Estimation of P$[Y < X]$ for Weibull Distribution

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Abstract

This paper deals with the estimation of $R = P[Y < X]$ when X and Y are two independent Weibull distributions with different scale parameters but having the same shape parameter. The maximum likelihood estimator and the approximate maximum likelihood estimator of R are proposed. We obtain the asymptotic distribution of the maximum likelihood estimator of R. Based on the asymptotic distribution, the confidence interval of R can be obtained. We also propose two bootstrap confidence intervals. We consider the Bayesian estimate of R and propose the corresponding credible interval for R. Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a real data set has also been presented for illustrative purposes.

Key Words and Phrases: Approximate maximum likelihood estimator; Bayes estimator; Bootstrap confidence intervals; Credible intervals; Maximum likelihood estimator; Stress-Strength model.

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1 Introduction

In this paper we consider the problem of estimating reliability in the stress strength setting when the strength of a unit or a system, X, has a cumulative distribution function (CDF) $F_1(x)$ and the ultimate stress to which it is subjected, Y has CDF $F_2(y)$. The main aim of this paper is to focus on the inference on $R = P(X > Y)$, where X and Y are independent Weibull distributions with the same shape parameter α but different scale parameters θ_1 and θ_2 respectively.

Note that the estimation of R is very common in the statistical literature. For example, if X is the strength of a system which is subjected to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. This particular problem was considered by McCool [23], but the treatment was not complete. He mainly obtained the MLE of a particular transformation of R and obtained its confidence interval. It may be mentioned that related problems have been widely used in the statistical literature. The maximum likelihood estimator (MLE) of R when X and Y have bivariate exponential distribution has been considered by Awad *et al.* [3]. Church and Harris [7], Downtown [12], Govidarajulu [17], Woodward and Kelley [32] and Owen, Cresswell and Hanson [24] considered the estimation of R when X and Y are normally distributed. Similar problem for multivariate normal distribution has been considered by Gupta and Gupta [18]. Kelley, Kelley and Schucany [20], Sathe and Shah [27], Tong [30, 31] considered the problem of estimating R when X and Y are independent exponential random variables. Constantine and Kerson [9] considered the estimation of R when X and Y are independent gamma random variables. Ahmad, Fakhry and Jaheen [2] and Surles and Padgett [29, 28] considered the estimation of R when X and Y are Burr type X random variables. Recently, Kundu and Gupta [22] and Raqab and Kundu [26] considered this problem when X and

Y are generalized exponential distributions and Burr type X distributions respectively. A comprehensive treatment of the different stress strength models can be found in the recent monograph of Kotz, Lumelskii and Pensky [21]

In this paper we obtain the MLE of R and it is observed that the MLE can be obtained by solving a non-linear equation. We propose a simple iterative scheme to find the MLE of R. Since the MLE can be obtained by using an iterative procedure, it needs an initial guess value to start the iterative process. To avoid this, we propose an approximate maximum likelihood estimator (AMLE) of R, which can be obtained without any iterative process. We obtain the asymptotic distribution of the MLE of R and based on the asymptotic distribution, we construct the asymptotic confidence interval of R. We also recommend two bootstrap confidence intervals of R. Bayes estimator of R and the corresponding credible interval using Gibbs sampling technique have been proposed. Different methods have been compared using Monte Carlo simulations and one data set has been used for illustrative purposes.

We use the following notation. Weibull distribution with the shape parameter α and scale parameter θ will be denoted by $WE(\alpha, \theta)$ and the corresponding density function $f(x; \alpha, \theta)$, for $\alpha > 0$ and $\theta > 0$, is as follows;

$$f(x; \alpha, \theta) = \frac{\alpha}{\theta} x^{\alpha-1} e^{-\frac{x^\alpha}{\theta}}; \qquad x > 0. \tag{1}$$

Moreover, the gamma density function with the shape and scale parameters a and b respectively will be denoted by $GA(a, b)$ and the corresponding density function, $f_{GA}(x; a, b)$, for $a, b > 0$ is

$$f_{GA}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \qquad x > 0. \tag{2}$$

If X follows $GA(a, b)$, then $1/X$ follows inverse gamma and it will be denoted as $IG(a, b)$.

