorem 2.2, we also calculate 95% confidence intervals of the LSEs. To see how the proposed model (1) performs as compared to the general MFM (5), we estimate the LSEs of the unknown parameters of MFM. We also obtain 95% confidence intervals using the asymptotic distribution (Kundu; 1997) of the LSEs of the parameters of MFM in case of each data set. In analyzing these data sets we use estimated error random variables to estimate  $\sigma^2 c_k(j)$ ,  $j = 1, \dots, q_k$ ;  $k = 1, \dots, M$ . We use run test (Draper and Smith; 1981) to test whether the estimated error is independent or not. For “eee” data set the estimated errors are independent for both models, whereas for all the other data sets, the test statistic value confirms that the errors are correlated. Using autocorrelation and partial autocorrelation function we model the error processes as different autoregressive (AR) processes in all such cases. In case of “aaa” with model (1) the error is modeled as an AR(3) process whereas with MFM, it is modeled as an AR(1) process. For “ahh”, “aww” and “uuu” the residuals are modeled as different AR(3) processes. We estimate the AR parameters using Yule-Walker equation. Finally, we again use the run test to verify whether the independence assumption on  $\hat{e}(t)$  is satisfied at 95% level of significance or not. We see that in all the cases  $\hat{e}(t)$  satisfies the independence assumption except “aww” data set when MFM is used for estimation. As the estimated error is used to estimate  $\sigma^2 c_k(j)$ 's for these data sets, in case of “aww” data, we apply run test to the  $\hat{e}(1), \dots, \hat{e}(256)$  and then independence is satisfied. For “aww” data, we have used the fact that frequencies appear as harmonics of the first fundamental frequency. In each case the roots of the characteristic equation of the estimated AR process are less than one in absolute value, so the estimated error is stationary and

can be expressed as linear process given in (4). The results for model (1) are provided in Tables 5, 7, 9, 11 and 13 and for MFM in Tables 6, 8, 10, 12 and 14.

The predicted signals say  $\hat{y}(t)$  for all data sets are provided in Figures 11, 12, 13, 14 and 15 for “eee”, “aaa”, “uuu”, “aww” and “ahh” respectively. For comparison, we have plotted the predicted values using LSEs of model (1), predicted values using LSEs of MFM and the original signal in the same figure. The fitted values match quite well with the original signal for all the cases.

We observe that in case of “eee” and “aaa” data sets the confidence intervals of all the parameters corresponding to first fundamental frequency  $\lambda_1^0$  and amplitudes corresponding to second fundamental frequency are slightly larger in case they are estimated using model (1) than those obtained with MFM. But the confidence lengths of  $\lambda_2^0$  and  $\omega_2^0$  and corresponding phases are much smaller in case of model (1). In case of “ahh” data set there is only one frequency and the confidence lengths of the phases are larger in case of MFM. For “aww” data set confidence intervals of amplitudes associated with  $\lambda_1^0$  is much higher in model (1), whereas for phases, it is the other way. The confidence interval for frequency  $\lambda_1$  (here it is used that  $\lambda_1 = \omega_1$ ) is much lower as compared to  $\beta_1$  of MFM. For second frequency, they are almost identical (but in this case there is only one frequency, so theoretically asymptotic variances are equal) for both the models. For “uuu” data set the confidence intervals for all the parameters are smaller in case of MFM. But in “uuu” data set like “ahh”,  $M = 1$  and the total number of parameters is ten. If we use the information that  $\lambda_1 = \omega_1$ , *i.e.* if the fundamental frequency model given in (6) is used, the number of non-linear parameters reduces to one from two. Using the asymptotic distribution of  $(\hat{\lambda}_1 - \hat{\omega}_1)$ , it is observed that the confidence interval of  $(\lambda_1 - \omega_1)$  is  $(-0.004721, 0.002557)$  which includes zero. Thus, we accept the hypothesis  $H_0 : \lambda_1 - \omega_1 = 0$ . So for this particular data set it is reasonable to use the fundamental frequency model rather than the proposed model (1). If we use the fundamental frequency model (6), then we see that the confidence intervals for phases and frequency  $\lambda_1$  is much lower as compared to MFM (not reported here). Note that the asymptotic variances of a particular frequency of MFM is inversely proportional to the square of the associated amplitude and they are independent of the other frequencies, which is not true in case of the proposed model. In this case the asymptotic variances of  $\lambda_k$  and  $\omega_k$  depend on all the amplitudes  $\rho_{k_j}$ ,  $j = 1, \dots, q_k$ . Thus, we have seen that several short-duration voiced speech data can be analyzed using the model (1). In analyzing these data sets, neither of the two models outperforms the other in all respects. But the advantage of using the proposed model is that the total number of non-linear parameters to be estimated, reduces as compared to

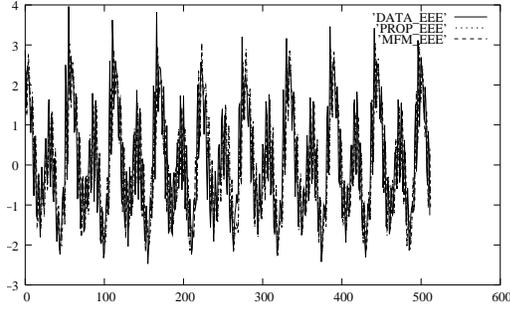


FIGURE 11. The plot of the observed (DATA\_EEE) and the fitted “eee” sound using LSEs of model (1) (PROP\_EEE) and MFM (MFM\_EEE).

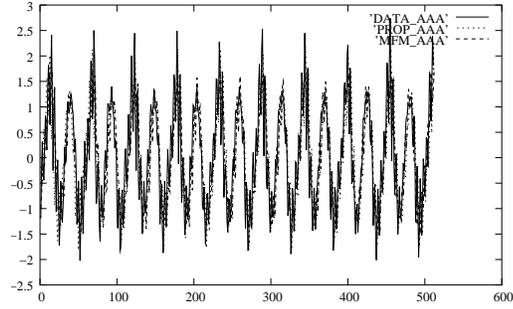


FIGURE 12. The plot of the observed (DATA\_AAA) and the fitted “aaa” sound using LSEs of model (1) (PROP\_AAA) and MFM (MFM\_AAA).

