Analysis of Type-II Progressively Hybrid Censored Competing Risks Data

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Abstract

In medical studies or in reliability analysis, it is quite common that the failure of any individual or any item may be attributable to more than one cause. Moreover, the observed data are often censored. Hybrid censoring scheme which is the mixture of conventional Type-I and Type-II censoring schemes is quite useful in life-testing or reliability experiments. Recently Type-II progressive censoring scheme becomes quite popular for analyzing highly reliable data. But in that case the length of the experiment can be quite large. Hence, in this paper we introduce a Type-II progressively hybrid censoring scheme for competing risks data, where the experiment terminates at a pre-specified time. We derive the likelihood inference of the unknown parameters under the assumptions that the lifetime distributions of the different causes are independent and exponentially distributed. We obtain the maximum likelihood estimators of the unknown parameters in exact forms. Asymptotic confidence intervals and two bootstrap confidence intervals are also proposed. Bayes estimates and credible intervals of the unknown parameters are obtained under the assumption of gamma priors on the unknown parameters. Different methods have been compared using Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

Keywords: Competing Risk; Maximum likelihood estimator; Type-I and Type-II censoring; Fisher Information matrix; Asymptotic distribution; Bayesian inference; Exponential distribution; Gamma distribution; Type-II progressive censoring scheme.

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1 Introduction

In medical studies or in reliability analysis, it is quite common that more than one cause or risk factor may be present at the same time. In analyzing the competing risks model, it is assumed that data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out under this assumption for both the parametric and the non-parametric set up. For the parametric set up it is assumed that different lifetime distributions follow some special parametric distribution, namely exponential, Weibull or gamma. Several authors, for example Berkson and Elveback [2], Cox [8], David and Moeschberger [10] consider the problem from the parametric point of view. In the non-parametric set up no specific life time distribution is assumed. Kaplan and Meier [21], Efron [11] and Peterson [23] analyzed the non-parametric version of this model.

The two most common censoring schemes namely Type-I and Type-II censoring schemes are widely used in practice. Briefly, they can be described as follows. Consider n items are under observations in a particular experiment. In the conventional Type-I censoring scheme, the experiment continues up to a pre-specified time T. On the other hand, the conventional Type-II censoring scheme requires the experiment to continue until a pre-specified number of failures $m \leq n$ occur. In this scenario, only the smallest lifetimes are observed. The mixture of Type-I and Type-II censoring schemes is known as the hybrid censoring scheme. This hybrid censoring scheme was first introduced by Epstein [13, 14]. But recently it becomes quite popular in the reliability and life-testing experiments. See for example the work of Chen and Bhattacharya [3], Childs *at al.* [4], Draper and Guttman [9], Fairbanks, Madasan and Dykstra [15], Gupta and Kundu [16] and Jeong, Park and Yum [19].

One of the drawbacks of the conventional Type-I, Type-II or hybrid censoring schemes is that they do not allow for removal of units at points other than the terminal point of the experiment. When the items are highly reliable it might be necessary to know the causes for which the items are failed and also necessary to remove items in between the experiment (at the time of each failure) for efficient estimation of the parameters. Because of this, one censoring scheme known as progressive censoring scheme under competing risks becomes very popular for the last few years. It can be described as follows: Consider n items in a study and assume that there are K causes of failure which are known. Suppose $m < n$ is fixed before the experiment. Moreover, m other integers, $R_1, \ldots, R_m$ are also fixed before so that $R_1 + \ldots + R_m + m = n$. At the time of the first failure $X_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $X_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the m-th failure $X_{m:m:n}$, the rest of the R_m units are removed. It is also known that the first failure takes place due to cause δ_1, similarly the second failure takes place due to cause δ_2 and so on, finally the m-th failure takes place due to cause δ_m. For an exhaustive list of references and further details on Type-II progressive censoring, the readers may refer to the book by Balakrishnan and Aggarwala [1].

In this paper, we introduce a Type-II progressively hybrid censoring scheme under competing risk. As the name suggests, it is a mixture of Type-II progressive and hybrid censoring schemes under the competing risk data. The detail description and its advantages will be described in the next section. In this new censoring scheme, we obtain the likelihood inference of the unknown parameters, under the assumptions that the lifetime distributions of the different causes are independent identically distributed (*i.i.d.*) exponential random variables. It is observed that the maximum likelihood estimators of the unknown parameters always exists and we obtain the explicit form of the maximum likelihood estimators (MLEs) of the unknown parameters. We also obtain the asymptotic confidence intervals and propose two bootstrap confidence intervals. Bayes estimates and credible intervals are also obtained under the assumption of the gamma priors on the unknown parameters. Different methods are compared using Monte Carlo simulations and for illustrative purposes we analyze one real data set.

The rest of the paper is organized as follows. We formulate the problem in Section 2 and provide the MLEs of the unknown parameters. Different confidence intervals are presented in Section 3. Bayesian analysis is provided in Section 4. Numerical results are presented in Section 5. One real data set has been analyzed in Section 6 and finally we conclude the paper in Section 7.

2 Model Description, Notation and MLE

2.1 Model Description and Notation

Suppose n identical items are put on a test and the lifetime distributions of the n items are denoted by $X_1, \ldots, X_n$. The integer $m < n$ is pre-fixed and also $R_1, \ldots, R_m$ are m pre-fixed integers satisfying $R_1 + \ldots + R_m + m = n$. T is a pre-fixed time point. At the time of first failure R_1 of the remaining units are randomly removed. Similarly at the time of the second failure R_2 of the remaining units are removed and so on. If the m-th failure occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$. On the other hand suppose the m-th failure does not occur before time point T and only J failures occur before the time point T, where $0 \leq J < m$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T. Note that $R_J^* = n - (R_1 + \ldots + R_J) - J$. We denote the two cases as Case I and Case II respectively and call this censoring scheme as the Type-II progressively hybrid censoring scheme under competing risk data. In presence of Type-II progressively hybrid censoring scheme under competing risks data, we have one of the following types of observations;

Case I: $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{m:m:n}, \delta_m, R_m)\}$; if $X_{m:m:n} < T$, or

Case II: $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{J:m:n}, \delta_J, R_J), (T, R_J^*)\}; \quad \text{if} \quad X_{J:m:n} < T < X_{J+1:m:n}$.

Note that for Case II, $X_{J:m:n} < T < X_{J+1:m:n} < \ldots < X_{m:m:n}$ and $X_{J+1:m:n} < \ldots < X_{m:m:n}$ are not observed.