The rest of the paper is organized as follows. In section 2, we derive the MLE of R. It is observed that the MLE can be obtained using an iterative procedure. In section 3, we propose an AMLE of R, which can be obtained explicitly. Different confidence intervals are presented in section 4. Bayesian solutions are presented in section 5. Different proposed methods are compared using Monte Carlo simulation and the results are presented in section 6. One data analysis has been carried out in section 7 and finally the conclusion appears in section 8.

2 Maximum Likelihood Estimator of R

Suppose X and Y follow $WE(\alpha, \theta_1)$ and $WE(\alpha, \theta_2)$ respectively and they are independent. Then it can be easily seen (see McCool [23]) that

$$R = P(Y < X) = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{3}$$

Now to compute the MLE of R, we need to compute the MLE of θ_1 and θ_2. We will see later that to compute the MLEs of θ_1 and θ_2, we need to compute the MLE of α also. Suppose $X_1, \ldots, X_n$ is a random sample from $WE(\alpha, \theta_1)$ and $Y_1, \ldots, Y_m$ is a random sample from $WE(\alpha, \theta_2)$, then the log-likelihood function of the observed sample is

$$\begin{aligned} L(\alpha, \theta_1, \theta_2) &= (m+n)\ln\alpha - n\ln\theta_1 - m\ln\theta_2 + (\alpha - 1)\left[\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j\right] \\ &\quad - \frac{1}{\theta_1}\sum_{i=1}^{n} x_i^{\alpha} - \frac{1}{\theta_2}\sum_{j=1}^{m} y_j^{\alpha}. \end{aligned} \tag{4}$$

The MLEs of α, θ_1 and θ_2, say $\hat{\alpha}_1$, $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively can be obtained as the solutions of

$$\frac{\partial L}{\partial \alpha} = \frac{m+n}{\alpha} + \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j - \frac{1}{\theta_1}\sum_{i=1}^{n} x_i^{\alpha} \ln x_i - \frac{1}{\theta_2}\sum_{j=1}^{m} y_j^{\alpha} \ln y_j = 0, \tag{5}$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{n}{\theta_1} + \frac{1}{\theta_1^2}\sum_{i=1}^{n} x_i^{\alpha} = 0, \tag{6}$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{m}{\theta_2} + \frac{1}{\theta_2^2}\sum_{j=1}^{m} y_j^{\alpha} = 0. \tag{7}$$

From (6) and (7) we obtain

$$\hat{\theta}_1(\alpha) = \frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha} \qquad \text{and} \qquad \hat{\theta}_2(\alpha) = \frac{1}{m}\sum_{j=1}^{m} y_j^{\alpha}. \tag{8}$$

Putting the values of $\hat{\theta}_1(\alpha)$ and $\hat{\theta}_2(\alpha)$ in (5), we obtain

$$\frac{m+n}{\alpha} + \frac{1}{\alpha}\left[\sum_{i=1}^{n} \ln x_i^{\alpha} + \sum_{j=1}^{m} \ln y_j^{\alpha}\right] - \frac{\sum_{i=1}^{n} x_i^{\alpha} \ln x_i}{\frac{1}{n}\sum_{k=1}^{n} x_k^{\alpha}} - \frac{\sum_{j=1}^{m} y_j^{\alpha} \ln y_j}{\frac{1}{m}\sum_{k=1}^{m} y_k^{\alpha}} = 0. \tag{9}$$

Therefore, $\hat{\alpha}$ can be obtained as a solution of the non-linear equation of the form

$$h(\alpha) = \alpha, \tag{10}$$

where

$$h(\alpha) = \frac{m+n+\sum_{i=1}^{n} \ln x_i^{\alpha} + \sum_{j=1}^{m} \ln y_j^{\alpha}}{\frac{\sum_{i=1}^{n} x_i^{\alpha} \ln x_i}{\frac{1}{n}\sum_{k=1}^{n} x_k^{\alpha}} + \frac{\sum_{j=1}^{m} y_j^{\alpha} \ln y_j}{\frac{1}{m}\sum_{k=1}^{m} y_k^{\alpha}}}. \tag{11}$$

Since $\hat{\alpha}$ is a fixed point solution of the non-linear equation (10), therefore, it can be obtained by using a simple iterative procedure as follows;

$$h(\alpha_{(j)}) = \alpha_{(j+1)}, \tag{12}$$

where $\alpha_{(j)}$ is the j-th iterate of $\hat{\alpha}$. The iterative procedure should be stopped when the absolute difference between $\alpha_{(j)}$ and $\alpha_{(j+1)}$ is sufficiently small. Note that the iterative process (12) is not the only iterative process for estimating α. Since the profile log-likelihood function of α, *i.e.*, $L(\alpha, \hat{\theta}_1(\alpha), \hat{\theta}_2(\alpha))$ is a unimodal concave function of α, therefore finding $\hat{\alpha}$ is not a very difficult problem. Most of the iterative process should work well.

