Point and Interval Estimations for the Two-parameter Birnbaum-Saunders Distribution based on Type-II Censored Samples

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Abstract

The maximum likelihood estimators, based on Type-II censored samples, of a two-parameter Birnbaum-Saunders distribution are discussed. We propose a simple bias-reduction method to reduce the bias of the maximum likelihood estimators. We also discuss a Monte Carlo EM-algorithm for the determination of the maximum likelihood estimators. Monte Carlo simulation is used to compare the performance of all these estimators. The probability coverages of confidence intervals based on inferential quantities associated with these estimators are evaluated using Monte Carlo simulations for small, moderate and large sample sizes, for various degrees of censoring. Two illustrative examples and some concluding remarks are finally presented.

Key words: Bias-corrected estimator; Monte Carlo simulation; Monte Carlo EM-algorithm; probability coverage; asymptotic distribution; confidence interval.

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Preprint submitted to Computational Statistics and Data Analysis 2 June 2005

1 Introduction

Birnbaum and Saunders (1969a) proposed a two-parameter failure time distribution for fatigue failure caused under cyclic loading. It was also assumed that the failure is due to the development and growth of a dominant crack. This distribution is so-called the two-parameter Birnbaum-Saunders distribution. A more general derivation was provided by Desmond (1985) based on a biological model. Desmond (1985) also strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders (1969a). Desmond (1986) investigated the relationship between the Birnbaum-Saunders distribution and the inverse Gaussian distribution. Some recent research works on Birnbaum-Saunders distribution can be found in Chang and Tang (1993, 1994), Dupuis and Mills (1998) and Rieck (1995, 1999), and a concise review of all these developments can be found in Johnson et al. (1995).

The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable T can be written as

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right], \qquad 0 < t < \infty, \qquad \alpha,\beta > 0, \quad (1)$$

where $\Phi(\cdot)$ is the standard normal CDF. The parameters α and β are the shape and the scale parameters, respectively. It is known that the density function of the Birnbaum-Saunders distribution is unimodal. Birnbaum and Saunders (1969b) discussed the maximum likelihood estimators (MLE's) and Engelhardt et al. (1981) obtained the asymptotic distributions of the MLE's. Recently, Ng et al. (2003) discussed the MLE's and a modification of the moment estimators (MME's) and proposed a bias-reduction method for these estimators. Ng et al. (2003) also derived the asymptotic distributions of the MME's and compare the performance of point and interval estimation based on the MLE's and MME's.

Most of the research works on two-parameter Birnbaum-Saunders distribution are based on complete samples. In this paper, we first discuss the maximum likelihood estimation of the parameters α and β based on Type-II right censored samples. We then derive the asymptotic variance-covariance matrix of the MLE's using which asymptotic confidence intervals for α and β are proposed with the use of asymptotic normality of the MLE's. We evaluate the performance of the MLE's through Monte Carlo simulations for various sample sizes and degrees of censoring (d.o.c.). Though the MLE's are asymptotically unbiased, these simulation results reveal that they are highly biased in case of small sample sizes particularly when the degree of censoring is high. We, therefore, propose a simple bias correction technique which performs quite well even for small sample sizes. Asymptotic confidence intervals for α and β based on these bias-corrected estimators are then proposed. Next, a Monte Carlo EM-algorithm for the determination of the MLE's is discussed. A comparison of all these estimators and the probability coverages of confidence intervals based on inferential quantities associated with these estimators is made using Monte Carlo simulations. We present two examples to illustrate all the methods of inference discussed here. Finally, we make some concluding remarks.

2 The Birnbaum-Saunders Distribution

The CDF of a two-parameter Birnbaum-Saunders random variable T is given by (1), and the corresponding probability density function (PDF) is

$$\begin{aligned} f_T(t;\alpha,\beta) = {} & \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] \\ & \times \exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right], \quad t>0, \alpha,\beta>0. \end{aligned} \tag{2}$$

Consider the monotone transformation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}}\right]$$

or

$$T = \beta\left(1+2X^2+2X(1+X^2)^{\frac{1}{2}}\right);$$

then, from (1), we have X to be normally distributed with mean zero and variance $\frac{1}{4}\alpha^2$. Using the above transformation, the expected value, variance, and coefficients of skewness and kurtosis can be easily obtained as

$$E(T) = \beta\left(1+\frac{1}{2}\alpha^2\right), \tag{3}$$

$$Var(T) = (\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right), \tag{4}$$

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2+6)}{(5\alpha^2+4)^3}, \tag{5}$$

$$\beta_2(T) = 3+\frac{6\alpha^2(93\alpha^2+41)}{(5\alpha^2+4)^2}. \tag{6}$$

Moreover, if T has a Birnbaum-Saunders distribution with parameters α and β, then T^{-1} also has a Birnbaum-Saunders distribution with the corresponding parameters α and β^{-1}, respectively (Birnbaum and Saunders, 1969a). Therefore, we also readily have

$$E(T^{-1}) = \beta^{-1}\left(1+\frac{1}{2}\alpha^2\right) \tag{7}$$

and

$$Var(T^{-1}) = \alpha^2\beta^{-2}\left(1+\frac{5}{4}\alpha^2\right). \tag{8}$$

3 Maximum Likelihood Estimators

Let $\{t_{(1)}, t_{(2)}, \ldots, t_{(r)}\}$ be an ordered Type-II right censored random sample obtained from n units placed on a life-testing experiment wherein each unit has its lifetime following the Birnbaum-Saunders distribution with PDF as given in (2), with the largest $(n-r)$ lifetimes having been censored. Then, the likelihood function is given by (Balakrishnan and Cohen, 1991)

$$\begin{aligned} L = & \frac{n!}{(n-r)!}\left\{1-\Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_{(r)}}{\beta}\right)\right]\right\}^{n-r} \\ & \times\left\{\frac{1}{(\sqrt{2\pi}\alpha\beta)^r}\left[\prod_{i=1}^{r}\xi'\left(\frac{t_{(i)}}{\beta}\right)\right]\exp\left[-\frac{1}{2\alpha^2}\sum_{i=1}^{r}\xi^2\left(\frac{t_{(i)}}{\beta}\right)\right]\right\}, \end{aligned} \tag{9}$$

and the log-likelihood function is

$$\begin{aligned} \ln L = & const. + (n-r)\ln\left\{1-\Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_{(r)}}{\beta}\right)\right]\right\} - r\ln\alpha - r\ln\beta \\ & +\sum_{i=1}^{r}\xi'\left(\frac{t_{(i)}}{\beta}\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{r}\xi^2\left(\frac{t_{(i)}}{\beta}\right); \end{aligned} \tag{10}$$

here,

$$\begin{aligned} \xi(t) &= t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \\ \xi^2(t) &= t + t^{-1} - 2, \\ \xi'(t) &= \frac{1}{2t}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) = \frac{1}{2\xi(t)}\left(1-\frac{1}{t^2}\right), \end{aligned}$$

$$\frac{t\xi''(t)}{\xi'(t)} = -\frac{1}{2} - \frac{1}{t+1}.$$

For notational convenience, let us denote

$$t^*_{(i)} = \frac{t_{(i)}}{\beta}$$

and the hazard function of the standard normal distribution as

$$H(x) = \frac{\phi(x)}{1-\Phi(x)}.$$

Then, we find from (10)

$$-\frac{\alpha^3}{r}\frac{\partial \ln L}{\partial \alpha} = -\frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]\xi\left(t^*_{(r)}\right) + \alpha^2 - \frac{1}{r}\sum_{i=1}^{r}\xi^2\left(t^*_{(i)}\right), \quad (11)$$

$$-\frac{\alpha^2\beta}{r}\frac{\partial \ln L}{\partial \beta} = -\frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] t^*_{(r)}\xi'\left(t^*_{(r)}\right) + \alpha^2$$
$$+\frac{\alpha^2}{r}\sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)} - \frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}\xi\left(t^*_{(i)}\right)\xi'\left(t^*_{(i)}\right). \quad (12)$$