The conventional Type-I progressive censoring scheme needs the pre-specification of $R_1, \ldots, R_m$ and also $T_1, \ldots, T_m$, see Cohen [5, 6] for details. The choices of $T_1, \ldots, T_m$ are not trivial. For the conventional Type-II progressive censoring scheme the experimental time is unbounded. In our proposed censoring scheme, the choice of T depends how much maximum experimental time the experimenter can afford to spend. Moreover, the experimental time is bounded.

Without loss of generality, we assume that there are only two independent causes of failure i.e. $K = 2$. It may be extended to the case of $K > 2$. Before progressing further, we introduce/ review the following notations;

X_{ji} :	lifetime of the i-th individual under cause j; for $j = 1, 2$ and $i = 1, \ldots, n$
$X_{i:m:n}$:	i-th observed failure time; $i = 1, \ldots, m$
$f(.)$:	probability density function (PDF) of X_i
$F(.)$:	cumulative distribution function (CDF) of X_i
$F_j(.)$:	cumulative distribution function (CDF) of X_{ji}
m_1 :	the number of failures observed before termination due to cause 1 for Case I
m_2 :	the number of failures observed before termination due to cause 2 for Case I
m :	total number of failures observed before termination for Case I; $i.e.$ $m = m_1 + m_2$
J_1 :	the number of failures observed before termination due to cause 1 for Case II
J_2 :	the number of failures observed before termination due to cause 2 for Case II

J :	total number of failures observed before termination for Case II; *i.e.* $J = J_1 + J_2$
D_1 :	the number of failures due to cause 1, *i.e.* $D_1 = m_1$ for Case I and $D_1 = J_1$ for Case II
D_2 :	the number of failures due to cause 2, *i.e.* $D_2 = m_2$ for Case I and $D_2 = J_2$ for Case II
D :	total number of failures, *i.e.* $D = m = m_1 + m_2$ for Case I and $D = J = J_1 + J_2$ for Case II
R_i :	the number of units removed at the time of i^{th} failure; $R_i \geq 0$
R_J^* :	the number of remaining units left at the time point T for Case II
δ_i :	indicator variable denoting the cause of failure of the i^{th} individual
$e(\lambda)$:	exponential random variable with PDF $\lambda e^{-\lambda x}$
$gamma(\alpha, \lambda)$:	gamma random variable with PDF $\dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

We assume that $(X_{1i}, X_{2i}), i = 1, \ldots, n$ are n *i.i.d.* exponential random variables. Further, X_{1i} and X_{2i} are independent for all $i = 1, \ldots, n$ and $X_i = min(X_{1i}, X_{2i})$. Now we provide the MLEs of the unknown parameters when X_{ji}'s (for $i = 1, \ldots, n$) are *i.i.d.* $\exp(\lambda_j)$, for $j = 1, 2$.

2.2 Maximum Likelihood Estimator

Based on the observations as discussed in the previous subsection, the log-likelihood function (without the constant term) can be written as;

$$L(\lambda_1, \lambda_2) = D_1 \ln \lambda_1 + D_2 \ln \lambda_2 - (\lambda_1 + \lambda_2) W, \tag{1}$$

where $D_1 = m_1$, $D_2 = m_2$, $W = \sum_{i=1}^{m}(1+R_i)x_{i:m:n}$ for Case I and $D_1 = J_1$, $D_2 = J_2$, $W = \sum_{i=1}^{J}(1+R_i)x_{i:m:n} + TR_J^*$ for Case II. From (1), it is clear that the MLEs of λ_1 and λ_2 always exists and they are

$$\hat{\lambda_1} = \frac{D_1}{W} \qquad \text{and} \qquad \hat{\lambda_2} = \frac{D_2}{W}. \tag{2}$$

It is not possible to obtain the exact distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ because of the complicated nature of the conditional distributions of $X_{1:m:n}, \ldots, X_{m:m:n}$ given $X_{m:m:n} < T$. Interestingly, the distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the mixture of discrete and continuous distributions. They have positive masses at the point 0 and have the bounded supports. Since, the exact distributions of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are not known, the exact confidence intervals also can not be obtained.

3 Confidence Intervals

In this section, we propose three different confidence intervals. One is based on the asymptotic distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ and two different bootstrap confidence intervals.

3.1 Asymptotic Confidence Interval

In this section, we present the Fisher information matrix of λ_1 and λ_2. Let $I(\lambda_1, \lambda_2) = (I_{ij}(\lambda_1, \lambda_2))$, $i.j$ =1, 2, denote the Fisher information matrix of the parameters λ_1 and λ_2, where

$$I_{ij}(\lambda_1, \lambda_2) = -E\left[\frac{\partial^2 L(\lambda_1, \lambda_2)}{\partial \lambda_i \lambda_j}\right] \tag{3}$$

From (1) it follows that

$$I_{11}(\lambda_1, \lambda_2) = \frac{E(D_1)}{\lambda_1^2}, \qquad I_{12}(\lambda_1, \lambda_2) = I_{21}(\lambda_1, \lambda_2) = 0 \quad \text{and} \quad I_{22}(\lambda_1, \lambda_2) = \frac{E(D_2)}{\lambda_2^2}.$$

Simple calculation shows that

$$E(D_1) = \sum_{i=1}^{m_1} P(X_{i:m:n} < T) \quad \text{and} \quad E(D_2) = \sum_{i=1}^{m_2} P(X_{i:m:n} < T).$$

It is not easy to compute $P(X_{i:m:n} < T)$ for general i, because $X_{i:m:n}$ is a sum of i independent but not identically distributed exponential random variables. Therefore, for $D_1 > 0$ and $D_2 > 0$, we propose the following approximate 100(1-α)% confidence interval for λ_1 and λ_2,

$$\hat{\lambda_1} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda_1}^2}{D_1}} \quad \text{and} \quad \hat{\lambda_2} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda_2}^2}{D_1}} \tag{4}$$

respectively.

3.2 Bootstrap Confidence Intervals

In this subsection we propose two confidence intervals based on the bootstrapping. The two bootstrap methods that are widely used in practice are;

(1) The percentile bootstrap (Boot-p) proposed by Efron [12], and

(2) The bootstrap-t method (Boot-t) proposed by Hall [17].

It is observed that in this type of situations (Kundu, Kannan and Balakrishnan, [20]), the non-parametric bootstrap method does not work well. Hence, we propose the following two parametric bootstrap confidence intervals for λ_1 and λ_2. We illustrate the procedure for the parameter λ_1. For the other parameter (λ_2) confidence interval may be constructed in an analogous manner.