Once we obtain $\hat{\alpha}$, the scale parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ can be obtained from (8) as $\hat{\theta}_1(\hat{\alpha})$ and $\hat{\theta}_2(\hat{\alpha})$ respectively. The MLE of R becomes

$$\hat{R} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}}}{\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}} + \frac{1}{m}\sum_{j=1}^{m} y_j^{\hat{\alpha}}}. \tag{13}$$

3 Approximate Maximum Likelihood Estimator of R

In the previous section, we propose the MLE of R and it is observed that the MLE of R can be obtained by an iterative procedure. In this section, we propose an estimator of R which can be obtained with out any iterative technique.

It is known that if the random variable U has a Weibull distribution, with the shape and scale parameters as α and θ respectively, *i.e.* the PDF of U is given by (1), then $V = \ln U$, has the extreme value distribution with PDF as

$$f_V(x;\mu,\sigma) = \frac{1}{\sigma} e^{\left\{\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}\right\}}, \qquad -\infty < x < \infty, \tag{14}$$

where $\mu = \frac{1}{\alpha}\ln\theta$ and $\sigma = \frac{1}{\alpha}$. The density function as described by (14) is known as the density function of an extreme value distribution, with location and scale parameters as μ and σ respectively.

Note that (1) and (14) are equivalent models in the sense, the procedures developed under one model can be easily used for the other model also. Although they are equivalent models, sometimes it is easier to work with the model (14) than (1), because in the model (14), the two parameters μ and σ appear as the location and scale parameters respectively. For $\mu = 0$ and $\sigma = 1$, the model (14) is known as the standard extreme value distribution and it has the following probability density function;

$$f_W(w;0,1) = e^{w-e^w}; \qquad -\infty < w < \infty. \tag{15}$$

We use the following notation for this subsection. $X_{1:n} < \ldots < X_{n:n}$ and $Y_{1:m} < \ldots < Y_{m:m}$ are the ordered X_i's and Y_i's respectively. $U_{i:n} = \ln X_{i:n}$, $V_{i:m} = \ln Y_{i:m}$, $\mu_1 = \frac{1}{\alpha}\ln\theta_1$, $\mu_2 = \frac{1}{\alpha}\ln\theta_2$, $\sigma = \frac{1}{\alpha}$, $a_{i:n} = \frac{U_{i:n}-\mu_1}{\sigma}$, $b_{i:m} = \frac{V_{i:m}-\mu_2}{\sigma}$.

The log-likelihood function of the observed data $U_{1:n} < \ldots < U_{n:n}$ and $V_{1:m} < \ldots < V_{m:m}$ without the constant is

$$L(\mu_1,\mu_2,\sigma) = -(m+n)\ln\sigma + \sum_{i=1}^{n}\ln g(a_{i:n}) + \sum_{i=1}^{m}\ln g(b_{i:m}), \tag{16}$$

where $g(x) = e^{x-e^x}$. Taking derivatives of (16) with respect to μ_1, μ_2 and σ we obtain the normal equations as;

$$\frac{\partial L}{\partial \mu_1} = -\frac{1}{\sigma}\sum_{i=1}^{n}\frac{g'(a_{i:n})}{g(a_{i:n})} = 0, \tag{17}$$

$$\frac{\partial L}{\partial \mu_2} = -\frac{1}{\sigma}\sum_{i=1}^{m}\frac{g'(b_{i:m})}{g(b_{i:m})} = 0, \tag{18}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{m+n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n}\frac{g'(a_{i:n})}{g(a_{i:n})}a_{i:n} - \frac{1}{\sigma}\sum_{i=1}^{m}\frac{g'(b_{i:m})}{g(b_{i:m})}b_{i:m} = 0. \tag{19}$$

Expanding $\frac{g'(a_{i:n})}{g(a_{i:n})}$ and $\frac{g'(b_{i:m})}{g(b_{i:m})}$ in Taylor series around the points $G^{-1}(p_i) = \ln(-\ln q_i) = c_i$ (say) and $G^{-1}(\bar{p}_i) = \ln(-\ln \bar{q}_i) = d_i$ (say) respectively, where $G^{-1}(x) = 1 - e^{-e^x}$, $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$, $\bar{p}_i = \frac{i}{m+1}$, $\bar{q}_i = 1 - \bar{p}_i$. See for reasoning David [10] and Arnold and Balakrishnan [1].