By equating (11) and (12) to zero, we obtain the likelihood equations as

$$\alpha^2 - g(\alpha,\beta)h_1(\beta) + h_2(\beta) = 0, \quad (13)$$
$$\alpha^2 - g(\alpha,\beta)h_3(\beta) + h_4(\beta) = 0, \quad (14)$$

where

$$g(\alpha,\beta) = \frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right],$$
$$h_1(\beta) = \xi\left(t^*_{(r)}\right),$$
$$h_2(\beta) = -\frac{1}{r}\sum_{i=1}^{r}\xi^2\left(t^*_{(i)}\right),$$
$$h_3(\beta) = \left[1 + \sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)}\right]^{-1} t^*_{(r)}\xi'\left(t^*_{(r)}\right),$$
$$h_4(\beta) = \left[1 + \sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)}\right]^{-1}\left[-\frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}\xi\left(t^*_{(i)}\right)\xi'\left(t^*_{(i)}\right)\right].$$

From (13) and (14), α can be written as a pure function of β as

$$\alpha^2 = \frac{h_2(\beta)h_3(\beta) - h_1(\beta)h_4(\beta)}{h_1(\beta) - h_3(\beta)} \quad (= \varphi^2(\beta),\ \text{say}). \tag{15}$$

Let

$$\begin{aligned}
u^* &= \frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}, \\
v^* &= \left[\frac{1}{r}\sum_{i=1}^{r} {t^*_{(i)}}^{-1}\right]^{-1}, \\
K^*(\beta) &= \left[\frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-1}\right]^{-1}, \\
K'^*(\beta) &= [K^*(\beta)]^2\left[\frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-2}\right];
\end{aligned}$$

then (12) can be rewritten as

$$\begin{aligned}
Q(\beta) = {} & \varphi^2(\beta)\left[\frac{1}{2} - \frac{1}{K^*(\beta)}\right] - \frac{u^*}{2} + \frac{1}{2v^*} \\
& - \frac{\varphi(\beta)(n-r)}{r} H\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right] t^*_{(r)}\xi'\left(t^*_{(r)}\right).
\end{aligned} \tag{16}$$

The maximum likelihood estimate of β is the solution of $Q(\beta) = 0$. Since $Q(\beta) = 0$ is a non-linear equation, one needs to use a numerical procedure to solve for β. Once we have the maximum likelihood estimate $\hat{\beta}$ of β, the maximum likelihood estimate $\hat{\alpha}$ of α can be obtained as the positive square root of the right-hand side of (15).

In order to construct confidence intervals for α and β using the MLE's, we need the variance-covariance matrix of the MLE's. We define

$$\begin{aligned}
Q'(\beta) = \frac{1}{\beta}\Bigg\{ & -\varphi^2(\beta)\left[\frac{1}{K^*(\beta)} - \frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-2}\right] + \frac{u^*}{2} + \frac{1}{2v^*} \\
& + \frac{\varphi(\beta)(n-r)}{r} t^*_{(r)}\Bigg[H\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right]\left[t^*_{(r)}\xi''\left(t^*_{(r)}\right) + \xi'\left(t^*_{(r)}\right)\right] \\
& + \frac{t^*_{(r)}}{\varphi(\beta)}\left[\xi'\left(t^*_{(r)}\right)\right]^2 H'\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right]\Bigg]\Bigg\}
\end{aligned}$$

$$= \frac{1}{\beta} A_{31} \text{ (say)},$$

where $H'(x) = -xH(x) + H^2(x)$. For the observed information matrix for α and β, we find

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \alpha^2} &= \frac{(n-r)}{\alpha^4} \xi\left(t^*_{(r)}\right) \left\{ 2\alpha H \left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] + \xi\left(t^*_{(r)}\right) H' \left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] \right\} \\
&\quad - \frac{r}{\alpha^2} + \frac{3}{\alpha^4} \sum_{i=1}^{r} \xi^2\left(t^*_{(r)}\right) = A_1, \\
-\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} &= \frac{1}{\alpha^3 \beta} \left\{ (n-r) \left[\alpha t^*_{(r)} \xi'\left(t^*_{(r)}\right) H \left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] \right.\right. \\
&\qquad \left. + t^*_{(r)} \xi\left(t^*_{(r)}\right) \xi'\left(t^*_{(r)}\right) H' \left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] \right] \\
&\qquad \left. + 2 \sum_{i=1}^{r} t^*_{(i)} \xi\left(t^*_{(i)}\right) \xi'\left(t^*_{(i)}\right) \right\} = \frac{1}{\beta} A_2, \\
-\frac{\partial^2 \ln L}{\partial \beta^2} &= \frac{r}{(\alpha\beta)^2} [Q(\beta) + A_{31}] = \frac{1}{\beta^2} A_3.
\end{aligned}$$

Then the observed information matrix is given by

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} = \begin{pmatrix} A_1 & \frac{1}{\beta} A_2 \\ \frac{1}{\beta} A_2 & \frac{1}{\beta^2} A_3 \end{pmatrix},$$

so that the variance-covariance matrix may be approximated as

$$\begin{aligned}
\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} &= \begin{pmatrix} A_1 & \frac{1}{\beta} A_2 \\ \frac{1}{\beta} A_2 & \frac{1}{\beta^2} A_3 \end{pmatrix}^{-1} \\
&= \frac{\beta^2}{A_1 A_3 - A_2^2} \begin{pmatrix} \frac{1}{\beta^2} A_3 & -\frac{1}{\beta} A_2 \\ -\frac{1}{\beta} A_2 & A_1 \end{pmatrix}.
\end{aligned}$$

The asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is then approximately bivariate normal, and is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \overset{appr.}{\sim} N \left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} \right]. \tag{17}$$