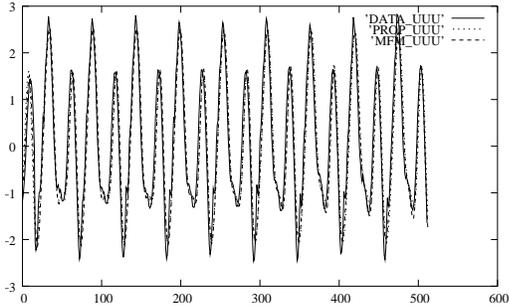


FIGURE 13. The plot of the observed (DATA\_UUU) and the fitted “uuu” sound using LSEs of model (1) (PROP\_UUU) and MFM (MFM\_UUU).

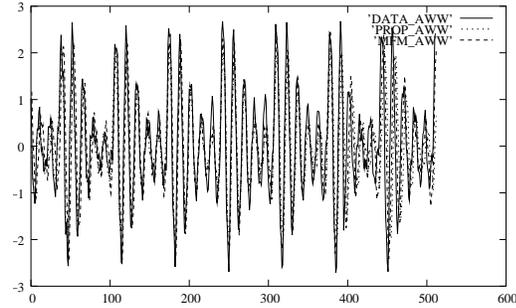


FIGURE 14. The plot of the observed (DATA\_AWW) and the fitted “aww” sound using LSEs of model (1) (PROP\_AWW) and MFM (MFM\_AWW).

the number of the effective frequencies. Several non-stationary data follow a particular relationship among the frequencies and it is captured by the proposed model.

## 5. CONCLUSIONS

In this paper we propose a new model with multiple fundamental frequencies in stationary noise. The model is a particular model of the multiple frequency model (5) and a generalization of the fundamental frequency model (6) as well as the harmonic model (7). It is observed that several non-stationary signals can be analyzed using this model. To

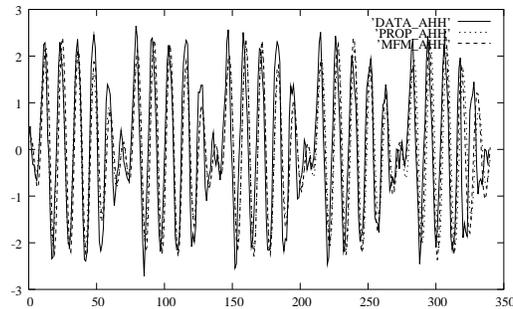


FIGURE 15. The plot of the observed (DATA\_AHH) and the fitted “ahh” sound using LSE of model (1) (PROP\_AHH) and MFM (MFM\_AHH).

analyze the data sets, the number of fundamental frequencies,  $M$  and the number of frequencies associated with  $k^{\text{th}}$  fundamental frequency  $q_k$ ,  $k = 1, \dots, M$  are estimated using the periodogram function. We have proposed the usual LSEs to estimate the unknown parameters. The estimators are strongly consistent and asymptotically normal. The asymptotic distribution indicates that the estimators of the unknown parameters corresponding to different fundamental frequencies are independent and amplitude estimators are independent of the corresponding phase and frequency estimators. The experimental results indicate that the asymptotic results can be used in making finite sample inferences. Several real data are analyzed and the estimated signals match quite well with the observed signals in each case. The asymptotic distribution is used to construct the confidence bounds of each parameter at 95% level of significance. In this paper, we have not considered the problem of estimating  $M$  and  $q_k$ . Some information theoretic criteria combined with the special structure of the proposed model may be used to estimate them. Further research is needed in this direction.

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TABLE 1. The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of the different parameters using the proposed model for  $\sigma^2 = 0.2$ .

Parameter	Average LSE (True Value)	MSE (Asym. Var.)	Av. Conf. Length (Ex. Conf. Length)	95% Cov. Prob.
$\rho_{11}$	0.125963062 (0.10)	3.95831512e-03 (4.3096542e-03)	0.337497294 (0.2573400)	0.95
$\rho_{12}$	0.455777884 (0.45)	4.15113848e-03 (4.1541611e-03)	0.326178163 (0.2526549)	0.94
$\rho_{13}$	0.404897749 (0.40)	3.96256289e-03 (3.9579375e-03)	0.315576911 (0.2466156)	0.95
$\phi_{11}$	0.494047403 (0.55)	0.688315213 (0.4414515)	4.24205685 (2.604519)	0.91
$\phi_{12}$	0.596532404 (0.60)	2.38425396e-02 (2.2643777e-02)	0.774129033 (0.5898757)	0.94
$\phi_{13}$	0.151415542 (0.15)	2.99340468e-02 (2.7628275e-02)	0.855129421 (0.6515728)	0.95
$\lambda_1$	0.439792693 (0.4398230)	1.55844623e-06 (1.0486136e-06)	5.46646677-03 (4.0141521e-03)	0.95
$\omega_1$	0.157105446 (0.1570796)	7.16287389e-07 (4.5592520e-07)	3.64642008-03 (2.6468716e-03)	0.95
$\rho_{21}$	0.210262716 (0.20)	3.31078214-03 (3.3515587e-03)	0.198635176 (0.2269392)	0.81
$\rho_{22}$	0.600659609 (0.60)	2.89968797-03 (2.9973800e-03)	0.189876512 (0.2146135)	0.82
$\rho_{23}$	0.404682457 (0.40)	2.62100901-03 (2.6255806e-03)	0.175994471 (0.2008625)	0.81
$\phi_{21}$	0.0978257582 (0.10)	9.84198451-02 (9.2979610e-02)	1.11533058 (1.195308)	0.83
$\phi_{22}$	0.501318812 (0.50)	9.60094389e-03 (1.0157371e-02)	0.359809935 (0.3950724)	0.85
$\phi_{23}$	0.203538164 (0.20)	1.92437172e-02 (2.0647796e-02)	0.506877661 (0.5632781)	0.84
$\lambda_2$	1.13098848 (1.1309734)	5.99011685e-07 (9.1906406e-07)	3.59018217-03 (3.7580191e-03)	0.95
$\omega_2$	0.188493758 (0.1884956)	3.12252752e-07 (4.8829617e-07)	2.60419305-03 (2.7392253e-03)	0.95

TABLE 2. The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of the different parameters using the proposed model for  $\sigma^2 = 0.4$ .