Boot-p Method:

[1] Estimate $\hat{\lambda_1}$ and $\hat{\lambda_2}$ from the sample using (2).

[2] Generate a bootstrap sample $\{X^*_{1:m:n}, \ldots, X^*_{D^*:m:n}\}$, using $\hat{\lambda}_1$, $\hat{\lambda}_2$, $R_1, \ldots, R_m$ and T. Obtain the bootstrap estimate of λ_1 say, $\hat{\lambda}_1^{\,*}$ using the bootstrap sample.

[3] Repeat Step [2] NBOOT times.

[4] Let $\widehat{CDF}(x) = P(\hat{\lambda}_1^{\,*} \le x)$, be the cumulative distribution function of $\hat{\lambda}_1^{\,*}$. Define $\hat{\lambda}_{1Boot-p}(x) = \widehat{CDF}^{-1}(x)$ for a given x. The approximate 100(1-α)% confidence interval for λ_1 is given by

$$\left(\hat{\lambda}_{1Boot-p}\left(\frac{\alpha}{2}\right), \hat{\lambda}_{1Boot-p}\left(1-\frac{\alpha}{2}\right)\right).$$

Boot-t Method:

[1] Estimate $\hat{\lambda}_1$ and $\hat{\lambda}_2$ from the sample using (2) as before.

[2] Generate a bootstrap sample $\{X^*_{1:m:n}, \ldots, X^*_{D^*:m:n}\}$, using $\hat{\lambda}_1$, $\hat{\lambda}_2$, $R_1, \ldots, R_m$ and T. Also compute $\hat{V}(\hat{\lambda}_1^{\,*}) = \frac{\hat{\lambda}_1^{*\,2}}{D_1^*}$ for $D_1^* > 0$.

[3] Determine the T_1^* statistic

$$T_1^* = \frac{\sqrt{D_1^*}(\hat{\lambda}_1^{\,*} - \hat{\lambda}_1)}{\sqrt{\hat{V}(\hat{\lambda}_1^{\,*})}}.$$

[4] Repeat Step [2] - [3] NBOOT times.

[5] Let $\widehat{CDF}(x) = P(T_1^* \le x)$, be the cumulative distribution function of T_1^*. For a given x, define $\hat{\lambda}_{1Boot-t}(x) = \hat{\lambda}_1 + D_1^{*-\frac{1}{2}}\sqrt{\hat{V}(\hat{\lambda}_1^{\,*})}\widehat{CDF}^{-1}(x)$. The approximate 100(1-$\alpha$)% confidence interval for λ_1 is given by

$$\left(\hat{\lambda}_{1Boot-t}\left(\frac{\alpha}{2}\right), \hat{\lambda}_{1Boot-t}\left(1-\frac{\alpha}{2}\right)\right).$$

4 Bayesian Analysis

In this section we approach the problem from the Bayesian point of view. In the context of exponential lifetimes λ_1 and λ_2 may be reasonably modeled by the gamma priors. We assume that λ_1 and λ_2 are independently distributed as $gamma(a_1, b_1)$ and $gamma(a_2, b_2)$ priors, respectively. The gamma parameters a_1, b_1, a_2 and b_2 are all assumed to be positive. When $a_1 = b_1 = 0$ ($a_2 = b_2 = 0$), we obtain the non-informative priors of λ_1 (λ_2). The posterior density of λ_1 and λ_2 based on the gamma priors is given by

$$l(\lambda_1, \lambda_2 | data) \propto \lambda_1^{D_1+a_1-1} \lambda_2^{D_2+a_2-1} e^{-\lambda_1(W+b_1)} e^{-\lambda_2(W+b_2)}. \tag{5}$$

From (5), it is clear that the posterior density functions of λ_1 and λ_2, say $l(\lambda_1|data)$ and $l(\lambda_2|data)$, respectively, are independent. Further, $l(\lambda_1|data)$ is the density function of a $gamma(D_1 + a_1, W + b_1)$ random variable, and $l(\lambda_2|data)$ is the density function of a $gamma(D_2 + a_2, W + b_2)$ random variable. Therefore, the Bayes estimates of λ_1 and λ_2 under squared error loss functions are

$$\hat{\lambda}_{1Bayes} = \frac{D_1 + a_1}{W + b_1} \qquad \text{and} \qquad \hat{\lambda}_{2Bayes} = \frac{D_2 + a_2}{W + b_2}, \tag{6}$$

respectively. Interestingly, when the non-informative priors $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimators coincide with the corresponding MLEs.

The credible intervals for λ_1 and λ_2 can be obtained using the posterior distributions of λ_1 and λ_2. Note that *a posteriori* $Z_1 = 2\lambda_1(W + b_1)$ and $Z_2 = 2\lambda_2(W + b_2)$ follow χ^2 distributions with $2(D_1 + a_1)$ and $2(D_2 + a_2)$ degrees of freedom respectively, provided both $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are positive integers. Therefore, 100(1-α)% credible intervals for λ_1 and λ_2 are

$$\left[\frac{\chi^2_{2(D_1+a_1),1-\frac{\alpha}{2}}}{2(W + b_1)}, \frac{\chi^2_{2(D_1+a_1),\frac{\alpha}{2}}}{2(W + b_1)} \right] \qquad \text{and} \qquad \left[\frac{\chi^2_{2(D_2+a_2),1-\frac{\alpha}{2}}}{2(W + b_1)}, \frac{\chi^2_{2(D_2+a_2),\frac{\alpha}{2}}}{2(W + b_2)} \right], \tag{7}$$

respectively for $(D_1 + a_1) > 0$ and $(D_2 + a_2) > 0$. Here $\chi^2_{k,\frac{\alpha}{2}}$ and $\chi^2_{k,1-\frac{\alpha}{2}}$ denote the lower and upper $\frac{\alpha}{2}$-th percentile points of a χ^2 distribution with 'k' degrees of freedom. Note that if $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are not integer values then gamma distribution can be used to construct the credible intervals. If no prior information is available, then non-informative priors can be used to compute the credible intervals for λ_1 and λ_2. Alternatively, using the suggestion of Congdon [7], very small positive values of a_1, b_1, a_2 and b_2 can be used to construct the Bayes estimates or the corresponding credible intervals.