Since $\mathbf{V}$ involves the parameters α and β, we replace the parameters by the corresponding MLE's in order to obtain an estimate of $\mathbf{V}$, which is denoted by

$$\hat{\mathbf{V}} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{12} & \hat{V}_{22} \end{pmatrix} = \frac{\hat{\beta}^2}{\hat{A}_1\hat{A}_3 - \hat{A}_2^2} \begin{pmatrix} \frac{1}{\hat{\beta}^2}\hat{A}_3 & -\frac{1}{\hat{\beta}}\hat{A}_2 \\ -\frac{1}{\hat{\beta}}\hat{A}_2 & \hat{A}_1 \end{pmatrix}.$$

By using (17), approximate $100(1-\gamma)\%$ confidence intervals for α and β are determined as

$$\left[\hat{\alpha} - z_{\gamma/2}\sqrt{\hat{V}_{11}}, \hat{\alpha} + z_{\gamma/2}\sqrt{\hat{V}_{11}}\right],$$

and

$$\left[\hat{\beta}\left(z_{\gamma/2}\sqrt{\frac{\hat{A}_1}{\hat{A}_1\hat{A}_3 - \hat{A}_2^2}} + 1\right)^{-1}, \hat{\beta}\left(z_{1-\gamma/2}\sqrt{\frac{\hat{A}_1}{\hat{A}_1\hat{A}_3 - \hat{A}_2^2}} + 1\right)^{-1}\right], \qquad (18)$$

where z_p is the upper p-th percentile of the standard normal distribution.

4 Bias-corrected Estimators

In general, the maximum likelihood estimator of α is biased, especially when the sample sizes are small or when the data is censored. In this section, we provide a simple bias-corrected estimator of α by using a regression curve fitted to the bias of the MLE of α based on Monte Carlo simulations. This technique is discussed and used by Hirose (1999) to obtain a simple bias correction formula for the maximum likelihood estimators of a two-parameter Weibull distribution.

Based on the results of an extensive Monte Carlo simulation study and inspecting the pattern of the bias of the MLE of α, we observed that

$$Bias(\hat{\alpha}) \approx -\frac{\alpha}{n}\left[1 + 2.5\left(1 - \frac{r}{n}\right)\right].$$

Then, by employing a standard bias-reduction method, we can simply construct an almost unbiased maximum likelihood estimator (UMLE, denoted by $\hat{\alpha}^*$) of α. The bias-corrected estimator thus obtained is given by

$$\hat{\alpha}^* = \hat{\alpha} \left\{1 - \frac{1}{n}\left[1 + 2.5\left(1 - \frac{r}{n}\right)\right]\right\}^{-1}. \tag{19}$$

From the distributional results presented in (17), we then readily have the asymptotic joint distribution of $\hat{\alpha}^*$ and $\hat{\beta}$ to be bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha}^* \\ \hat{\beta} \end{pmatrix} \overset{appr.}{\sim} N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \mathbf{\Sigma}\right], \tag{20}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \left\{1 - \frac{1}{n}\left[1 + 2.5\left(1 - \frac{r}{n}\right)\right]\right\}^{-2} V_{11} & \left\{1 - \frac{1}{n}\left[1 + 2.5\left(1 - \frac{r}{n}\right)\right]\right\}^{-1} V_{12} \\ \left\{1 - \frac{1}{n}\left[1 + 2.5\left(1 - \frac{r}{n}\right)\right]\right\}^{-1} V_{12} & V_{22} \end{pmatrix}.$$

Remark: Though the results presented here are for right-censored samples, they are also useful when the observed sample is Type-II left-censored. This is due to the fact that if $t_{(n-r+1)}$, $t_{(n-r+2)}$, $\ldots$, $t_{(n)}$ is a left-censored sample from Birnbaum-Saunders distribution with parameters α and β, then $t_1^* = \frac{1}{t_{(n)}}$, $t_2^* = \frac{1}{t_{(n-1)}}$, $\ldots$, $t_r^* = \frac{1}{t_{(n-r+1)}}$ form a right-censored sample from Birnbaum-Saunders distribution with parameters α and $\frac{1}{\beta}$.

5 Monte-Carlo EM-Algorithm

The Type-II right censored data can be viewed as an incomplete data problem, and consequently the EM-algorithm is applicable to determine the maximum likelihood estimators of the parameters. First of all, denote the observed and censored data by $\mathbf{T} = (T_{(1)}, T_{(2)}, \ldots, T_{(r)})$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{n-r})$, respectively. The censored data vector $\mathbf{Y}$ can be thought of as the missing data. Combine $\mathbf{T}$ and $\mathbf{Y}$ to form $\mathbf{X}$ which is the complete data set. In order to facilitate the EM-algorithm, the conditional distribution of $\mathbf{Y}$, conditional on $\mathbf{T}$ and the current guess of the parameters, needs to be determined. Due to the Markovian property of order statistics, the conditional density of $Y_j, j = 1, 2\ldots, n-r$, conditioned on $T_{(1)} = t_{(1)}, \ldots, T_{(r)} = t_{(r)}$, is (see Arnold et al., 1992 and David and Nagaraja, 2003)

$$\begin{aligned} & f_{Y|T}(y_j \mid T_{(1)} = t_{(1)}, T_{(2)} = t_{(2)}, T_{(r)} = t_{(r)}) \\ = & f_{Y|T}(y_j \mid T_{(r)} = t_{(r)}) = \frac{f_T(y_j)}{[1 - F_T(t_{(r)})]}, \qquad y_j > t_{(r)}; \end{aligned} \tag{21}$$

in other words, given $T_{(r)} = t_{(r)}$, $\mathbf{Y}$ forms a random sample from the left-truncated Birnbaum-Saunders distribution, and hence the expectations of functions of $\mathbf{Y}$ can be derived.

Let

$$s = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right]^{-1},$$

$$K(\beta) = \left[\frac{1}{n}\sum_{i=1}^{n}(\beta + x_i)^{-1}\right]^{-1} \qquad \text{for } \beta \geq 0,$$

and

$$K'(\beta) = [K(\beta)]^2 \left[\frac{1}{n}\sum_{i=1}^{n}(\beta + x_i)^{-1}\right]^{-2} \qquad \text{for } \beta \geq 0.$$

For the complete sample case, the maximum likelihood estimate of β can be obtained as the unique positive root of the equation (see Birnbaum and Saunders, 1969b and Engelhardt et al., 1981)

$$g(\beta) = \beta^2 - \beta[2v + K(\beta)] + v[u + K(\beta)] = 0. \tag{22}$$

Once $\hat{\beta}$ is obtained as a solution of (22), the MLE of α can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{u}{\hat{\beta}} + \frac{\hat{\beta}}{v} - 2\right]^{\frac{1}{2}}.$$

If Newton-Raphson method is instead used to solve (22), we would require

$$g'(\beta) = 2\beta_{(h)} - 2\hat{v}_{(h)} + (\hat{v}_{(h)} - \beta_{(h)})\hat{K}'(\beta_{(h)}) - \hat{K}(\beta_{(h)}).$$

E-step

In the (h+1)-th iteration, we need

$$E_1 = E\left[Y \,|\, t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right],$$
$$E_2 = E\left[\frac{1}{Y}\middle|\, t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right],$$

$$E_3 = E\left[\left.\frac{1}{(\beta_{(h)}+Y)}\right| t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right],$$
$$E_4 = E\left[\left.\frac{1}{(\beta_{(h)}+Y)^2}\right| t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right],$$

which can be estimated as follows using the Monte Carlo method:

1. Simulate l random variates from the left-truncated Birnbaum-Saunders distribution with parameters $\alpha = \alpha_{(h)}$ and $\beta = \beta_{(h)}$ and truncation point at $t_{(r)}$, and denote them by $Y_1, Y_2, \ldots, Y_l$.