Parameter	Average LSE (True Value)	MSE (Asym. Var.)	Av. Conf. Length (Ex. Conf. Length)	95% Cov. Prob.
$\rho_{1_1}$	0.151604086 (0.10)	8.27635452e-03 (8.6193085e-03)	0.476105064 (0.3639337)	0.95
$\rho_{1_2}$	0.46168986 (0.45)	8.26609321e-03 (8.3083222e-03)	0.461868048 (0.3573080)	0.94
$\rho_{1_3}$	0.411106139 (0.40)	7.85315502e-03 (7.9158749e-03)	0.446626604 (0.3487671)	0.95
$\phi_{1_1}$	0.411025107 (0.55)	1.22210026 (0.8829030)	5.07852125 (3.683347)	0.88
$\phi_{1_2}$	0.594926059 (0.60)	4.85680364e-02 (4.5287553e-02)	1.11277223 (0.8342102)	0.94
$\phi_{1_3}$	0.152366251 (0.15)	6.14459403e-02 (5.5256549e-02)	1.23698533 (0.9214631)	0.95
$\lambda_1$	0.439781189 (0.4398230)	3.03807201e-06 (2.0972273e-06)	7.8823017e-03 (5.6768684e-03)	0.95
$\omega_1$	0.157114819 (0.1570796)	1.42341173e-06 (9.1185041e-07)	5.31302486e-03 (3.7432418e-03 )	0.95
$\rho_{2_1}$	0.220800266 (0.20)	6.48095924e-03 (6.7031174e-03)	0.281173646 (0.3209405)	0.81
$\rho_{2_2}$	0.602861106 (0.60)	5.78639749e-03 (5.9947600e-03)	0.268543154 (0.3035093)	0.82
$\rho_{2_3}$	0.409267187 (0.40)	5.22927288e-03 (5.2511613e-03)	0.249112844 (0.2840624)	0.81
$\phi_{2_1}$	0.0954661742 (0.10)	0.232197076 (0.1859592)	1.66008222 (1.690421)	0.82
$\phi_{2_2}$	0.501951993 (0.50)	1.93329826e-02 (2.0314742e-02)	0.514545619 (0.5587168)	0.85
$\phi_{2_3}$	0.20508866 (0.20)	3.9053704e-02 (4.1295592e-02)	0.72747165 (0.7965956)	0.85
$\lambda_2$	1.13099003 (1.1309734)	1.19849585e-06 (1.8381281e-06)	5.12786489e-03 (5.3146416e-03)	0.94
$\omega_2$	0.188494474 (0.1884956)	6.31346211e-07 (9.7659233e-07)	3.73536511e-03 (3.8738493e-03)	0.94

TABLE 3. The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of different parameters using MFM for  $\sigma^2 = 0.2$ .

Parameter	Average LSE (True Value)	MSE (Asym. Var.)	Av. Conf. Length (Ex. Conf. Length)	95% Cov. Prob.
$a_1$	0.100683041	6.62101229e-5 (4.30965424e-3)	0.220231146 (0.257339984)	1.0
$a_2$	0.451257676	3.44792061e-4 (4.15416108e-3)	0.216402516 (0.25265491)	1.0
$a_3$	0.400900543	2.59834371e-4 (3.95793747e-3)	0.212928489 (0.246615589)	1.0
$a_4$	0.200601995	9.68176464e-5 (3.35155893e-3)	0.152528748 (0.226939201)	1.0
$a_5$	0.601447225	5.48608368e-4 (2.99737975e-3)	0.138830096 (0.214613453)	0.95
$a_6$	0.400943607	2.53332662e-4 (2.62558088e-3)	0.135277435 (0.200862452)	0.98
$\phi_1$	0.551747084	4.09032963e-3 (1.72386169)	4.39787626 (5.14680004)	1.0
$\phi_2$	0.60472846	6.82766503e-3 (8.20575058e-2)	0.960425317 (1.12291074)	0.99
$\phi_3$	0.151506901	3.14068445e-3 (9.8948434e-2)	1.06390452 (1.233078)	1.0
$\phi_4$	0.0995254889	1.77992496e-3 (0.335155904)	1.52432573 (2.26939178)	1.0
$\phi_5$	0.501218498	5.90951648e-3 (3.33042182e-2)	0.462355971 (0.715378225)	0.95
$\phi_6$	0.200857386	2.44277902e-3 (6.56395182e-2)	0.675922155 (1.00431228)	0.99
$\beta_1$	0.439534605	5.60617264e-05 (1.29289634e-4)	3.80868129e-2 (4.45725955e-2)	0.95
$\beta_2$	0.596982002	2.0185114e-06 (6.15431281e-06)	8.31752364e-3 (9.72469151e-3)	0.97
$\beta_3$	0.754041851	2.13030876e-06 (7.42113252e-6)	9.21367202e-3 (1.06787682e-2)	0.98
$\beta_4$	1.1309551	9.4032639e-06 (2.51366928e-5)	1.3201056e-2 (1.96535103e-2)	0.93
$\beta_5$	1.31950402	9.81566586e-07 (2.49781647e-6)	4.00412921e-3 (6.1953566e-3)	0.89
$\beta_6$	1.50798428	1.40446798e-06 (4.92296385e-6)	5.85365482e-3 (8.69759917e-3)	0.94

TABLE 4. The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of different parameters using MFM for  $\sigma^2 = 0.4$ .

Parameter	Average LSE (True Value)	MSE (Asym. Var.)	Av. Conf. Length (Ex. Conf. Length)	95% Cov. Prob.
$a_1$	0.103271469	3.12825432e-4 (8.61930847e-3)	0.310845464 (0.363933712)	1.0
$a_2$	0.453495115	8.75412254e-4 (8.30832217e-3)	0.303890526 (0.357308)	0.99
$a_3$	0.402786911	6.71417103e-4 (7.91587494e-3)	0.299821109 (0.348767132)	1.0
$a_4$	0.20187901	3.19670828e-4 (6.70311786e-3)	0.213473439 (0.320940495)	0.99
$a_5$	0.603555858	1.3390953e-3 (5.9947595e-3)	0.195176035 (0.303509265)	0.95
$a_6$	0.402185053	6.61559461e-4 (5.25116175e-3)	0.190110669 (0.284062415)	0.97
$\phi_1$	0.553592145	8.26491881e-3 (3.44772339)	6.60993528 (7.27867413)	1.0
$\phi_2$	0.610250294	1.67299565e-2 (0.164115012)	1.345227 (1.5880357)	0.99
$\phi_3$	0.152599573	7.37958541e-3 (0.197896868)	1.49412251 (1.74383569)	1.0
$\phi_4$	.0999217778	3.84876621e-3 (0.670311809)	2.12972307 (3.20940495)	1.0
$\phi_5$	0.50403589	1.38942935e-2 (6.66084364e-2)	0.648625791 (1.01169753)	0.95
$\phi_6$	0.202204913	6.29422953e-3 (0.131279036)	0.948941886 (1.42031193)	0.99
$\beta_1$	0.439267337	9.31811592e-5 (2.58579268e-4)	5.72436824e-2 (6.30351603e-2)	0.95
$\beta_2$	0.597042143	4.41594557e-6 (1.23086256e-5)	1.16500212e-2 (1.37527911e-2)	0.97
$\beta_3$	0.754063606	4.70306395e-6 (1.4842265e-5)	1.2939482e-2 (1.51020577e-2)	0.97
$\beta_4$	1.13074279	2.5412297e-5 (5.02733856e-5)	1.84439197e-2 (2.77942587e-2)	0.90
$\beta_5$	1.31954181	2.17460683e-6 (4.99563293e-6)	5.61727071e-3 (8.76155775e-3)	0.88
$\beta_6$	1.50802243	3.02026615e-6 (9.84592771e-6)	8.2180649e-3 (1.23002622e-2)	0.93