Note that

$$\begin{aligned}
\frac{g'(a_{i:n})}{g(a_{i:n})} &\approx \alpha_i - \beta_i a_{i:n}, \qquad i = 1, \ldots, n, \\
\frac{g'(b_{i:m})}{g(b_{i:m})} &\approx \bar{\alpha}_i - \bar{\beta}_i b_{i:m}, \qquad i = 1, \ldots, m,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_i &= \frac{g'(c_i)}{g(c_i)} - c_i\left[\frac{g''(c_i)}{g(c_i)} - \left(\frac{g'(c_i)}{g(c_i)}\right)^2\right] = 1 + \ln q_i(1 - \ln(-\ln q_i)), \\
\beta_i &= \left[\left(\frac{g'(c_i)}{g(c_i)}\right)^2 - \frac{g''(c_i)}{g(c_i)}\right] = -\ln q_i, \\
\bar{\alpha}_i &= \frac{g'(d_i)}{g(d_i)} - d_i\left[\frac{g''(d_i)}{g(d_i)} - \left(\frac{g'(d_i)}{g(d_i)}\right)^2\right] = 1 + \ln \bar{q}_i(1 - \ln(-\ln \bar{q}_i)), \\
\bar{\beta}_i &= \left[\left(\frac{g'(d_i)}{g(d_i)}\right)^2 - \frac{g''(d_i)}{g(d_i)}\right] = -\ln \bar{q}_i.
\end{aligned}$$

Therefore, (17), (18) and (19) can be approximated as

$$\frac{\partial L}{\partial \mu_1} \approx \frac{\partial L^*}{\partial \mu_1} = -\frac{1}{\sigma}\sum_{i=1}^{n}(\alpha_i - \beta_i a_{i:n}) = 0, \tag{20}$$

$$\frac{\partial L}{\partial \mu_2} \approx \frac{\partial L^*}{\partial \mu_2} = -\frac{1}{\sigma}\sum_{i=1}^{m}\left(\bar{\alpha}_i - \bar{\beta}_i b_{i:m}\right) = 0, \tag{21}$$

$$\frac{\partial L}{\partial \sigma} \approx \frac{\partial L^*}{\partial \sigma} = -\frac{1}{\sigma}\left[(m+n) + \sum_{i=1}^{n}(\alpha_i - \beta_i a_{i:n})\, a_{i:n} + \sum_{i=1}^{m}\left(\bar{\alpha}_i - \bar{\beta}_i b_{i:m}\right) b_{i:m}\right] = 0. \tag{22}$$

If we denote $\tilde{\mu_1}$, $\tilde{\mu_2}$ and $\tilde{\sigma} > 0$ as the solutions of (20), (21) and (22), then observe that

$$\tilde{\mu_1} = A_1 - B_1\tilde{\sigma}, \tag{23}$$

$$\tilde{\mu_2} = A_2 - B_2\tilde{\sigma}, \tag{24}$$

where

$$A_1 = \frac{\sum_{i=1}^{n} \beta_i u_{i:n}}{\sum_{i=1}^{n} \beta_i}, \quad A_2 = \frac{\sum_{i=1}^{m} \bar{\beta}_i v_{i:m}}{\sum_{i=1}^{m} \bar{\beta}_i}, \quad B_1 = \frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} \beta_i}, \quad B_2 = \frac{\sum_{i=1}^{m} \bar{\alpha}_i}{\sum_{i=1}^{m} \bar{\beta}_i}.$$