2. Compute the average values of the required functions as estimates of these functions

$$\hat{E}_1 = \frac{1}{l}\sum_{i=1}^{l} Y_i, \quad \hat{E}_2 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{Y_i},$$
$$\hat{E}_3 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{(\beta_{(h)}+Y_i)}, \quad \hat{E}_4 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{(\beta_{(h)}+Y_i)^2}.$$

3. Compute

$$\hat{u}_{(h)} = \frac{1}{n}\left[\sum_{i=1}^{r} t_i + (n-r)\hat{E}_1\right],$$
$$\hat{v}_{(h)} = \left\{\frac{1}{n}\left[\sum_{i=1}^{r} t_i^{-1} + (n-r)\hat{E}_2\right]\right\}^{-1},$$
$$\hat{K}(\beta_{(h)}) = \left\{\frac{1}{n}\left[\sum_{i=1}^{r}(\beta_{(h)}+t_i)^{-1} + (n-r)\hat{E}_3\right]\right\}^{-1},$$
$$\hat{K}'(\beta_{(h)}) = \frac{\left[\hat{K}(\beta_{(h)})\right]^2}{n}\left[\sum_{i=1}^{r}(\beta_{(h)}+t_i)^{-2} + (n-r)\hat{E}_4\right].$$

<u>M-step</u>

In the (h+1)-th iteration, $\beta_{(h+1)}$ can be obtained as the unique positive root of the equation [see Eq. (22)]

$$g(\beta) = \beta^2 - \beta[2\hat{v}_{(h)} + K(\beta)] + \hat{v}_{(h)}[\hat{u}_{(h)} + K(\beta)] = 0. \tag{23}$$

Once $\beta_{(h+1)}$ is obtained, $\alpha_{(h+1)}$ can be obtained explicitly as

$$\alpha_{(h+1)} = \left[\frac{\hat{u}_{(h)}}{\beta_{(h+1)}} + \frac{\beta_{(h+1)}}{\hat{v}_{(h)}} - 2\right]^{\frac{1}{2}}.$$

If Newton-Raphson method is instead used to solve Eq. (22), we would require

$$g'(\beta) = 2\beta - 2v + (v - \beta)K'(\beta) - K(\beta).$$

Birnbaum and Saunders (1969b) showed that the maximum likelihood estimates are equivalent to the modified moment estimates under certain conditions, in the case of complete samples. Therefore, instead of solving Eq. (23) by a numerical procedure, we can obtain the estimates for α and β (using the modified moment estimates) in the M-step as

$$\alpha_{(h+1)} = \left\{2\left[\left(\frac{\hat{u}_{(h)}}{\hat{v}_{(h)}}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}},$$
$$\beta_{(h+1)} = \left(\hat{u}_{(h)}\hat{v}_{(h)}\right)^{\frac{1}{2}}.$$

The EM iterations are repeated for $I = 3000$ times and the burn-in period is taken as $J = 800$, with the final estimates of α and β taken as

$$\hat{\alpha} = \frac{1}{I - J}\sum_{i=J+1}^{I} \alpha_{(i)},$$
$$\hat{\beta} = \frac{1}{I - J}\sum_{i=J+1}^{I} \beta_{(i)}.$$

6 Monte Carlo Simulation Results

In order to compare the performance of all the above estimators, we performed a simulation study for different sample sizes and different parameter values with varying degrees of censoring. We took the sample size as n = 20, 30, 50, the shape parameter as α = .10, .30, .50, 1.00, 2.00, and the degree of censoring as 0(10)60 %. The scale parameter β was kept fixed at 1.0, without loss of any generality. All the results were based on 10,000 Monte Carlo runs.

In Tables 1-3, we have presented the average values, standard errors and covariances of the estimates $\hat{\alpha}$, $\hat{\alpha}^*$ and $\hat{\beta}$. From these tables, we readily observe that while the bias of $\hat{\alpha}$ increases as the degree of censoring increases for all sample sizes considered, the simple bias-corrected estimator $\hat{\alpha}^*$ remains almost unbiased in all cases. The unbiasedness of $\hat{\alpha}^*$ comes with a slightly larger standard deviation than that of $\hat{\alpha}$. Also, the estimator $\hat{\beta}$ is almost unbiased in all cases considered.

In Tables 4 and 5, we have presented the probability coverages of 90% and 95% confidence intervals for α and β based on MLE's as well as UMLE's. From these tables, we note that the probability coverages of confidence intervals based on MLE's become considerably smaller than the nominal levels particularly when the degree of censoring increases; however, the probability coverages of confidence intervals based on UMLE's remain close to the nominal levels.

7 Illustrative Examples

Practical application of the inferential methods discussed in the preceding sections is illustrated here with two examples, with one involving a large sample and the other with a small sample.

7.1 Example 1

The data set is given by Birnbaum and Saunders (1969b) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31000 psi. The data are presented in Table 6.

Table 6. Fatigue lifetime data presented by Birnbaum and Saunders (1969b)

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

Under different censoring proportions, the point estimates of α and β obtained by Newton-Raphson method and the Monte Carlo EM-algorithm (using MLE's and MME's) are summarized in Table 7. From (17), the asymptotic variances of the estimators can be obtained, and also the asymptotic confidence intervals for α and β based on the MLE's can be readily constructed using (18). The results so obtained are presented in Table 8.

Table 7. Point Estimates of α and β for Example 1

	Newton-Raphson			MCEM (MLE)		MCEM (MME)	
r	$\hat{\alpha}$	$\hat{\alpha}^*$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
101	0.1704	0.1721	131.8188	0.1704	131.8188	0.1704	131.8188
100	0.1698	0.1716	131.7871	0.1698	131.7876	0.1698	131.7871
95	0.1691	0.1710	131.7686	0.1691	131.7691	0.1691	131.7687
90	0.1706	0.1728	131.8776	0.1706	131.8781	0.1706	131.8776
80	0.1751	0.1777	132.2527	0.1751	132.2531	0.1751	132.2525
70	0.1735	0.1766	132.1069	0.1735	132.1074	0.1735	132.1062
60	0.1829	0.1866	133.2411	0.1829	133.2416	0.1829	133.2403
50	0.1848	0.1891	133.5043	0.1848	133.5051	0.1848	133.5038
40	0.2112	0.2165	137.5925	0.2111	137.5921	0.2111	137.5879

Table 8. Standard Deviations of Estimates and Confidence Intervals for α and β for Example 1