TABLE 5. Results for “eee” data set using the proposed model.

**Data Set:** “eee”.  $M = 2$ ,  $q_1 = 3$ ,  $q_2 = 6$ .

Parameter	Estimate	Lower Bound	Upper Bound
$\rho_{1_1}$	0.792912781	0.745130777	0.840694785
$\rho_{1_2}$	1.41441321	1.36663115	1.46219528
$\rho_{1_3}$	0.203153163	0.15537113	0.250935197
$\phi_{1_1}$	0.681846261	0.434020698	0.929671824
$\phi_{1_2}$	0.913759112	0.747117102	1.08040118
$\phi_{1_3}$	-0.0415502712	-0.298853695	0.215753138
$\lambda_1$	0.1140192	0.113080189	0.114958212
$\omega_1$	0.113874406	0.11353147	0.114217341
$\rho_{2_1}$	0.110554226	0.0627721995	0.158336252
$\rho_{2_2}$	0.708103955	0.660321951	0.755885959
$\rho_{2_3}$	0.170492932	0.122710906	0.218274966
$\rho_{2_4}$	0.141166449	0.0933844224	0.188948482
$\rho_{2_5}$	0.344861776	0.297079742	0.392643809
$\rho_{2_6}$	0.16529128	0.117509253	0.213073313
$\phi_{2_1}$	0.427151084	-0.0344740637	0.888776243
$\phi_{2_2}$	0.479345709	0.34814921	0.610542178
$\phi_{2_3}$	-2.71241069	-3.01004577	-2.41477561
$\phi_{2_4}$	0.815808356	0.451096743	1.18051994
$\phi_{2_5}$	-0.773955345	-1.01284397	-0.535066724
$\phi_{2_6}$	-0.577183068	-0.966903508	-0.187462628
$\lambda_2$	1.3514533	1.35081983	1.35208678
$\omega_2$	0.11346291	0.113172941	0.113752879

$$\hat{X}(t) = e(t).$$

**Run Test:**  $z$  for  $\hat{e}(t) = -1.10903132$ .

Residual sum of squares: 0.154122174.

#### APPENDIX

For notational convenience, first we prove the results for  $M = 1$ , *i.e.* the model has only one fundamental frequency and then we sketch the outline of the proof for general

TABLE 6. LSE and confidence intervals for “eee” data with multiple frequency model.

Parameters	LSE	Lower Limit	Upper Limit
$a_1$	0.792754471	0.745822906	0.839686036
$a_2$	1.41381097	1.36687934	1.46074259
$a_3$	0.203779265	0.1568477	0.250710845
$a_4$	0.111959115	0.0650275424	0.158890679
$a_5$	0.7039783	0.657046735	0.750909865
$a_6$	0.172563389	0.125631824	0.219494954
$a_7$	0.145034194	0.0981026217	0.191965759
$a_8$	0.355013132	0.308081567	0.401944697
$a_9$	0.176291764	0.129360199	0.223223329
$\phi_1$	0.687658191	0.569256902	0.80605948
$\phi_2$	0.912467241	0.846077085	0.978857398
$\phi_3$	0.0317988843	-0.428812951	0.492410719
$\phi_4$	0.628177047	-0.21019274	1.46654689
$\phi_5$	0.495831549	0.362499118	0.62916398
$\phi_6$	2.82846522	2.28453088	3.37239957
$\phi_7$	0.847214043	0.200034678	1.49439347
$\phi_8$	-0.360235542	-0.624628961	-0.0958421007
$\phi_9$	0.00214044028	-0.530290186	0.534571111
$\beta_1$	0.114038095	0.113637552	0.114438638
$\beta_2$	0.227885813	0.227661222	0.228110403
$\beta_3$	0.342097998	0.340539783	0.343656212
$\beta_4$	1.35217214	1.34933603	1.35500824
$\beta_5$	1.46495438	1.46450329	1.46540546
$\beta_6$	1.5753814	1.57354128	1.57722151
$\beta_7$	1.69190764	1.68971825	1.69409704
$\beta_8$	1.80700564	1.80611122	1.80790007
$\beta_9$	1.92089081	1.91908967	1.92269194

$$\hat{X}(t) = e(t).$$

**Run Test:** z for  $\hat{e}(t) = -1.03238821$ .

Residual sum of squares: 0.148744926.

TABLE 7. Results for “aaa” data set using the proposed model.

Data Set: “aaa”.  $M = 2$ ,  $q_1 = 4$ ,  $q_2 = 5$ .

Parameter	Estimate	Lower Bound	Upper Bound
$\rho_{1_1}$	0.220683411	0.16870077	0.272666067
$\rho_{1_2}$	1.21162212	1.16022503	1.2630192
$\rho_{1_3}$	0.160044715	0.109573394	0.210516036
$\rho_{1_4}$	0.138260633	0.088989988	0.187531278
$\phi_{1_1}$	2.11266017	1.35893762	2.8663826
$\phi_{1_2}$	2.74804449	2.53007388	2.9660151
$\phi_{1_3}$	-0.129586205	-0.89653182	0.637359381
$\phi_{1_4}$	0.377879053	-1.02523708	1.78099513
$\lambda_1$	0.113897234	0.11110048	0.116693988
$\omega_1$	0.113964394	0.111329563	0.116599225
$\rho_{2_1}$	0.146804646	0.109372251	0.184237033
$\rho_{2_2}$	0.245368242	0.209165379	0.28157112
$\rho_{2_3}$	0.377789408	0.342714161	0.412864655
$\rho_{2_4}$	0.0969588906	0.0629100502	0.131007731
$\rho_{2_5}$	0.133374184	0.100253083	0.166495293
$\phi_{2_1}$	2.94918633	2.42440987	3.47396278
$\phi_{2_2}$	0.0518640578	-0.357301384	0.4610295
$\phi_{2_3}$	1.36409092	0.98188448	1.74629736
$\phi_{2_4}$	2.10673928	1.55073655	2.66274214
$\phi_{2_5}$	-0.706613243	-1.30023837	-0.112988077
$\lambda_2$	1.36052275	1.35873103	1.36231446
$\omega_2$	0.11399442	0.113307588	0.1146212

$$\hat{X}(t) = 0.696457803 \hat{X}(t-1) - 0.701408327 \hat{X}(t-2) + 0.618664384 \hat{X}(t-3) + e(t).$$

**Run Test:**  $z$  for  $\hat{X}(t) = -5.87313509$ ,

$z$  for  $\hat{e}(t) = 0.628699183$ .