5 Numerical Results and Discussions

Since the performance of the different methods can not compared theoretically, we use Monte Carlo simulations to compare different methods for different parameter values and for different sampling schemes. The term *different sampling schemes* means for different sets of R_i's and for different T values. All the computations are performed using Pentium IV processor and using the random number generation algorithm RAN2 of Press *et al.* [22]. All the programs are written in FORTRAN and they can be obtained from the authors on request.

Before progressing further, first we describe how we generate Type-II progressively hybrid censored competing risk data for a given set n, m, $R_1, \ldots, R_m$ and T. We use the following transformation suggested in Balakrishnan and Aggarwala [1].

$$\begin{aligned} Z_1 &= nX_{1:m:n} \\ Z_2 &= (n - R_1 - 1)(X_{2:m:n} - X_{1:m:n}) \\ &\vdots \\ Z_m &= (n - R_1 - \ldots - R_{m-1} - m + 1)(X_{m:m:n} - X_{m-1:m:n}). \end{aligned} \tag{8}$$

It is known that if X_i's are *i.i.d* $exp(\lambda_1+\lambda_2)$, then the spacings Z_i's are also *i.i.d* $exp(\lambda_1+\lambda_2)$

random variables. From (8) it follows that

$$
\begin{aligned}
X_{1:m:n} &= \frac{1}{n} Z_1 \\
X_{2:m:n} &= \frac{1}{n - R_1 - 1} Z_2 + \frac{1}{n} Z_1 \\
&\vdots \\
X_{m:m:n} &= \frac{1}{n - R_1 - \ldots - R_{m-1} - m + 1} Z_m + \ldots + \frac{1}{n} Z_1.
\end{aligned} \tag{9}
$$

Using (9), Type-II progressively hybrid censored competing risk data can be easily generated as follows. For a given n, m, $R_1, \ldots, R_m$, we generate $X_{1:m:n}, \ldots, X_{m:m:n}$ using (9). Again using the random number generation algorithm RAN2 of Press *et al.* [22], we generate new random variable $U(i)$, for $i = 1 \ldots m$. Now if $U(i) < \frac{\lambda_1}{\lambda_1 + \lambda_2}$ then assign $\delta_i = 1$ otherwise $\delta_i = 2$. If $X_{m:m:n} < T$, then we have Case I and the corresponding sample is $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{m:m:n}, \delta_m, R_m)\}$ otherwise we have Case II and we find J, such that $X_{J:m:n} < T < X_{J+1:m:n}$. The corresponding sample is $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{J:m:n}, \delta_J, R_J), (T, R_J^*)\}$, where R_J^* is same as defined before.

We consider different n, m, T, λ_1, λ_2 and R_i's. In all our simulation experiments, we take $\lambda_1 = 1.0$ and $\lambda_2 = 0.8$. We take $n = 15, 25, 50, 100$, $m = 5, 10, 15$, $T = 0.25, 0.50, 1.00, 2.00$ and three different sampling schemes. Scheme 1: $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - m$. Scheme 2: $R_1 = n - m$ and $R_1 = \ldots = R_m = 0$. Scheme 3: $R_1 = \ldots = R_{m-1} = 1$ and $R_m = n - 2m + 1$. For each case we compute the MLEs and the 95% confidence intervals of λ_1 and λ_2 using all the three proposed method. For comparison purposes we also compute the 95% credible intervals using non-informative prior. We replicate the process 1000 times in each case and report the average bias, mean squared errors and the coverage percentages. The results are reported in Tables 1 - 9.

Some of the important observations are as follows. For fixed n as m increases the biases and MSEs of both λ_1 and λ_2 decrease for all cases as expected. But interestingly for fixed m

as n increases the biases increase and the MSEs decrease for both λ_1 and λ_2. This phenomena is quite counter intuitive and we can not find a proper explanation for this. Now comparing different confidence intervals in terms of their average lengths and coverage percentages, it is observed that the MLEs, BOOT-T confidence intervals and Bayes credible intervals behave quite satisfactory unless the T is very small. Otherwise most of cases these three confidence intervals maintain the nominal coverage probabilities. Since BOOT-T method is involved numerically and the confidence intervals based on the asymptotic distributions are slightly larger than the Bayes credible intervals we recommend to use the Bayes credible intervals for all cases. Among the different schemes it is observed that the scheme 1 produces the smallest confidence intervals followed by scheme 3 and scheme 2.

6 Data Analysis

In this section we consider one real-life dataset originally analyzed by Hoel [18]. The data arose from a laboratory experiment in which male mice received a radiation dose of 300 roentgens at 5 to 6 weeks of age. The cause of death for each mouse was determined by autopsy to be thymic lymphoma, reticulum cell sarcoma, or other causes. For the purpose of analysis, we consider reticulum cell sarcoma as cause 1 and combine the other causes of death as cause 2. There were n = 77 observations in the data. We generated a progressively type-II censored sample from the original measurements.

Example 1: In this case $n = 77$ and we take $m = 25$, $T = 700$, $R_1 = R_2 = \ldots = R_{24} = 2$ and $R_{25} = 4$. Thus the Type II progressively hybrid censored sample is : (40, 2), (42, 2), (62, 2), (163, 2), (179,2), (206, 2), (222, 2), (228, 2), (252, 2), (259, 2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (507, 2), (517, 2), (524, 2), (525, 1), (528, 1), (536, 1), (605, 1), (612, 1), (620, 2), (621, 1).

In this case, $D_1 = 7$, $D_2 = 18$ and $W = \sum_{i=1}^{25}(1 + R_i)x_{i:m:n} = 28962$. Therefore,

$$\hat{\lambda_1} = \frac{7}{28962} = 2.41696 \times 10^{-4} \quad \text{and} \quad \hat{\lambda_2} = \frac{18}{28962} = 6.21504 \times 10^{-4}.$$

Now we report the 95% asymptotic, Boot-P, Boot-t confidence intervals and also the 95% credible intervals of λ_1 and λ_2 in Table 10.

TABLE 10

Methods ↓	λ_1	λ_2
Asymptotic	$(0.62645 \times 10^{-4}, 4.20747 \times 10^{-4})$	$(3.34384 \times 10^{-4}, 9.08624 \times 10^{-4})$
Boot-p	$(0.76099 \times 10^{-4}, 4.52108 \times 10^{-4})$	$(3.47439 \times 10^{-4}, 10.52984 \times 10^{-4})$
Boot-t	$(0.58039 \times 10^{-4}, 4.26943 \times 10^{-4})$	$(2.71588 \times 10^{-4}, 9.46895 \times 10^{-4})$
Credible	$(0.97174 \times 10^{-4}, 4.50918 \times 10^{-4})$	$(3.60913 \times 10^{-4}, 9.31153 \times 10^{-4})$

It is clear that although all of them provided almost similar confidence/ credible intervals, but Bayes credible intervals have the smallest lengths. Now we generate the data using $T = 600$ instead of $T = 700$, while m and $R(i)$'s are same as before.