Moreover, $\tilde{\sigma} > 0$ can be obtained as the unique solution of the quadratic equation

$$C\sigma^2 + D\sigma - E = 0,$$

here

$$\begin{aligned}
C &= (m+n) + B_1 \sum_{i=1}^{n} \alpha_i + B_2 \sum_{i=1}^{m} \bar{\alpha}_i - B_1^2 \sum_{i=1}^{n} \beta_i - B_2^2 \sum_{i=1}^{m} \bar{\beta}_i = m+n, \\
D &= \sum_{i=1}^{n} \alpha_i (u_{i:n} - A_1) - 2B_1 \sum_{i=1}^{n} \beta_i (u_{i:n} - A_1) + \sum_{i=1}^{m} \bar{\alpha}_i (v_{i:m} - A_2) - 2B_2 \sum_{i=1}^{m} \bar{\beta}_i (v_{i:m} - A_2), \\
E &= \sum_{i=1}^{n} \beta_i \left(u_{i:n} - A_1\right)^2 + \sum_{i=1}^{m} \bar{\beta}_i \left(v_{i:m} - A_2\right)^2 > 0.
\end{aligned}$$

Therefore,

$$\tilde{\sigma} = \frac{-D + \sqrt{D^2 + 4E(m+n)}}{2(m+n)}. \tag{25}$$

Once $\tilde{\sigma}$ is obtained, $\tilde{\mu_1}$ and $\tilde{\mu_2}$ can be obtained using (23) and (24). Therefore, we use the approximate MLEs of α, θ_1, θ_2 and R as

$$\tilde{\alpha} = \frac{1}{\tilde{\sigma}}, \qquad \tilde{\theta}_1 = e^{\frac{1}{\tilde{\sigma}}(A_1 - B_1\tilde{\sigma})}, \qquad \tilde{\theta}_2 = e^{\frac{1}{\tilde{\sigma}}(A_2 - B_2\tilde{\sigma})}, \qquad \text{and} \qquad \tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1 + \tilde{\theta}_2}.$$

4 Asymptotic Distribution of $\hat{R}$ and Different Confidence Intervals

In this section, first we obtain the asymptotic distribution of $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\theta}_1, \hat{\theta}_2)$ and then derive the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, we obtain the asymptotic confidence interval of R. Let us denote the Fisher information matrix of $\boldsymbol{\theta} = (\alpha, \theta_1, \theta_2)$ as $\mathbf{I}(\boldsymbol{\theta}) = ((I_{ij}(\boldsymbol{\theta})))$ for $i, j = 1, 2, 3$. Therefore,

$$\mathbf{I}(\boldsymbol{\theta}) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \theta_1}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \theta_2}\right) \\ E\left(\frac{\partial^2 L}{\partial \theta_1 \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 L}{\partial \theta_2 \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \theta_2 \partial \theta_1}\right) & E\left(\frac{\partial^2 L}{\partial \theta_2^2}\right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad \text{(say)}.$$

Moreover

$$\begin{aligned}
E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) &= -I_{11} = -\frac{m+n}{\alpha^2} - \frac{1}{\theta_1} E \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 - \frac{1}{\theta_2} E \sum_{j=1}^{m} y_j^{\alpha} (\ln y_j)^2 \\
&= -\frac{1}{\alpha^2}\left[(m+n)(1+\Gamma''(2)) + 2\Gamma'(2)(n \ln \theta_1 + m \ln \theta_2) + n(\ln \theta_1)^2 + m(\ln \theta_2)^2\right] \\
E\left(\frac{\partial^2 L}{\partial \theta_1^2}\right) &= -\frac{n}{\theta_1^2} = -I_{22}, \qquad E\left(\frac{\partial^2 L}{\partial \theta_2^2}\right) = -\frac{m}{\theta_2^2} = -I_{33}, \\
E\left(\frac{\partial^2 L}{\partial \alpha \partial \theta_1}\right) &= E\left(\frac{\partial^2 L}{\partial \theta_1 \partial \alpha}\right) = \frac{1}{\theta_1^2} E \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i) = \frac{n}{\alpha \theta_1}\left[\ln \theta_1 + \Gamma'(2)\right] = -I_{12} = -I_{21} \\
E\left(\frac{\partial^2 L}{\partial \alpha \partial \theta_2}\right) &= E\left(\frac{\partial^2 L}{\partial \theta_2 \partial \alpha}\right) = \frac{1}{\theta_2^2} E \sum_{j=1}^{m} y_j^{\alpha} (\ln y_j) = \frac{m}{\alpha \theta_2}\left[\ln \theta_2 + \Gamma'(2)\right] = -I_{13} = -I_{31}.
\end{aligned}$$

Since the two-parameter Weibull family satisfies all the regularity conditions, therefore we have the following result.