		α			β		
r	Estimator	S.D.	90% C.I.	95% C.I.	S.D.	90% C.I.	95% C.I.
101	MLE	0.0120	(0.1507,0.1901)	(0.1469,0.1939)	2.2267	(128.2552,135.5861)	(127.5944,136.3325)
	UMLE	0.0124	(0.1517,0.1925)	(0.1478,0.1964)	2.2492	(128.2203,135.6251)	(127.5532,136.3796)
100	MLE	0.0120	(0.1500,0.1897)	(0.1462,0.1934)	2.2203	(128.2336,135.5432)	(127.5746,136.2873)
	UMLE	0.0124	(0.1512,0.1919)	(0.1473,0.1958)	2.2432	(128.1979,135.5831)	(127.5325,136.3354)
95	MLE	0.0125	(0.1486,0.1895)	(0.1446,0.1935)	2.2213	(128.2136,135.5265)	(127.5543,136.2710)
	UMLE	0.0128	(0.1499,0.1921)	(0.1459,0.1961)	2.2470	(128.1735,135.5713)	(127.5071,136.3250)
90	MLE	0.0131	(0.1491,0.1921)	(0.1450,0.1962)	2.2596	(128.2627,135.7021)	(127.5927,136.4602)
	UMLE	0.0136	(0.1503,0.1952)	(0.1460,0.1995)	2.2886	(128.2176,135.7526)	(127.5396,136.5210)
80	MLE	0.0145	(0.1512,0.1989)	(0.1466,0.2035)	2.3763	(128.4561,136.2801)	(127.7536,137.0799)
	UMLE	0.0151	(0.1529,0.2025)	(0.1482,0.2073)	2.4124	(128.4001,136.3432)	(127.6876,137.1560)
70	MLE	0.0156	(0.1478,0.1992)	(0.1429,0.2041)	2.4435	(128.2056,136.2516)	(127.4845,137.0756)
	UMLE	0.0163	(0.1498,0.2034)	(0.1446,0.2086)	2.4856	(128.1405,136.3253)	(127.4077,137.1645)
60	MLE	0.0182	(0.1530,0.2128)	(0.1473,0.2185)	2.7506	(128.8646,137.9236)	(128.0589,138.8586)
	UMLE	0.0191	(0.1552,0.2180)	(0.1492,0.2240)	2.8018	(128.7858,138.0139)	(127.9663,138.9677)
50	MLE	0.0204	(0.1512,0.2185)	(0.1448,0.2249)	3.0475	(128.6725,138.7120)	(127.7866,139.7565)
	UMLE	0.0216	(0.1536,0.2246)	(0.1468,0.2314)	3.1049	(128.5848,138.8140)	(127.6836,139.8799)
40	MLE	0.0270	(0.1667,0.2555)	(0.1582,0.2640)	4.1182	(131.1319,144.7126)	(129.9636,146.1625)
	UMLE	0.0286	(0.1694,0.2636)	(0.1604,0.2726)	4.1876	(131.0283,144.8390)	(129.8424,146.3162)

7.2 Example 2

This example is from McCool (1974) giving the fatigue life in hours of ten bearings of a certain type. These data were used as an illustrative example for the three-parameter Weibull distribution by Cohen et al. (1984). The data are presented in Table 9.

Table 9. Fatigue lifetime data presented by McCool (1974)

152.7	172.0	172.5	173.3	193.0
204.7	216.5	234.9	262.6	422.6

In this case, if we assume a Type-II right censored sample of size $r = 8$, we obtain the maximum likelihood estimates of α and β to be $\hat{\alpha} = 0.1792$ and $\hat{\beta} = 200.7262$, and the bias-corrected estimate of α to be $\hat{\alpha}^* = 0.2108$. The corresponding standard deviations and the confidence intervals are presented in Table 10.

Table 10. Standard Deviations of Estimates and Confidence Intervals for α and β for Example 2 with $r = 8$

	α			β		
Estimator	S.D.	90% C.I.	95% C.I.	S.D.	90% C.I.	95% C.I.
MLE	0.0471	(0.1017,0.2566)	(0.0868,0.2715)	11.6871	(183.1828,221.9857)	(180.1662,226.5831)
UMLE	0.0719	(0.0925,0.3290)	(0.06984,0.3517)	13.7416	(180.4109,226.1973)	(176.9795,231.8331)

8 Concluding Remarks

In this paper, we have discussed the MLE's of parameters α and β of the Birnbaum-Saunders distribution based on Type-II right-censored samples. As the estimator of α becomes quite biased especially when the degree of censoring is high, we have proposed a simple bias-corrected estimator based on MLE which has been shown to be almost unbiased in all situations considered. We have also described the Monte Carlo EM-algorithm for the determination of the MLE's. The probability coverages of confidence intervals based on these estimators have been evaluated. While the probability coverages in the case of MLE's turn out to be considerably below the nominal levels, the confidence intervals based on bias-corrected MLE's seem to be have probability coverages much closer to the nominal levels.

Table 1. Means, standard deviations and covariances of estimates based on Monte Carlo simulations for $n = 20$ ($\beta = 1.0$)