EXAMPLE 2: In this case the progressively hybrid censored sample obtained as: (40, 2), (42, 2), (62, 2), (163, 2), (179,2), (206, 2), (222, 2), (228, 2), (252, 2), (259, 2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (507, 2), (517, 2), (524, 2), (525, 1), (528, 1), (536, 1).

Here $D_1 = 4$, $D_2 = 17$ and $W = \sum_{i=1}^{21}(1 + R_i)x_{i:m:n} = 20346$. Therefore, we obtain

$$\hat{\lambda_1} = \frac{4}{28746} = 1.39150 \times 10^{-4} \quad \text{and} \quad \hat{\lambda_2} = \frac{17}{28746} = 20.23809 \times 10^{-4}.$$

In this case, we report the 95% asymptotic, Boot-P, Boot-t confidence intervals and also the 95% credible intervals of λ_1 and λ_2 in Table 11.

Table 11

Methods ↓	λ_1	λ_2
Asymptotic	$(0.02783 \times 10^{-4}, 2.75517 \times 10^{-4})$	$(10.61752 \times 10^{-4}, 29.85867 \times 10^{-4})$
Boot-p	$(0.00000 \times 10^{-4}, 3.02527 \times 10^{-4})$	$(14.13159 \times 10^{-4}, 32.89348 \times 10^{-4})$
Boot-t	$(-0.26530 \times 10^{-4}, 3.63490 \times 10^{-4})$	$(11.92432 \times 10^{-4}, 27.94359 \times 10^{-4})$
Credible	$(0.37913 \times 10^{-4}, 3.04992 \times 10^{-4})$	$(3.37047 \times 10^{-4}, 8.95152 \times 10^{-4})$

From Table 11, it is observed that T plays a major role for the estimation of λ's and for the construction of the corresponding confidence intervals. It is also important to note that Boot-p and Boot-t are the most affected due to T and the Bayes confidence intervals are the least affected. Therefore, Bayes confidence intervals are quite robust also with respect to T.

7 Conclusions

In this paper we discuss a new censoring scheme namely the Type II progressively hybrid censoring scheme under competing risks data. Assuming that the lifetime distributions are exponentially distributed we obtain the maximum likelihood estimators of the unknown parameter and propose different confidence intervals using asymptotic distributions as well as using bootstrap methods. Bayesian estimates of the unknown parameters are also proposed and it is observed that the Bayes credible intervals with respect to non-informative prior work quite well in this case and it has several desirable properties. Although we have assumed that the lifetime distributions are exponential but most of the methods may be extended for other distributions also, like Weibull distribution or gamma distribution. The work is in progress and it will be reported elsewhere.

References

[1] Balakrishnan, N. and Aggarwala, R. (2000), *Progressive Censoring: Theory, Methods and Applications*, Boston: Birkhäuser.

[2] Berkson, J. and Elveback, L. (1960), "Competing exponential risks with particular inference to the study of smoking lung cancer", *Journal of the American Statistical Association*, vol. 87, 84-89.

[3] Chen, S.M. and Bhattacharya, G.K. (1988), "Exact confidence bounds for an exponential parameter hybrid censoring", *Communications in Statistics - Theory and Methods*, vol. 17, 1858 - 1870.

[4] Childs, A., Chandrasekhar, B., Balakrishnan, N. and Kundu, D. (2003), "Exact Likelihood Inference Based on Type-I and Type-II Hybrid Censored Samples from the Exponential Distribution",*Annals of the Institute of Statistical Mathematics*, vol. 55, 319-330.

[5] Cohen, A.C. (1963), "Progressively censored samples in life testing", *Technometrics*, vol. 5, 327 - 329.

[6] Cohen, A.C. (1966), "Life-testing and early failure", *Technometrics*, vol. 8, 539 - 549.

[7] Congdon, P. (2001), *Bayesian Statistical Modeling*, John Wiley, New York.

[8] Cox, D. R. (1959), "The analysis of exponentially distributed lifetimes with two types of failures", *Journal of Royal Statistics*, B 21, 411-421.

[9] Draper, N. and Guttman, T. (1987), "Bayesian analysis of hybrid life-test with exponential failure times", *Annals of the Institute of Statistical Mathematics*, vol. 39, 219 - 255.

[10] David, H. A. and Moeschberger, M. L. (1978), "The Theory of Competing Risks", *Griffin*, London.

[11] Efron, B. (1967),"The two sample problem with censored data", *Proceedings of the Fifth Berkeley Symposium Math. Stat. Prob.*, pp. 831 - 853.

[12] Efron, B. (1982), *The Jackknife, the Bootstrap and Other Re-sampling Plans*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 38, SIAM, Philadelphia, PA.

[13] Epstein B. (1954), "Truncated life-tests in the exponential case", *Annals of Mathematical Statistics*, vol. 25, 555 - 564.

[14] Epstein, B. (1960), "Estimation from life-test data", *Technometrics*, vol. 2, 447 - 454.

[15] Fairbanks, K., Madasan, R. and Dykstra, R. (1982), "A confidence interval for an exponential parameter from hybrid life-test", *Journal of the American Statistical Association*, vol. 77, 137 - 140.

[16] Gupta, R.D. and Kundu, D. (1998), "Hybrid censoring schemes with exponential failure distribution", *Communications in Statistics - Theory and Methods*, vol. 27, 3065 - 3083.

[17] Hall, P. (1988), "Theoretical comparison of bootstrap confidence intervals", *Annals of Statistics*, vol. 16, 927 - 953.

[18] Hoel, D. G. (1972), "A representation of mortality data by competing risks", *Biometrics*, 28, 475 - 488.

[19] Jeong, H.S., Park, J.I. and Yum, B.J. (1996), "Development of (r, T) hybrid sampling plans for exponential lifetime distributions", *Journal of Applied Statistics*, vol. 23, 601-607.

[20] Kundu, D., Kannan, N. and Balakrishnan, N. (2004), "Analysis of progressively censored competing risks data", *Handbook of Statistics*, vol. 23, eds., Balakrishnan, N. and Rao, C.R., Elsevier, New York.