THEOREM 1: As $n \to \infty$ and $m \to \infty$ and $\frac{n}{m} \to p$, then

$$\left(\sqrt{m}(\hat{\alpha} - \alpha), \sqrt{n}(\hat{\theta}_1 - \theta_1), \sqrt{n}(\hat{\theta}_2 - \theta_2)\right) \xrightarrow{d} N_3\left(\mathbf{0}, \mathbf{A}^{-1}(\alpha, \theta_1, \theta_2\right),$$

where

$$\mathbf{A}(\alpha, \theta_1, \theta_2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$$

and

$$a_{11} = \frac{1}{\alpha^2}\left[(1+p)(1+\Gamma''(2)) + 2\Gamma'(2)(p \ln \theta_1 + \ln \theta_2) + p(\ln \theta_1)^2 + (\ln \theta_2)^2\right] = \lim_{n,m \to \infty} \frac{I_{11}}{m},$$

$$a_{22} = \frac{1}{\theta_1^2} = \lim_{n,m \to \infty} \frac{1}{n} I_{22}, \qquad a_{12} = -\frac{\sqrt{p}}{\alpha \theta_1}\left[\ln \theta_1 + \Gamma'(2)\right] = a_{21} = \lim_{n,m \to \infty} \frac{\sqrt{p}}{n} I_{12},$$

$$a_{33} = \frac{1}{p\theta_2^2} = -\lim_{n,m\to\infty} \frac{1}{n} I_{33}, \qquad a_{13} = -\frac{1}{\sqrt{p}\alpha\theta_2}\left[\ln\theta_2 + \Gamma'(2)\right] = a_{31} = \lim_{n,m\to\infty} \frac{1}{m\sqrt{p}} I_{13}.$$

PROOF: The proof follows by expanding the derivative of the log-likelihood function using Taylor series and using the Central limit theorem.

Now we have the main result:

THEOREM 2: As $m \to \infty$ and $n \to \infty$, so that $\frac{n}{m} \to p$, then

$$\sqrt{m}\left(\hat{R} - R\right) \to N(0, B),$$

where $B = \frac{c}{u}\mathbf{b}^T\mathbf{G}\mathbf{b}$, and

$$u = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}, \qquad c = \left(1 + \frac{\theta_2}{\theta_1}\right)^{-4}\left(\frac{\theta_2}{\theta_1}\right)^2 = \frac{\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4},$$

$$\mathbf{b}^T = \left(0, \frac{1}{\theta_1}, -\frac{1}{\theta_2}\right),$$

$$\mathbf{G} = \begin{bmatrix} a_{22}a_{33} & -a_{21}a_{33} & -a_{22}a_{31} \\ -a_{21}a_{13} & a_{11}a_{33} - a_{13}^2 & a_{12}a_{31} \\ -a_{22}a_{31} & a_{12}a_{31} & a_{11}a_{22} - a_{12}^2 \end{bmatrix}.$$

PROOF: It follows from Theorem 1.

REMARK 1: The asymptotic distribution of $\hat{R}$ is independent of α.

REMARK 2: In Theorem 1, the normalizing constants $\sqrt{m}$ and $\sqrt{n}$ can be interchanged and the necessary changes are required in the corresponding dispersion matrix.

REMARK 3: Theorem 2 can be used to construct asymptotic confidence interval of R. To compute the confidence interval of R, the variance B needs to be estimated. We recommend to use the empirical Fisher information matrix and the MLE estimates of α, θ_1 and θ_2 to estimate B, which is very convenient.

REMARK 4: Theoretically it is very difficult to compute the asymptotic distributions of the AMLEs of the unknown parameters. In our simulation experiments we observe that for large sample sizes the MLEs and AMLEs match very well. Although we could not prove it theoretically but we believe that the MLEs and AMLEs are asymptotically equivalent. It needs further investigation.

It is observed that the asymptotic confidence intervals do not perform very well for small sample sizes. We propose the following two bootstrap confidence intervals mainly for small sample sizes, which might be computationally very demanding for large samples.