α	d.o.c.	$\hat{\alpha}$	SD($\hat{\alpha}$)	$\hat{\alpha}^*$	SD($\hat{\alpha}^*$)	$\hat{\beta}$	SD($\hat{\beta}$)	Cov($\hat{\alpha}$, $\hat{\beta}$)	Cov($\hat{\alpha}^*$, $\hat{\beta}$)
0.1	0%	0.0964	0.0155	0.1014	0.0164	1.0001	0.0225	0.0000	0.0000
	10%	0.0956	0.0169	0.1020	0.0180	0.9997	0.0227	0.0000	0.0000
	20%	0.0949	0.0182	0.1026	0.0197	0.9991	0.0233	0.0001	0.0001
	30%	0.0941	0.0198	0.1031	0.0217	0.9984	0.0241	0.0001	0.0001
	40%	0.0930	0.0218	0.1034	0.0243	0.9974	0.0255	0.0002	0.0002
	50%	0.0914	0.0243	0.1030	0.0273	0.9956	0.0275	0.0003	0.0003
	60%	0.0887	0.0272	0.1014	0.0311	0.9923	0.0310	0.0005	0.0005
0.3	0%	0.2889	0.0466	0.3041	0.0491	1.0018	0.0671	0.0000	0.0000
	10%	0.2866	0.0506	0.3057	0.0540	1.0005	0.0678	0.0002	0.0002
	20%	0.2845	0.0546	0.3076	0.0591	0.9990	0.0694	0.0005	0.0006
	30%	0.2822	0.0595	0.3092	0.0652	0.9970	0.0720	0.0010	0.0011
	40%	0.2790	0.0658	0.3100	0.0731	0.9940	0.0762	0.0017	0.0019
	50%	0.2742	0.0733	0.3089	0.0826	0.9891	0.0823	0.0027	0.0031
	60%	0.2661	0.0826	0.3041	0.0944	0.9800	0.0923	0.0043	0.0050
0.5	0%	0.4811	0.0776	0.5065	0.0817	1.0053	0.1101	0.0000	0.0000
	10%	0.4773	0.0843	0.5091	0.0899	1.0030	0.1116	0.0006	0.0007
	20%	0.4738	0.0912	0.5122	0.0986	1.0007	0.1146	0.0016	0.0017
	30%	0.4698	0.0996	0.5148	0.1091	0.9977	0.1193	0.0029	0.0032
	40%	0.4645	0.1107	0.5161	0.1229	0.9932	0.1267	0.0050	0.0056
	50%	0.4565	0.1240	0.5144	0.1397	0.9857	0.1372	0.0081	0.0091
	60%	0.4430	0.1405	0.5063	0.1606	0.9719	0.1542	0.0129	0.0147
1.0	0%	0.9596	0.1554	1.0101	0.1635	1.0190	0.2053	-0.0004	-0.0005
	10%	0.9514	0.1690	1.0148	0.1803	1.0140	0.2108	0.0030	0.0032
	20%	0.9439	0.1839	1.0204	0.1988	1.0098	0.2197	0.0076	0.0082
	30%	0.9356	0.2030	1.0253	0.2225	1.0051	0.2323	0.0142	0.0156
	40%	0.9249	0.2295	1.0276	0.2550	0.9988	0.2514	0.0247	0.0275
	50%	0.9092	0.2632	1.0244	0.2965	0.9883	0.2783	0.0405	0.0456
	60%	0.8826	0.3080	1.0087	0.3520	0.9695	0.3221	0.0670	0.0766
2.0	0%	1.9054	0.3177	2.0057	0.3344	1.0462	0.3288	-0.0035	-0.0037
	10%	1.8925	0.3436	2.0150	0.3666	1.0383	0.3496	0.0116	0.0132
	20%	1.8719	0.3791	2.0254	0.4124	1.0329	0.3748	0.0340	0.0378
	30%	1.8481	0.4297	2.0213	0.4722	1.0208	0.4094	0.0678	0.0746
	40%	1.8207	0.5054	2.0106	0.5584	1.0079	0.4576	0.1268	0.1343
	50%	1.7886	0.6158	1.9976	0.6842	1.0058	0.5578	0.2342	0.2529
	60%	1.7493	0.8341	1.9992	0.9532	1.0481	0.9194	0.6046	0.6910

Table 2. Means, standard deviations and covariances of estimates based on Monte Carlo simulations for $n = 30$ ($\beta = 1.0$)

α	d.o.c.	$\hat{\alpha}$	SD($\hat{\alpha}$)	$\hat{\alpha}^*$	SD($\hat{\alpha}^*$)	$\hat{\beta}$	SD($\hat{\beta}$)	Cov($\hat{\alpha}$, $\hat{\beta}$)	Cov($\hat{\alpha}^*$, $\hat{\beta}$)
0.1	0%	0.0975	0.0127	0.1008	0.0132	1.0000	0.0185	0.0000	0.0000
	10%	0.0970	0.0138	0.1012	0.0144	0.9997	0.0187	0.0000	0.0000
	20%	0.0965	0.0150	0.1016	0.0158	0.9993	0.0191	0.0000	0.0000
	30%	0.0961	0.0163	0.1020	0.0173	0.9989	0.0198	0.0001	0.0001
	40%	0.0953	0.0180	0.1021	0.0193	0.9982	0.0209	0.0001	0.0001
	50%	0.0941	0.0200	0.1017	0.0216	0.9969	0.0226	0.0002	0.0002
	60%	0.0924	0.0227	0.1008	0.0247	0.9948	0.0258	0.0003	0.0004
0.3	0%	0.2923	0.0382	0.3024	0.0395	1.0010	0.0551	0.0000	0.0000
	10%	0.2910	0.0415	0.3036	0.0433	1.0002	0.0557	0.0001	0.0002
	20%	0.2894	0.0451	0.3047	0.0475	0.9991	0.0569	0.0004	0.0004
	30%	0.2881	0.0490	0.3060	0.0520	0.9980	0.0590	0.0007	0.0007
	40%	0.2859	0.0542	0.3063	0.0581	0.9959	0.0624	0.0011	0.0012
	50%	0.2822	0.0605	0.3051	0.0654	0.9922	0.0676	0.0018	0.0020
	60%	0.2771	0.0688	0.3022	0.0751	0.9864	0.0773	0.0031	0.0034
0.5	0%	0.4869	0.0636	0.5034	0.0663	1.0032	0.0904	0.0000	0.0000
	10%	0.4846	0.0691	0.5055	0.0720	1.0018	0.0916	0.0004	0.0005
	20%	0.4821	0.0753	0.5080	0.0787	1.0001	0.0940	0.0011	0.0011
	30%	0.4799	0.0821	0.5089	0.0867	0.9984	0.0978	0.0020	0.0021
	40%	0.4761	0.0911	0.5090	0.0968	0.9952	0.1038	0.0034	0.0035
	50%	0.4700	0.1023	0.5066	0.1094	0.9895	0.1128	0.0055	0.0058
	60%	0.4614	0.1173	0.5023	0.1261	0.9810	0.1296	0.0092	0.0097
1.0	0%	0.9721	0.1273	1.0056	0.1317	1.0119	0.1674	-0.0001	-0.0001
	10%	0.9671	0.1387	1.0091	0.1447	1.0090	0.1722	0.0022	0.0023
	20%	0.9616	0.1522	1.0122	0.1602	1.0058	0.1792	0.0054	0.0057
	30%	0.9571	0.1676	1.0164	0.1780	1.0036	0.1897	0.0098	0.0104
	40%	0.9494	0.1895	1.0172	0.2031	0.9989	0.2051	0.0169	0.0181
	50%	0.9372	0.2176	1.0132	0.2353	0.9904	0.2278	0.0276	0.0298
	60%	0.9207	0.2591	1.0044	0.2826	0.9802	0.2712	0.0484	0.0528
2.0	0%	1.9388	0.2585	2.0056	0.2674	1.0300	0.2579	-0.0011	-0.0012
	10%	1.9274	0.2811	2.0109	0.2929	1.0222	0.2767	0.0088	0.0101
	20%	1.9123	0.3142	2.0168	0.3279	1.0152	0.2992	0.0249	0.0263
	30%	1.8985	0.3567	2.0141	0.3770	1.0124	0.3324	0.0490	0.0517
	40%	1.8788	0.4231	2.0090	0.4474	1.0092	0.3813	0.0910	0.0932
	50%	1.8536	0.5198	1.9966	0.5573	1.0084	0.4634	0.1673	0.1784
	60%	1.8369	0.7114	2.0039	0.7761	1.0418	0.7461	0.4301	0.4692

Table 3. Means, standard deviations and covariances of estimates based on Monte Carlo simulations for $n = 50$ ($\beta = 1.0$)