Residual sum of squares: 0.101041436.

TABLE 8. LSE and confidence intervals for “aaa” data with multiple frequency model.

Parameters	LSE	Lower Limit	Upper Limit
$a_1$	0.226719305	0.190961957	0.262476653
$a_2$	1.21300876	1.17725146	1.24876606
$a_3$	0.163902849	0.128145501	0.199660197
$a_4$	0.137947291	0.102189943	0.173704639
$a_5$	0.143389478	0.10763213	0.179146826
$a_6$	0.266157418	0.230400071	0.301914752
$a_7$	0.38930583	0.353548497	0.425063163
$a_8$	0.101906128	0.0661487803	0.137663469
$a_9$	0.134322852	0.098565504	0.1700802
$\phi_1$	2.1619103	1.84647751	2.47734308
$\phi_2$	2.54826617	2.48930979	2.60722256
$\phi_3$	-0.0690812841	-0.505404949	0.367242366
$\phi_4$	0.371035159	-0.147385299	0.889455616
$\phi_5$	3.04547048	2.54672599	3.54421496
$\phi_6$	-0.602789819	-0.871483028	-0.334096611
$\phi_7$	0.90565449	0.721956491	1.08935249
$\phi_8$	1.58608973	0.884319425	2.28785992
$\phi_9$	-1.12101579	-1.65342474	-0.588606775
$\beta_1$	0.114094153	0.113027073	0.115161233
$\beta_2$	0.227092206	0.226892769	0.227291644
$\beta_3$	0.34200263	0.340526581	0.34347868
$\beta_4$	0.455799341	0.454045564	0.457553118
$\beta_5$	1.36083913	1.35915196	1.3625263
$\beta_6$	1.47200274	1.47109377	1.47291172
$\beta_7$	1.58692384	1.5863024	1.58754528
$\beta_8$	1.70071459	1.69834054	1.70308864
$\beta_9$	1.81501627	1.81321514	1.8168174

$$\hat{X}(t) = 0.31577149\hat{X}(t-1) + e(t).$$

**Run Test:** z for  $\hat{X}(t) = -3.13001537$ ,

z for  $\hat{e}(t) = 0.26275149$ .

Residual sum of squares: 0.0874886289.

TABLE 9. Results for “uuu” data set using the proposed model.

**Data Set: “uuu”.**  $M = 1$ ,  $q_1 = 4$ .

Parameter	Estimate	Lower Bound	Upper Bound
$\rho_{1_1}$	0.633288026	0.544679284	0.721896768
$\rho_{1_2}$	1.70542562	1.61684871	1.79400253
$\rho_{1_3}$	0.419593066	0.331111431	0.508074701
$\rho_{1_4}$	0.343196869	0.254873455	0.431520283
$\phi_{1_1}$	-2.61283183	-3.14323187	-2.08243179
$\phi_{1_2}$	1.33654237	1.08859777	1.58448696
$\phi_{1_3}$	-2.88876963	-3.43958592	-2.33795333
$\phi_{1_4}$	-2.69739866	-3.66095066	-1.73384655
$\lambda_1$	0.113104612	0.111106128	0.115103096
$\omega_1$	0.114186443	0.112432718	0.115940168

$$\hat{X}(t) = 1.2119236 \hat{X}(t-1) - 0.518095672 \hat{X}(t-2) \\ + 0.0920835361 \hat{X}(t-2) + e(t).$$

**Run Test:** z for  $\hat{X}(t) = -14.5708122$ ,  
z for  $\hat{e}(t) = -1.03364444$ .

Residual sum of squares: 0.0981521457.

$M$ . For  $M = 1$ ,  $\Psi = \theta_1 = \theta$ ,  $q_1 = q$ , say and let us write  $\theta = (\rho_1, \dots, \rho_q, \phi_1, \dots, \phi_q, \lambda, \omega)$ .  $\theta^0 = (\rho_1^0, \dots, \rho_q^0, \phi_1^0, \dots, \phi_q^0, \lambda^0, \omega^0)$  and  $\hat{\theta} = (\hat{\rho}_1, \dots, \hat{\rho}_q, \hat{\phi}_1, \dots, \hat{\phi}_q, \hat{\lambda}, \hat{\omega})$  denote the true value of  $\theta$  and the LSE of  $\theta^0$  respectively. We need the following lemmas to prove the theorems.

**Lemma 1.** *If  $X(t)$  satisfies assumption 4, then*

$$\lim_{N \rightarrow \infty} \sup_{\theta} \left| \frac{1}{N^{L+1}} \sum_{t=1}^N U(t) t^L \cos(\theta t) \right| = 0 \quad a.s.,$$

$$\lim_{N \rightarrow \infty} \sup_{\theta} \left| \frac{1}{N^{L+1}} \sum_{t=1}^N U(t) t^L \sin(\theta t) \right| = 0 \quad a.s.,$$

for  $L = 0, 1, 2, \dots$

TABLE 10. LSE and confidence intervals for “uuu” data with multiple frequency model.

Parameters	LSE	Lower Limit	Upper Limit
$a_1$	0.63175	0.57488	0.68861
$a_2$	1.71710	1.66195	1.77225
$a_3$	0.43196	0.37941	0.48452
$a_4$	0.35917	0.30972	0.40861
$\phi_1$	-2.41835	-2.59838	-2.23832
$\phi_2$	1.52894	1.46471	1.59318
$\phi_3$	-2.30741	-2.55075	-2.06407
$\phi_4$	-2.08317	-2.35851	-1.80782
$\beta_1$	0.11390	0.11329	0.11451
$\beta_2$	0.22804	0.22782	0.22825
$\beta_3$	0.34376	0.34294	0.34458
$\beta_4$	0.45793	0.45700	0.45886

$$\hat{X}(t) = 1.09189606 \hat{X}(t-1) - 0.502943218 \hat{X}(t-2) + 0.102927223 \hat{X}(t-2) + e(t).$$

**Run Test:**  $z$  for  $\hat{X}(t) = -12.1590595$ ,  
 $z$  for  $\hat{e}(t) = -1.48203266$ .

Residual sum of squares: 0.0620818324.