4.1 Bootstrap Confidence Intervals

In this subsection, we propose to use two confidence intervals based on the non-parametric bootstrap methods; (i) percentile bootstrap method (we call it from now on as Boot-p) based on the idea of Efron (1982), (ii) bootstrap-t method (we refer it as Boot-t from now on) based on the idea of Hall (1988). We illustrate briefly how to estimate confidence intervals of R using both methods.

Boot-p Method:

Step 1: Generate a bootstrap sample of size n, $\{x_1^*, \ldots x_n^*\}$ from $\{x_1, \ldots x_n\}$ and generate a bootstrap sample of size m, $\{y_1^*, \ldots, y_m^*\}$ from $\{y_1, \ldots, y_m\}$. Based on $\{x_1^*, \ldots x_n^*\}$ and $\{y_1^*, \ldots, y_m^*\}$ compute the bootstrap estimate of R, say $\hat{R}^*$, using (13).

Step 2: Repeat step 1, NBOOT times.

Step 3: Let $G(x) = P(\hat{R}^* \leq x)$, be the cumulative distribution function of $\hat{R}^*$. Define $\hat{R}_{Boot-p}(x) = G^{-1}(x)$ for a given x. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-p}(\frac{\gamma}{2}), \hat{R}_{Boot-p}(1-\frac{\gamma}{2})\right). \tag{26}$$

Boot-t Method:

Step 1: From the sample $\{x_1, \ldots x_n\}$ and $\{y_1, \ldots, y_m\}$ compute $\hat{R}$.

Step 2: Same as in Boot-p method first generate bootstrap sample $\{x_1^*, \ldots x_n^*\}$, $\{y_1^*, \ldots, y_m^*\}$ and then compute $\hat{R}^*$, the bootstrap estimate of R. Also compute the following statistic:

$$T^* = \frac{\sqrt{m}(\hat{R}^* - \hat{R})}{\sqrt{V(\hat{R}^*)}}.$$

Compute $V(\hat{R}^*)$ using Remark 2.

Step 3: Repeat step 2, NBOOT times.

Step 4: From the NBOOT T* values obtained, determine the upper and lower bound of the $100(1-\gamma)\%$ confidence interval of R as follows: Let $H(x) = P(T^* \leq x)$ be the cumulative distribution function of T^*. For a given x, define

$$\hat{R}_{Boot-t} = \hat{R} + m^{-\frac{1}{2}}\sqrt{V(\hat{R})}H^{-1}(x).$$

Here also, $V(\hat{R})$ can be computed as mentioned in Remark 2. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-t}(\frac{\gamma}{2}), \hat{R}_{Boot-t}(1-\frac{\gamma}{2})\right).$$

5 Bayesian Inference on R

In this section, we consider Bayesian inference on R. We mainly obtain the Bayes estimate of R under the squared error loss and the corresponding credible interval by Gibbs sampling technique.

5.1 The Prior and the Posterior Distributions

Following the approach of Berger and Sun [5], it is assumed that the prior density of θ_j is inverted $IG(a_j, b_j)$. Therefore, the prior density function of θ_j, for $j = 1, 2$, becomes

$$\pi_{1j}(\theta_j) = \pi_{1j}(\theta_j|a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)}\theta_j^{-(1+a_j)}e^{-\frac{b_j}{\theta_j}}, \tag{27}$$

where $a_j > 0$ and $b_j > 0$. Moreover, it is assumed that θ_1 and θ_2 are independent. The joint density function of θ_1 and θ_2 is

$$\pi_1(\theta_1, \theta_2) = \prod_{j=1}^{2} \pi_{1j}(\theta_j).$$

No specific form of prior, $\pi_2(\alpha)$ on α is assumed here. It is only assumed that the support of $\pi_2(\alpha)$ is $(0, \infty)$ and it is independent of θ_1 and θ_2. Based on the above assumptions, we have the likelihood function of the observed data as

$$l(data|\alpha, \theta_1, \theta_2) = \alpha^{m+n}\theta_1^{-n}\theta_2^{-m}\prod_{i=1}^{n} x_i^{\alpha-1}\prod_{j=1}^{m} y_j^{\alpha-1}e^{-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i^{\alpha}}e^{-\frac{1}{\theta_2}\sum_{j=1}^{m} y_j^{\alpha}}. \tag{28}$$