α	d.o.c.	$\hat{\alpha}$	SD($\hat{\alpha}$)	$\hat{\alpha}^*$	SD($\hat{\alpha}^*$)	$\hat{\beta}$	SD($\hat{\beta}$)	Cov($\hat{\alpha}$, $\hat{\beta}$)	Cov($\hat{\alpha}^*$, $\hat{\beta}$)
0.1	0%	0.0984	0.0099	0.1004	0.0101	0.9999	0.0142	0.0000	0.0000
	10%	0.0981	0.0108	0.1006	0.0111	0.9997	0.0144	0.0000	0.0000
	20%	0.0975	0.0127	0.1008	0.0121	0.9992	0.0152	0.0000	0.0000
	30%	0.0970	0.0139	0.1010	0.0132	0.9988	0.0160	0.0001	0.0000
	40%	0.4848	0.0703	0.1011	0.0145	0.9963	0.0796	0.0020	0.0001
	50%	0.0964	0.0157	0.1009	0.0164	0.9981	0.0174	0.0001	0.0001
	60%	0.0953	0.0177	0.1003	0.0187	0.9967	0.0199	0.0002	0.0002
0.3	0%	0.2950	0.0298	0.3010	0.0304	1.0002	0.0423	0.0000	0.0000
	10%	0.2942	0.0324	0.3017	0.0332	0.9997	0.0428	0.0001	0.0001
	20%	0.2933	0.0351	0.3024	0.0362	0.9991	0.0437	0.0002	0.0002
	30%	0.2923	0.0383	0.3029	0.0397	0.9982	0.0454	0.0004	0.0004
	40%	0.2910	0.0418	0.3031	0.0436	0.9970	0.0479	0.0007	0.0007
	50%	0.2891	0.0473	0.3027	0.0496	0.9951	0.0523	0.0011	0.0012
	60%	0.2858	0.0538	0.3008	0.0566	0.9914	0.0599	0.0019	0.0020
0.5	0%	0.4915	0.0497	0.5023	0.0509	1.0013	0.0693	0.0001	0.0000
	10%	0.4901	0.0540	0.5037	0.0549	1.0004	0.0704	0.0003	0.0002
	20%	0.4886	0.0587	0.5050	0.0602	0.9994	0.0721	0.0007	0.0006
	30%	0.4870	0.0642	0.5059	0.0668	0.9981	0.0752	0.0012	0.0012
	40%	0.4848	0.0703	0.5055	0.0738	0.9963	0.0796	0.0020	0.0020
	50%	0.4817	0.0801	0.5041	0.0830	0.9934	0.0873	0.0034	0.0034
	60%	0.4761	0.0918	0.5019	0.0961	0.9879	0.1006	0.0057	0.0058
1.0	0%	0.9820	0.0994	1.0020	0.1015	1.0059	0.1273	0.0001	0.0001
	10%	0.9789	0.1083	1.0040	0.1111	1.0041	0.1314	0.0015	0.0016
	20%	0.9757	0.1186	1.0059	0.1223	1.0023	0.1370	0.0034	0.0035
	30%	0.9722	0.1313	1.0074	0.1361	1.0003	0.1455	0.0062	0.0064
	40%	0.9677	0.1464	1.0080	0.1525	0.9975	0.1570	0.0101	0.0105
	50%	0.9616	0.1710	1.0070	0.1791	0.9938	0.1764	0.0172	0.0180
	60%	0.9508	0.2033	1.0008	0.2140	0.9868	0.2099	0.0297	0.0313
2.0	0%	1.9643	0.2003	2.0044	0.2044	1.0184	0.1940	-0.0006	-0.0006
	10%	1.9537	0.2191	2.0087	0.2220	1.0108	0.2069	0.0065	0.0053
	20%	1.9447	0.2448	2.0112	0.2497	1.0068	0.2259	0.0163	0.0151
	30%	1.9342	0.2807	2.0117	0.2906	1.0035	0.2525	0.0316	0.0315
	40%	1.9222	0.3293	2.0053	0.3438	1.0016	0.2904	0.0558	0.0573
	50%	1.9113	0.4162	1.9984	0.4265	1.0054	0.3568	0.1073	0.1066
	60%	1.9029	0.5656	2.0068	0.5880	1.0262	0.5133	0.2437	0.2501

Table 4. Probability coverages of 90% confidence intervals based on Monte Carlo simulations ($\beta = 1.0$)

		Probability coverages for α						Probability coverages for β					
		$n = 20$		$n = 30$		$n = 50$		$n = 20$		$n = 30$		$n = 50$	
α	d.o.c.	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE
0.1	0%	85.65	93.06	87.19	92.01	87.56	90.58	87.09	88.85	87.67	88.97	88.58	89.40
	10%	84.56	93.08	85.90	91.98	87.45	91.17	86.64	89.04	87.41	89.06	88.21	89.18
	20%	83.80	93.77	85.61	92.46	86.64	91.50	86.14	88.77	87.02	88.90	87.73	89.00
	30%	82.92	94.02	85.36	92.96	86.41	92.17	85.51	88.49	86.50	88.48	87.81	88.95
	40%	81.02	93.81	83.54	92.88	86.34	92.28	84.53	88.10	86.35	88.43	87.80	88.93
	50%	79.18	93.48	82.40	92.68	85.19	92.08	83.25	86.37	84.88	87.24	87.03	88.28
	60%	75.90	91.23	80.60	91.86	83.81	91.54	80.91	84.04	83.37	85.43	85.79	86.93
0.3	0%	85.59	93.00	87.17	92.05	87.52	90.56	87.05	89.02	87.62	88.86	88.67	89.39
	10%	84.51	93.11	85.89	91.95	87.43	91.15	86.62	88.89	87.56	88.88	88.26	89.19
	20%	83.74	93.78	85.57	92.44	87.17	91.47	85.99	88.81	87.02	88.70	88.06	89.04
	30%	82.90	93.95	85.34	92.95	86.63	92.08	85.38	88.55	86.46	88.59	87.75	88.88
	40%	80.95	93.76	83.44	92.83	86.38	92.28	84.67	88.12	86.51	88.58	87.74	89.00
	50%	79.07	93.41	82.31	92.61	85.24	92.04	83.30	86.56	85.20	87.25	87.02	88.28
	60%	75.67	91.04	80.50	91.68	83.77	91.43	81.27	84.25	83.67	85.64	85.99	86.87
0.5	0%	85.49	92.97	87.06	91.80	87.49	91.03	87.15	89.00	87.56	89.63	88.79	89.78
	10%	84.35	93.04	85.81	92.38	87.36	91.73	86.66	89.00	87.33	89.49	88.36	90.04
	20%	83.57	93.76	85.43	92.88	87.11	92.01	86.05	88.79	86.83	89.60	88.06	90.13
	30%	82.71	93.94	85.13	93.00	86.69	91.95	85.31	88.75	86.42	89.14	87.81	89.74
	40%	80.86	93.67	83.38	92.93	86.34	92.03	84.84	88.01	86.52	88.93	87.80	89.54
	50%	78.98	93.24	82.12	92.24	85.08	91.86	83.44	86.79	85.31	87.73	86.96	88.99
	60%	75.33	90.71	80.35	90.91	83.61	91.05	81.52	84.38	83.84	85.91	85.99	87.60
1.0	0%	85.25	92.78	86.66	91.81	87.31	90.53	86.99	89.00	87.29	88.74	88.75	89.49
	10%	83.80	92.84	85.50	91.70	87.10	90.94	86.83	89.13	87.41	88.91	88.45	89.47
	20%	82.96	93.50	84.97	92.23	86.90	91.26	86.03	88.89	86.84	88.67	88.10	89.05
	30%	82.33	93.73	84.72	92.81	86.41	91.95	85.23	88.47	86.54	88.62	87.83	88.78
	40%	80.19	93.02	82.97	92.56	86.12	91.99	84.99	87.99	86.45	88.69	87.62	88.90
	50%	77.94	92.29	81.23	91.73	84.64	91.49	83.69	86.59	85.37	87.10	87.17	88.32
	60%	74.02	89.36	78.94	90.11	83.07	90.46	81.94	84.01	84.25	85.51	86.18	86.71
2.0	0%	83.28	91.32	85.51	91.38	87.47	90.82	86.90	89.04	87.40	89.37	88.41	89.61
	10%	82.81	91.99	85.04	91.70	86.62	91.48	86.61	89.26	87.39	89.42	88.36	89.90
	20%	81.59	91.96	84.05	91.89	86.37	91.64	86.46	89.08	87.14	89.69	88.26	89.77
	30%	80.63	91.77	83.63	91.78	85.93	91.18	85.58	88.37	86.51	89.03	87.92	89.76
	40%	78.39	90.40	82.19	90.83	85.58	91.07	84.64	87.68	86.06	88.54	87.69	89.29
	50%	76.21	88.45	79.81	88.18	84.43	89.66	83.77	85.81	85.38	86.88	87.36	88.84
	60%	73.19	84.66	78.67	85.38	83.52	87.08	81.64	81.26	84.84	83.77	86.33	85.93