[21] Kaplan, E. L. and Meier, P. (1958), "Non-parametric estimation from incomplete observation", *Journal of the American Statistical Association*, vol. 53, 457 - 481.

[22] Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. (1991), *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, U.K.

[23] Peterson Jr., A. P. (1977), "Expressing the Kaplan-Meier estimator as a function of empirical survival functions", *Journal of the American Statistical Association*, vol. 72, 854 - 858.

Table 1: Here $n = 15, m = 5$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.2406 (1.2953)	0.2834 (1.2330)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1422 (0.6589)	0.1754 (0.6266)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.8876 (86.4)	2.9185 (93.3)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4473 (90.5)	2.4790 (89.6)	2.4801 (89.6)	2.4801 (89.6)
1	BOOT-P	λ_1	4.0095 (88.3)	4.0829 (91.1)	4.0721 (91.6)	4.0717 (91.6)
		λ_2	3.2510 (87.0)	3.3224 (89.1)	3.3175 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.6389 (87.7)	2.8758 (90.7)	2.9050 (90.6)	2.9055 (90.6)
		λ_2	2.1035 (89.8)	2.3166 (88.7)	2.3436 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.7977 (93.1)	2.8322 (93.8)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3545 (88.9)	2.3885 (91.6)	2.3895 (91.6)	2.3895 (91.6)
		λ_1	0.2280 (1.7153)	0.2247 (1.3883)	0.2417 (1.2802)	0.2759 (1.2423)
		λ_2	0.1689 (1.0298)	0.1461 (0.7663)	0.1475 (0.6577)	0.1706 (0.6320)
	MLE	λ_1	3.6133 (79.0)	3.1929 (88.3)	2.9571 (90.7)	2.9142 (92.8)
		λ_2	3.0330 (69.5)	2.6902 (81.5)	2.5017 (87.5)	2.4762 (89.2)
2	BOOT-P	λ_1	4.1914 (77.3)	4.0090 (85.5)	4.0136 (90.7)	4.0654 (89.9)
		λ_2	3.3645 (67.7)	3.2375 (79.9)	3.2395 (86.2)	3.3093 (88.9)
	BOOT-T	λ_1	3.3581 (78.7)	2.9655 (87.4)	2.8422 (91.3)	2.8636 (90.8)
		λ_2	2.6215 (69.4)	2.3683 (80.9)	2.2597 (88.1)	2.3070 (89.0)
	BAYES	λ_1	3.4450 (77.3)	3.0707 (87.1)	2.8612 (92.9)	2.8273 (93.6)
		λ_2	2.8805 (67.8)	2.5721 (80.6)	2.4046 (88.0)	2.3851 (91.0)
		λ_1	0.2199 (1.3079)	0.2804 (1.2382)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1269 (0.6734)	0.1725 (0.6300)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9090 (89.5)	2.9144 (92.6)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4540 (87.9)	2.4755 (89.3)	2.4801 (89.6)	2.4801 (89.6)
3	BOOT-P	λ_1	3.9577 (89.2)	4.0778 (90.5)	4.0734 (91.6)	4.0717 (91.6)
		λ_2	3.2041 (85.2)	3.3183 (88.9)	3.3180 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.6347 (91.1)	2.8461 (90.7)	2.9038 (90.6)	2.9055 (90.6)
		λ_2	2.0913 (88.2)	2.2907 (88.6)	2.3413 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8142 (92.0)	2.8282 (93.7)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3580 (86.2)	2.3848 (91.1)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 2: Here $n = 25, m = 5$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.2825 (1.2347)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1741 (0.6284)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9170 (93.1)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4770 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
1	BOOT-P	λ_1	4.0845 (90.8)	4.0726 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3214 (89.3)	3.3178 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.8529 (90.8)	2.9056 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.2954 (88.9)	2.3428 (88.7)	2.3437 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8308 (93.6)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3864 (91.2)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
		λ_1	0.2370 (1.6967)	0.2279 (1.3813)	0.2414 (1.2803)	0.2759 (1.2423)
		λ_2	0.1712 (1.0103)	0.1482 (0.7633)	0.1483 (0.6561)	0.1715 (0.6314)
	MLE	λ_1	3.6058 (80.1)	3.1899 (88.8)	2.9538 (90.9)	2.9139 (92.8)
		λ_2	3.0232 (70.7)	2.6895 (81.9)	2.5017 (87.7)	2.4777 (89.3)
2	BOOT-P	λ_1	4.2070 (78.3)	4.0052 (85.3)	4.0114 (90.8)	4.0654 (90.0)
		λ_2	3.3690 (68.8)	3.2410 (79.5)	3.2438 (86.4)	3.3097 (88.9)
	BOOT-T	λ_1	3.4596 (79.9)	2.9826 (87.5)	2.8495 (90.8)	2.8646 (90.7)
		λ_2	2.6999 (69.9)	2.3953 (81.5)	2.2670 (88.0)	2.3073 (89.0)
	BAYES	λ_1	3.4403 (78.2)	3.0685 (87.7)	2.8583 (93.0)	2.8271 (93.6)
		λ_2	2.8724 (69.2)	2.5718 (81.3)	2.4047 (88.2)	2.3866 (91.1)
		λ_1	0.2812 (1.2368)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1718 (0.6308)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9159 (92.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4744 (89.3)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
3	BOOT-P	λ_1	4.0860 (90.7)	4.0736 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3216 (89.1)	3.3181 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.8364 (90.4)	2.9047 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.2802 (88.8)	2.3412 (88.7)	2.3437 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8297 (94.2)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3838 (90.8)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 3: Here $n = 25, m = 10$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.0812 (0.3105)	0.1225 (0.2790)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0560 (0.2404)	0.0882 (0.2188)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8802 (90.8)	1.8411 (94.0)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6573 (92.5)	1.6259 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	BOOT-P	λ_1	2.1524 (91.4)	2.1440 (94.0)	2.1319 (94.1)	2.1317 (94.1)
		λ_2	1.8623 (88.6)	1.8597 (91.8)	1.8537 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.7514 (92.6)	1.8218 (93.7)	1.8341 (93.7)	1.8340 (93.7)
		λ_2	1.5029 (89.7)	1.5810 (90.8)	1.5951 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.8460 (92.8)	1.8120 (94.3)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.6194 (91.1)	1.5932 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0753 (0.5199)	0.0778 (0.3620)	0.0984 (0.3136)	0.1181 (0.2821)
		λ_2	0.0400 (0.4258)	0.0497 (0.2902)	0.0733 (0.2355)	0.0828 (0.2208)
	MLE	λ_1	2.5991 (90.3)	2.1705 (91.5)	1.9260 (92.9)	1.8488 (93.7)
		λ_2	2.2059 (85.2)	1.8888 (87.7)	1.7022 (91.6)	1.6304 (92.7)
2	BOOT-P	λ_1	2.7334 (91.7)	2.3661 (92.2)	2.1893 (93.5)	2.1398 (93.9)
		λ_2	2.2943 (85.3)	2.0360 (92.0)	1.8917 (89.8)	1.8541 (91.3)
	BOOT-T	λ_1	2.4446 (91.5)	2.0895 (91.9)	1.8889 (93.4)	1.8255 (93.8)
		λ_2	2.0044 (85.7)	1.7540 (91.0)	1.6192 (89.9)	1.5852 (91.1)
	BAYES	λ_1	2.5100 (90.7)	2.1177 (92.9)	1.8908 (93.4)	1.8191 (94.4)
		λ_2	2.1189 (83.9)	1.8330 (92.0)	1.6633 (92.9)	1.5971 (93.4)
		λ_1	0.0752 (0.3272)	0.1142 (0.2855)	0.1226 (0.2788)	0.1225 (0.2789)
		λ_2	0.0445 (0.2500)	0.0823 (0.2222)	0.0890 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.9918 (90.5)	1.8449 (94.0)	1.8407 (93.9)	1.8406 (93.9)
		λ_2	1.7386 (88.3)	1.6301 (92.3)	1.6261 (92.7)	1.6261 (92.7)
3	BOOT-P	λ_1	2.2036 (92.2)	2.1502 (93.5)	2.1335 (94.1)	2.1317 (94.1)
		λ_2	1.9051 (89.8)	1.8606 (91.3)	1.8547 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.8715 (92.3)	1.8015 (93.6)	1.8326 (93.7)	1.8340 (93.7)
		λ_2	1.5931 (89.6)	1.5596 (91.0)	1.5940 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.9504 (92.7)	1.8152 (94.0)	1.8117 (94.1)	1.8116 (94.1)
		λ_2	1.6939 (90.7)	1.5968 (93.7)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 4: Here $n = 50, m = 5$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