**Proof of lemma 1:** For  $L = 0$ , the result is available in Kundu (1997). For general  $L$ , the result follows using the fact that  $\frac{t}{N} < 1$ . The lemma also follows from Theorem 4.5.1 in Brillinger (1981; page 98). ■

**Comment:** Lemma 1 is a very strong results and has been proved under different conditions. Walker (1971) proved for *i.i.d.* errors. Hannan (1973) proved it under ergodic and purely non-deterministic conditions. Kundu (1997) provided the proof for stationary linear processes, Brillinger (1986) and Nandi, Iyer and Kundu (2002) proved a version of this lemma for spatial point processes and for *i.i.d.* stable processes respectively.

**Lemma 2.** *Let us define*

$$S_{\delta,K} = \{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta \text{ or } |\omega - \omega^0| > \delta \text{ or } |\rho_j - \rho_j^0| > \delta \text{ or } |\phi_j - \phi_j^0| > \delta, \\ \text{for any } j = 1, \dots, q, \text{ and } \rho_k \leq K, \text{ for all } k = 1, \dots, q\}.$$

TABLE 11. Results for “**aww**” data set using the proposed model.

**Data Set:** “**aww**”.  $M = 2$ ,  $q_1 = 6$ ,  $q_2 = 1$ .

Parameter	Estimate	Lower Bound	Upper Bound
$\rho_{1_1}$	0.127831191	0.0166739132	0.238988474
$\rho_{1_2}$	0.164348915	0.0448353849	0.283862442
$\rho_{1_3}$	0.242797658	0.107381746	0.378213555
$\rho_{1_4}$	0.490236014	0.328855962	0.651616037
$\rho_{1_5}$	1.11415863	0.920606256	1.30771101
$\rho_{1_6}$	0.362586141	0.164035648	0.561136603
$\phi_{1_1}$	-2.36564922	-3.23676491	-1.49453342
$\phi_{1_2}$	0.397528142	-0.337061763	1.13211799
$\phi_{1_3}$	-2.53827643	-3.11740494	-1.95914781
$\phi_{1_4}$	0.833990335	0.444618791	1.22336185
$\phi_{1_5}$	-1.08417308	-1.39682209	-0.771524072
$\phi_{1_6}$	-2.51970673	-3.14991474	-1.88949871
$\lambda_1$	0.0922982097	0.0920951292	0.0925012901
$\rho_{2_1}$	0.168908238	0.14887704	0.188939437
$\phi_{2_1}$	0.61500001	0.377815574	0.852184474
$\lambda_2$	1.66371846	1.66291606	1.66452086

$$\hat{X}(t) = 1.31350672 \hat{X}(t-1) - 0.511514068 \hat{X}(t-2) - 0.103843123 \hat{X}(t-2) + e(t).$$

**Run Test:** z for  $\hat{X}(t) = -15.0958452$ ,

z for  $\hat{e}(t) = -0.89579612$ .

Residual sum of squares: 0.484496683.

If for any  $\delta > 0$  and for some  $0 < K < \infty$ ,

$$(11) \quad \liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in S_{\delta, K}} \frac{1}{N} [Q_N(\boldsymbol{\theta}) - Q_N(\boldsymbol{\theta}^0)] > 0 \quad a.s.$$

then  $\hat{\boldsymbol{\theta}}$  which minimizes (8) (when  $M = 1$ ,  $\boldsymbol{\Psi} = \boldsymbol{\theta}$ ), is a strongly consistent estimator of  $\boldsymbol{\theta}^0$ .

TABLE 12. LSE and confidence intervals for “**aww**” data with multiple frequency model.

Parameters	LSE	Lower Limit	Upper Limit
$a_1$	0.120224684	0.0445712507	0.195878118
$a_2$	0.152249157	0.0754743591	0.229023963
$a_3$	0.238393679	0.160011902	0.316775471
$a_4$	0.514508545	0.434411645	0.594605446
$a_5$	1.3179431	1.23688722	1.39899898
$a_6$	0.456269532	0.376150042	0.536388993
$a_7$	0.172196656	0.154326186	0.190067127
$\phi_1$	-2.42837739	-3.68691158	-1.16984332
$\phi_2$	0.851819396	-0.156722158	1.86036098
$\phi_3$	-2.27399015	-2.93157291	-1.61640739
$\phi_4$	1.53354931	1.22219622	1.8449024
$\phi_5$	0.00387152098	-0.119132072	0.126875117
$\phi_6$	-1.31130302	-1.66249669	-0.960109353
$\phi_7$	0.664068937	0.456509978	0.871627867
$\beta_1$	0.0920180827	0.0877605751	0.0962755904
$\beta_2$	0.186266482	0.182854667	0.189678296
$\beta_3$	0.278118253	0.275893718	0.280342788
$\beta_4$	0.372019708	0.370966434	0.373072982
$\beta_5$	0.46590662	0.46549052	0.46632272
$\beta_6$	0.558983684	0.557795644	0.560171723
$\beta_7$	1.66409254	1.6633904	1.66479468

$$\hat{X}(t) = 1.21117878 \hat{X}(t-1) - 0.603570342 \hat{X}(t-2) + 0.0734062716 \hat{X}(t-2) + e(t).$$

**Run Test:** z for  $\hat{X}(t) = -13.1944799$  (using whole data set),  
z for  $\hat{X}(t) = -8.97054672$  (using first 256 data points).  
z for  $\hat{e}(t) = -1.72880948$  (using first 256 data points).  
Residual sum of squares: 0.193716243.

**Proof of lemma 2:** In this proof we denote  $\hat{\theta}$  by  $\hat{\theta}_N = (\hat{\rho}_{1N}, \dots, \hat{\rho}_{qN}, \hat{\phi}_{1N}, \dots, \hat{\phi}_{qN}, \hat{\lambda}_N, \hat{\omega}_N)$  just to emphasize that it depends on  $N$ . Suppose  $\hat{\theta}_N$  is not consistent, then we can have one of the following two cases.