Table 5. Probability coverages of 95% confidence intervals based on Monte Carlo simulations ($\beta = 1.0$)

		Probability coverages for α						Probability coverages for β					
		$n = 20$		$n = 30$		$n = 50$		$n = 20$		$n = 30$		$n = 50$	
α	d.o.c.	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE	MLE	UMLE
0.1	0%	90.49	95.87	91.89	95.68	92.42	93.95	92.90	94.08	93.25	95.18	93.81	94.26
	10%	89.26	96.00	90.75	95.70	92.37	94.07	92.45	93.98	93.05	95.36	93.68	94.31
	20%	88.69	96.25	90.26	95.92	91.63	94.01	91.90	93.78	92.67	95.43	93.05	94.28
	30%	87.84	96.20	89.96	95.89	91.26	93.70	91.13	93.30	92.06	95.89	92.76	93.84
	40%	86.07	95.92	88.44	95.71	91.18	93.09	90.25	92.73	91.59	95.80	92.86	93.67
	50%	83.78	95.41	87.14	95.16	90.38	92.23	88.69	91.27	90.44	95.43	92.30	93.13
	60%	80.90	93.80	85.60	94.26	89.03	90.27	86.13	88.37	88.79	94.57	90.78	91.73
0.3	0%	90.42	95.87	91.83	95.65	92.42	95.16	92.78	94.03	93.14	94.03	93.85	94.21
	10%	89.25	96.00	90.73	95.68	92.40	95.31	92.33	94.06	93.02	94.13	93.77	94.37
	20%	88.63	96.21	90.25	95.87	91.82	95.42	91.90	93.82	92.65	94.06	93.49	94.42
	30%	87.70	96.17	89.92	95.85	91.62	95.88	91.13	93.47	92.12	93.70	93.18	93.93
	40%	85.99	95.86	88.46	95.73	91.24	95.76	90.43	92.86	91.70	93.17	92.71	93.75
	50%	83.77	95.32	87.04	95.07	90.27	95.36	89.07	91.62	90.64	92.40	92.38	93.21
	60%	80.68	93.70	85.40	94.20	88.87	94.47	86.49	88.78	89.08	90.47	91.04	91.90
0.5	0%	90.28	95.82	91.78	95.56	92.35	95.38	92.59	93.90	93.07	94.20	93.77	94.59
	10%	89.17	95.92	90.67	95.87	92.28	95.81	92.41	94.06	93.00	94.37	93.80	94.58
	20%	88.43	96.19	90.10	95.96	91.78	95.65	91.71	93.83	92.68	94.38	93.61	94.82
	30%	87.65	96.12	89.84	95.87	91.58	95.65	91.21	93.50	92.16	94.20	93.14	94.60
	40%	85.87	95.75	88.30	95.39	91.18	95.47	90.64	92.97	91.73	93.71	92.86	94.36
	50%	83.50	95.20	86.77	95.04	90.08	95.32	89.23	91.71	90.82	92.48	92.32	94.01
	60%	80.32	93.38	85.16	93.94	88.54	94.38	86.88	89.04	89.41	91.06	91.27	92.36
1.0	0%	89.94	95.72	91.59	95.54	92.25	95.03	92.34	93.71	92.72	93.83	93.62	94.20
	10%	88.73	95.68	90.33	95.61	92.17	95.18	92.02	93.75	92.56	93.76	93.70	94.33
	20%	88.14	96.04	89.69	95.58	91.68	95.34	91.45	93.66	92.56	93.91	93.54	94.26
	30%	86.97	95.77	89.47	95.55	91.24	95.64	91.22	93.41	92.26	93.60	93.04	94.11
	40%	85.19	95.22	87.70	95.20	90.73	95.47	90.74	92.99	91.94	93.47	92.96	93.80
	50%	82.37	94.47	85.70	94.35	89.62	94.71	89.66	91.81	90.97	92.30	92.58	93.37
	60%	78.74	92.10	83.71	92.93	87.49	93.59	87.41	89.17	89.78	90.75	91.55	92.06
2.0	0%	88.58	94.59	90.70	95.11	92.40	95.14	91.61	93.12	92.35	93.44	93.51	94.13
	10%	87.92	94.79	89.67	95.38	91.80	95.58	91.86	93.47	92.31	93.70	93.70	94.30
	20%	86.93	95.03	88.83	95.31	91.16	95.36	91.74	93.55	92.28	94.00	93.46	94.42
	30%	85.10	94.30	88.23	94.76	90.49	95.03	91.00	93.23	92.26	94.00	93.12	94.41
	40%	83.09	93.07	86.17	93.41	89.78	94.05	90.58	92.71	91.96	93.71	93.07	94.30
	50%	80.47	91.61	84.10	91.50	88.60	92.61	89.71	90.89	91.16	92.35	92.68	93.73
	60%	77.44	88.31	82.97	88.86	87.72	90.25	87.43	87.23	90.67	89.81	91.86	91.57

Acknowledgments

The authors are grateful to the editor and the referee for their valuable suggestions that enhanced the paper. The authors also thank the Natural Sciences and Engineering Research Council of Canada for funding this research. The first author wishes to express his gratitude to Ms. Sheron Ng for her support.

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