1	BOOT-P	λ_1	4.0723 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3176 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.9049 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3430 (88.7)	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
		λ_1	0.2378 (1.6791)	0.2302 (1.3733)	0.2427 (1.2795)	0.2757 (1.2485)
		λ_2	0.1761 (1.0055)	0.1494 (0.7596)	0.1493 (0.6548)	0.1716 (0.6312)
	MLE	λ_1	3.5945 (80.7)	3.1875 (89.5)	2.9530 (90.8)	2.9136 (92.8)
		λ_2	3.0208 (71.5)	2.6866 (82.2)	2.5029 (87.8)	2.4777 (89.3)
2	BOOT-P	λ_1	4.2231 (78.9)	4.0181 (85.7)	4.0113 (90.4)	4.0653 (90.1)
		λ_2	3.3637 (69.2)	3.2376 (79.8)	3.2436 (86.2)	3.3096 (88.9)
	BOOT-T	λ_1	3.4955 (80.4)	2.9977 (87.6)	2.8515 (90.9)	2.8656 (90.7)
		λ_2	2.7151 (70.4)	2.3951 (81.7)	2.2697 (87.8)	2.3087 (89.0)
	BAYES	λ_1	3.4304 (78.9)	3.0669 (88.0)	2.8577 (92.8)	2.8267 (93.6)
		λ_2	2.8714 (70.1)	2.5696 (81.4)	2.4060 (88.5)	2.3866 (91.0)
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
3	BOOT-P	λ_1	4.0726 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3178 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.9056 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3428 (88.7)	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 5: Here $n = 50, m = 10$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.1226 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0890 (0.2183)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8408 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	BOOT-P	λ_1	2.1406 (94.0)	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8576 (91.7)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.8280 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5886 (91.1)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.8118 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0812 (0.5127)	0.0794 (0.3626)	0.1002 (0.3127)	0.1183 (0.2816)
		λ_2	0.0405 (0.4190)	0.0510 (0.2876)	0.0733 (0.2343)	0.0831 (0.2204)
	MLE	λ_1	2.5875 (90.1)	2.1628 (91.3)	1.9254 (93.4)	1.8488 (93.6)
		λ_2	2.1918 (85.7)	1.8825 (87.8)	1.7004 (91.7)	1.6306 (92.9)
2	BOOT-P	λ_1	2.7158 (92.1)	2.3613 (92.3)	2.1873 (93.3)	2.1396 (93.8)
		λ_2	2.3004 (86.0)	2.0385 (91.6)	1.8924 (90.2)	1.8550 (91.3)
	BOOT-T	λ_1	2.4721 (91.7)	2.0908 (91.5)	1.8900 (93.3)	1.8256 (93.8)
		λ_2	2.0481 (86.1)	1.7653 (90.9)	1.6233 (90.3)	1.5857 (91.1)
	BAYES	λ_1	2.5003 (91.0)	2.1106 (92.4)	1.8904 (93.5)	1.8191 (94.5)
		λ_2	2.1061 (84.8)	1.8274 (91.9)	1.6616 (93.0)	1.5972 (93.6)
		λ_1	0.1225 (0.2790)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0882 (0.2188)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8411 (94.0)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6259 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
3	BOOT-P	λ_1	2.1440 (94.0)	2.1319 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8597 (91.8)	1.8537 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.8218 (93.7)	1.8341 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5810 (90.8)	1.5951 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.8120 (94.3)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5932 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 6: Here $n = 50, m = 15$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.0800 (0.1570)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0336 (0.1174)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4553 (93.5)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2720 (93.1)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
1	BOOT-P	λ_1	1.6128 (93.6)	1.5828 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4223 (93.1)	1.4045 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	BOOT-T	λ_1	1.4274 (94.0)	1.4515 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2578 (93.0)	1.2819 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	BAYES	λ_1	1.4400 (94.0)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2545 (95.9)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)
		λ_1	0.0746 (0.3559)	0.0651 (0.2411)	0.0682 (0.1739)	0.0819 (0.1545)
		λ_2	0.0313 (0.2689)	0.0270 (0.1677)	0.0275 (0.1314)	0.0332 (0.1180)
	MLE	λ_1	2.1969 (87.6)	1.7837 (90.7)	1.5448 (93.3)	1.4626 (94.1)
		λ_2	1.8902 (90.7)	1.5599 (92.3)	1.3513 (92.6)	1.2771 (92.9)
2	BOOT-P	λ_1	2.2113 (91.7)	1.8593 (94.5)	1.6663 (94.0)	1.5974 (94.7)
		λ_2	1.8917 (91.8)	1.6091 (92.0)	1.4683 (94.4)	1.4134 (93.4)
	BOOT-T	λ_1	2.0680 (91.0)	1.7434 (94.6)	1.5346 (93.4)	1.4580 (93.9)
		λ_2	1.7138 (91.4)	1.4864 (91.5)	1.3445 (93.0)	1.2842 (93.3)
	BAYES	λ_1	2.1411 (93.0)	1.7534 (92.2)	1.5258 (93.6)	1.4471 (94.3)
		λ_2	1.8314 (92.3)	1.5262 (93.1)	1.3298 (94.4)	1.2594 (95.2)
		λ_1	0.0686 (0.1630)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0241 (0.1216)	0.0365 (0.1151)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4702 (93.2)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2846 (93.1)	1.2687 (93.6)	1.2687 (93.7)	1.2687 (93.7)
3	BOOT-P	λ_1	1.6215 (93.1)	1.5844 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4262 (93.3)	1.4056 (93.4)	1.4043 (93.5)	1.4043 (93.5)
	BOOT-T	λ_1	1.4336 (94.1)	1.4499 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2587 (93.3)	1.2813 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	BAYES	λ_1	1.4539 (93.7)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2660 (94.9)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 7: Here $n = 100, m = 5$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
1	BOOT-P	λ_1	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
		λ_1	0.2398 (1.6732)	0.2317 (1.3679)	0.2428 (1.2792)	0.2759 (1.2422)
		λ_2	0.1783 (1.0011)	0.1500 (0.7576)	0.1512 (0.6542)	0.1715 (0.6313)
	MLE	λ_1	3.5902 (80.8)	3.1872 (89.8)	2.9520 (90.7)	2.9141 (92.7)
		λ_2	3.0201 (71.6)	2.6851 (82.3)	2.5047 (87.9)	2.4775 (89.3)
2	BOOT-P	λ_1	4.2216 (78.9)	4.0150 (85.8)	4.0098 (90.5)	4.0650 (90.1)
		λ_2	3.3769 (69.5)	3.2425 (79.8)	3.2461 (86.2)	3.3100 (88.9)
	BOOT-T	λ_1	3.4957 (80.4)	2.9995 (87.4)	2.8521 (90.9)	2.8666 (90.7)
		λ_2	2.7357 (71.0)	2.4007 (81.6)	2.2715 (87.9)	2.3092 (89.0)
	BAYES	λ_1	3.4270 (78.9)	3.0669 (88.4)	2.8568 (92.8)	2.8272 (93.6)
		λ_2	2.8711 (70.6)	2.5683 (81.5)	2.4079 (88.5)	2.3865 (91.0)
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
3	BOOT-P	λ_1	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	BOOT-T	λ_1	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	BAYES	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 8: Here $n = 100, m = 10$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	BOOT-P	λ_1	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0833 (0.5097)	0.0795 (0.3643)	0.1005 (0.3126)	0.1182 (0.2817)
		λ_2	0.0418 (0.4155)	0.0512 (0.2890)	0.0729 (0.2342)	0.0830 (0.2204)
	MLE	λ_1	2.5789 (90.0)	2.1578 (91.4)	1.9246 (93.5)	1.8485 (93.6)
		λ_2	2.1851 (86.0)	1.8791 (87.9)	1.6989 (91.7)	1.6303 (92.9)
2	BOOT-P	λ_1	2.7055 (91.9)	2.3619 (92.4)	2.1864 (93.3)	2.1397 (93.9)
		λ_2	2.3012 (86.6)	2.0384 (91.4)	1.8924 (90.3)	1.8552 (91.3)
	BOOT-T	λ_1	2.4757 (91.7)	2.0947 (91.7)	1.8898 (93.3)	1.8258 (93.9)
		λ_2	2.0653 (86.3)	1.7689 (90.7)	1.6233 (90.5)	1.5857 (91.1)
	BAYES	λ_1	2.4928 (91.4)	2.1060 (92.5)	1.8896 (93.7)	1.8189 (94.5)
		λ_2	2.1004 (85.2)	1.8243 (91.8)	1.6603 (93.0)	1.5970 (93.6)
		λ_1	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
3	BOOT-P	λ_1	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	BOOT-T	λ_1	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	BAYES	λ_1	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 9: Here $n = 100, m = 15$.*