TABLE 13. Results for “**ahh**” data set using the proposed model.

**Data Set: “ahh”.**  $M = 1$ ,  $q_1 = 6$ .

Parameter	Estimate	Lower Bound	Upper Bound
$\rho_{1_1}$	0.154038683	0.00933268107	0.298744678
$\rho_{1_2}$	0.110609308	-0.038386818	0.259605438
$\rho_{1_3}$	0.20130381	0.044975698	0.357631922
$\rho_{1_4}$	0.219141886	0.0524176359	0.385866135
$\rho_{1_5}$	0.973635674	0.794148386	1.1531229
$\rho_{1_6}$	1.39281392	1.20093036	1.58469748
$\phi_{1_1}$	1.51953971	0.227566779	2.81151271
$\phi_{1_2}$	1.95636904	0.578811526	3.33392644
$\phi_{1_3}$	2.49770141	1.72301531	3.2723875
$\phi_{1_4}$	-2.94550371	-3.68023229	-2.21077514
$\phi_{1_5}$	-2.23193026	-2.43851805	-2.02534246
$\phi_{1_6}$	0.776610613	-0.572394848	2.12561607
$\lambda_1$	0.0916645825	0.079089947	0.104239218
$\omega_1$	0.0923735052	0.0873730332	0.0973739773

$$\hat{X}(t) = 1.2019335 \hat{X}(t-1) - 0.639286041 \hat{X}(t-2) + 0.049728144 \hat{X}(t-2) + e(t).$$

**Run Test:**  $z$  for  $\hat{X}(t) = -10.5448523$ ,  
 $z$  for  $\hat{e}(t) = -0.152799755$ .  
Residual sum of squares: 0.460617959.

**Case 1:** For all subsequences  $\{N_k\}$  of  $\{N\}$ , at least one  $|\hat{\rho}_{j_{N_k}}|$  tends to  $\infty$ .

**Case 2:** There exists a  $\delta > 0$ , a  $0 < K < \infty$  and a subsequence  $\{N_k\}$  of  $\{N\}$  such that  $\hat{\theta}_{N_k} \in S_{\delta, K}$  for all  $k = 1, 2, \dots$ .

Now for both the cases, under the definition of  $Q_N(\theta)$  (see (8)) and because of (11), there exists a  $K^0$ , such that for all  $k > K^0$ ,

$$Q_{N_k}(\hat{\theta}_{N_k}) - Q_{N_k}(\theta^0) > 0 \quad a.s.$$

This contradicts the fact that  $\hat{\theta}_{N_k}$  minimizes  $Q_{N_k}(\theta)$ . ■

TABLE 14. LSE and confidence intervals for “**ahh**” data with multiple frequency model.

Parameters	LSE	Lower Limit	Upper Limit
$a_1$	0.15321	0.04255	0.26386
$a_2$	0.12292	0.01293	0.23291
$a_3$	0.19762	0.08844	0.30680
$a_4$	0.26646	0.15817	0.37476
$a_5$	1.05137	0.94365	1.15908
$a_6$	1.47706	1.36958	1.58455
$\phi_1$	1.57744	0.13292	3.02196
$\phi_2$	2.56365	0.77409	4.35321
$\phi_3$	2.45601	1.35103	3.56099
$\phi_4$	-1.93182	-2.74466	-1.11898
$\phi_5$	-1.58160	-1.78651	-1.37670
$\phi_6$	1.38753	1.24199	1.53307
$\beta_1$	0.09222	0.08486	0.09958
$\beta_2$	0.18906	0.17995	0.19818
$\beta_3$	0.27671	0.27108	0.28234
$\beta_4$	0.37503	0.37089	0.37917
$\beta_5$	0.46543	0.46439	0.46647
$\beta_6$	0.55728	0.55654	0.55802

$$\hat{X}(t) = 1.01746476 \hat{X}(t-1) - 0.636902988 \hat{X}(t-2) + 0.176133722 \hat{X}(t-2) + e(t).$$

**Run Test:** z for  $\hat{X}(t) = -7.69534779$ ,

z for  $\hat{e}(t) = -0.594967306$ .

Residual sum of squares: 0.239770621.

**Proof of theorem 2.1:** Let us write

$$S_{\delta,K} = P_1 \cup P_2 \cup \dots \cup P_q \cup \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_q \cup \Lambda \cup \Omega,$$

where for  $j = 1, \dots, q$ ,

$$\begin{aligned} P_j &= \{\boldsymbol{\theta} : |\rho_j - \rho_j^0| > \delta, \rho_k \leq K, \text{ for all } k = 1, \dots, q\}, \\ \Phi_j &= \{\boldsymbol{\theta} : |\phi_j - \phi_j^0| > \delta, \rho_k \leq K, \text{ for all } k = 1, \dots, q\}, \\ \Lambda &= \{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta, \rho_k \leq K, \text{ for all } k = 1, \dots, q\}, \\ \text{and } \Omega &= \{\boldsymbol{\theta} : |\omega - \omega^0| > \delta, \rho_k \leq K, \text{ for all } k = 1, \dots, q\}. \end{aligned}$$

Now observe that

$$\begin{aligned} & \frac{1}{N} [Q_N(\boldsymbol{\theta}) - Q_N(\boldsymbol{\theta}^0)] \\ &= \frac{1}{N} \sum_{t=1}^N \left\{ \sum_{j=1}^q \rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\} - \sum_{j=1}^q \rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\} \right\}^2 \\ & \quad + \frac{2}{N} \sum_{t=1}^N X(t) \left\{ \sum_{j=1}^q \rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\} - \sum_{j=1}^q \rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\} \right\} \\ &= f_N(\boldsymbol{\theta}) + g_N(\boldsymbol{\theta}) \quad (\text{say}). \end{aligned}$$

For any  $\delta > 0$  and a fixed  $0 < K < \infty$ ,

$$\begin{aligned} & \liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in P_i} f_N(\boldsymbol{\theta}) \\ &= \liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in P_i} \frac{1}{N} \sum_{t=1}^N \left\{ \sum_{j=1}^q \rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\} \right. \\ & \quad \left. - \sum_{j=1}^q \rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\} \right\}^2 \\ &= \liminf_{N \rightarrow \infty} \inf_{|\rho_i - \rho_i^0| > \delta} \frac{1}{N} \sum_{t=1}^N [(\rho_i^0 - \rho_i) \cos\{(\lambda^0 + (i-1)\omega^0)t - \phi_i^0\}]^2 \\ &= \inf_{|\rho_i - \rho_i^0| > \delta} \frac{1}{2} (\rho_i - \rho_i^0)^2 > \frac{1}{2} \delta^2 > 0 \quad \text{a.s.}, \quad i = 1, \dots, q. \end{aligned}$$