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
1	BOOT-P	λ_1	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4044 (93.5)	1.4043 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	BOOT-T	λ_1	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2818 (93.5)	1.2817 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	BAYES	λ_1	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)
		λ_1	0.0739 (0.3503)	0.0675 (0.2395)	0.0678 (0.1735)	0.0819 (0.1545)
		λ_2	0.0343 (0.2643)	0.0264 (0.1671)	0.0275 (0.1315)	0.0332 (0.1180)
	MLE	λ_1	2.1841 (87.9)	1.7816 (90.9)	1.5434 (93.3)	1.4625 (94.2)
		λ_2	1.8860 (90.7)	1.5555 (92.0)	1.3503 (92.4)	1.2770 (92.9)
2	BOOT-P	λ_1	2.2098 (92.0)	1.8572 (94.6)	1.6646 (94.0)	1.5972 (94.7)
		λ_2	1.8977 (91.8)	1.6063 (92.6)	1.4677 (94.4)	1.4136 (93.4)
	BOOT-T	λ_1	2.0764 (91.3)	1.7421 (94.2)	1.5339 (93.3)	1.4576 (93.9)
		λ_2	1.7271 (91.6)	1.4871 (91.7)	1.3446 (93.1)	1.2843 (93.3)
	BAYES	λ_1	2.1292 (92.6)	1.7515 (91.8)	1.5245 (93.7)	1.4469 (94.3)
		λ_2	1.8280 (92.5)	1.5221 (93.0)	1.3289 (94.4)	1.2593 (95.2)
		λ_1	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
3	BOOT-P	λ_1	1.5828 (94.3)	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4045 (93.5)	1.4043 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	BOOT-T	λ_1	1.4515 (93.9)	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2819 (93.5)	1.2817 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	BAYES	λ_1	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.