Similarly it can be proved that

$$\liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in \Phi_i} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}, \quad i = 1, \dots, q$$

and

$$\liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in \Lambda} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}, \quad \liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in \Omega} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}$$

This proves that

$$(12) \quad \liminf_{N \rightarrow \infty} \inf_{\boldsymbol{\theta} \in S_{\delta, K}} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}$$

Using lemma 1, it follows that

$$(13) \quad \lim_{N \rightarrow \infty} \sup_{\boldsymbol{\theta} \in S_{\delta, K}} g_N(\boldsymbol{\theta}) = 0 \quad a.s.$$

Now using (12) and (13) in lemma 2, the theorem follows.  $\blacksquare$

**Proof of theorem 2.2:** Let  $Q'_N(\boldsymbol{\theta})$ , the first derivative vector of  $Q_N(\boldsymbol{\theta})$  be defined as follows:

$$Q'_N(\boldsymbol{\theta}) = \left( \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \rho_1}, \dots, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \rho_q}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \phi_1}, \dots, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \phi_q}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \lambda}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \omega} \right).$$

Similarly  $Q''_N(\boldsymbol{\theta})$ , the  $(2q+2) \times (2q+2)$  matrix of second derivatives of  $Q_N(\boldsymbol{\theta})$  is also defined. Expanding  $Q'_N(\boldsymbol{\theta})$  at  $\hat{\boldsymbol{\theta}}$  around  $\boldsymbol{\theta}^0$  using Taylor Series expansion, we have

$$(14) \quad Q'_N(\hat{\boldsymbol{\theta}}) - Q'_N(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^T Q''_N(\bar{\boldsymbol{\theta}}),$$

where  $\bar{\boldsymbol{\theta}} = \alpha \hat{\boldsymbol{\theta}} + (1 - \alpha)\boldsymbol{\theta}^0$  for some  $0 < \alpha < 1$ . Consider a  $(2q+2) \times (2q+2)$  diagonal matrix  $\mathbf{D}_1$  ( $= \mathbf{D}$  with  $M = 1$  and  $q_1 = q$ ) as follows.

$$\mathbf{D}_1 = \begin{pmatrix} N^{-1/2} \mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N^{-1/2} \mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & N^{-3/2} & 0 \\ \mathbf{0} & \mathbf{0} & 0 & N^{-3/2} \end{pmatrix}.$$

Therefore, (14) can be written as

$$(15) \quad (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^T \mathbf{D}_1^{-1} = - [Q'_N(\boldsymbol{\theta}^0) \mathbf{D}_1] [\mathbf{D}_1 Q''_N(\bar{\boldsymbol{\theta}}) \mathbf{D}_1]^{-1}.$$

As  $\hat{\boldsymbol{\theta}}$  is a strongly consistent estimator of  $\boldsymbol{\theta}^0$  and  $\bar{\boldsymbol{\theta}}$  lies between  $\hat{\boldsymbol{\theta}}$  and  $\boldsymbol{\theta}^0$ , it can be shown that

$$(16) \quad \lim_{N \rightarrow \infty} [\mathbf{D}_1 Q''_N(\bar{\boldsymbol{\theta}}) \mathbf{D}_1] = \lim_{N \rightarrow \infty} [\mathbf{D}_1 Q''_N(\boldsymbol{\theta}^0) \mathbf{D}_1] = \boldsymbol{\Sigma},$$

where  $\boldsymbol{\Sigma}$  is same as defined in the statement of theorem 2.2. Now using the Central Limit Theorem of a linear process (see Fuller; 1976, page 251-256) it can be proved that

$$(17) \quad Q'_N(\boldsymbol{\theta}^0) \mathbf{D}_1 \rightarrow N_{(2q+2)}(\mathbf{0}, 2\sigma^2 \mathbf{G}),$$

and  $\mathbf{G}$  ( $= \mathbf{G}_1$ ) is same as defined earlier. Now theorem 2.2 follows immediately using (16) and (17) in (14).  $\blacksquare$

OUTLINE OF THE PROOFS OF THE RESULTS WHEN MORE THAN ONE FUNDAMENTAL FREQUENCY IS PRESENT IN THE MODEL ( $M > 1$ )

The consistency of  $\hat{\Psi}$ , the least squares estimator of  $\Psi^0$  (when  $M > 1$  in model (1)) follows exactly the same way as the proof of theorem 2.1, considering the entire set of parameters *i.e.* considering  $\hat{\Psi}$ ,  $\Psi^0$  and  $\Psi$  instead of  $\hat{\theta}$ ,  $\theta^0$  and  $\theta$  respectively.

The asymptotic normality of the LSEs of the model (1) ( $M > 1$ ) can be obtained along the same line as the proof of theorem 2.2. Expanding  $Q'_N(\hat{\Psi})$  by Taylor series similarly as (14), an equivalent expression to (15)

$$(18) \quad (\hat{\Psi} - \Psi^0)^T \mathbf{D}^{-1} = - [Q'_N(\Psi^0) \mathbf{D}] [\mathbf{D} Q''_N(\bar{\Psi}) \mathbf{D}]^{-1}.$$

can be obtained for the general model having more than one fundamental frequency. The left hand side of (18) is a  $1 \times R$  ( $R = 2 \sum_{k=1}^M q_k + 2M$ ) random vector whereas the right hand side is a product of  $1 \times R$  ( $Q'_N(\Psi^0) \mathbf{D}$ ) random vector and a  $R \times R$  ( $\mathbf{D} Q''_N(\bar{\Psi}) \mathbf{D}$ ) random matrix. Using similar techniques, the  $R \times R$  matrix converges to a block diagonal matrix  $\Sigma$  of  $M$  blocks with  $k^{th}$  block as  $\Sigma_k$  of order  $2q_k + 2$ . The  $1 \times R$  random vector  $Q'_N(\Psi^0) \mathbf{D}$  converges to a  $R$ -variate normal distribution with mean vector zero and the dispersion matrix  $2\sigma^2 \mathbf{G}$ , having a block-diagonal form with  $k^{th}$  diagonal block as  $2\sigma^2 \mathbf{G}_k$ . Therefore, asymptotic distribution of  $\hat{\Psi}$  is the same as given in theorem 2.2. ■

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