Therefore, the joint density of the *data*, α, θ_1 and θ_2 can be obtained as

$$l(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1, \theta_2) \times \pi_2(\alpha). \tag{29}$$

Based on (29), we obtain the joint posterior density of α, θ_1 and θ_2 given the data as

$$l(\alpha, \theta_1, \theta_2|data) = \frac{l(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1, \theta_2) \times \pi_2(\alpha)}{\int_0^\infty \int_0^\infty \int_0^\infty l(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1, \theta_2) \times \pi_2(\alpha)d\alpha d\theta_1 d\theta_2}. \tag{30}$$

Since (30) can not be obtained analytically, we adopt the Gibbs sampling technique to compute the Bayes estimate of R and the corresponding credible interval of R. To perform the Gibbs sampling, in this case we further assume that $\pi_2(\alpha)$ is log-concave. It may be mentioned that most common densities are log-concave, for example gamma, Weibull, normal, log-normal all have log-concave densities. Another natural concern is how to choose (a_j, b_j) for the conditional prior distribution of θ_j. Two methods as suggested by Berger and Sun

[5], (a) specify the first two marginal moments for θ_j; (b) specify the two marginal quantiles of θ_j, and solve for (a_j, b_j), can be adopted in this set up also. It is not pursued here. Based on the above priors, we provide different posterior densities.

Theorem 3: The conditional density of θ_1, given α and *data* is

$$\pi_1(\theta_1|\alpha, data) = \frac{(b_1 + \sum_{i=1}^n x_i^\alpha)^{a_1+n}}{\Gamma(a_1+n)} \theta_1^{-(1+a_1+n)} e^{-\frac{1}{\theta_1}(b_1 + \sum_{i=1}^n x_i^\alpha)}, \tag{31}$$

similarly, the conditional density of θ_2, given α and *data* is

$$\pi_2(\theta_2|\alpha, data) = \frac{(b_2 + \sum_{j=1}^m y_j^\alpha)^{a_2+m}}{\Gamma(a_2+m)} \theta_2^{-(1+a_2+m)} e^{-\frac{1}{\theta_2}(b_2 + \sum_{j=1}^m y_j^\alpha)}, \tag{32}$$

and they are independent.

Proof: See in the appendix.

Theorem 4: The conditional density of α given the *data* is log-concave.

Proof: See in the appendix.

We use the method proposed by Devroye [11] to generate sample from a log-concave density function for Gibbs sampling purposes. The details are provided below.

5.2 Bayes Estimate and Credible Interval

We use the idea of Geman and Geman [16] to generate sample from the conditional posterior densities. We adopt the following scheme:

- Step 1: Generate α_1, from the log-concave density $l(.|data)$, as given in (35).
- Step 2: Generate $\theta_{1,1}$ and $\theta_{2,1}$ from $\pi_1(.|\alpha_1, data)$ and $\pi_2(.|\alpha_1, data)$ as given in (31) and (32) respectively. Calculate R.
- Step 3: Repeat Steps 1 and 2, M times.

Now the approximate posterior mean and posterior variance of R becomes

$$\hat{E}(R|data) = \frac{1}{M}\sum_{k=1}^{M}\frac{\theta_{1,k}}{\theta_{1,k}+\theta_{2,k}}$$

and

$$\hat{V}(R|data) = \frac{1}{M}\sum_{k=1}^{M}\left(\frac{\theta_{1,k}}{\theta_{1,k}+\theta_{2,k}} - (\hat{E}(R|data))\right)^2,$$

respectively. Based on M, R values, using the method proposed by Chen and Shao [6] the approximate highest posterior density (HPD) credible of R can be easily constructed.

6 Numerical Experiments and Discussions

In this section we mainly present some Monte Carlo simulation results, which we performed to observe the behavior of the different methods for different sample sizes and for different parameter values. We mainly compare the performances of the MLEs, AMLEs and the Bayes estimates with respect to the squared error loss function in terms of biases and mean squares errors (MSEs). We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLEs and two different bootstrap confidence intervals in terms of the average confidence lengths and coverage percentages. Since it is observed that for large sample sizes the MLEs and AMLEs match very well, we also compute the confidence intervals based on the asymptotic distributions of MLEs but replacing the MLEs by AMLEs. For comparison purposes we also compute the HPD credible intervals of R, under the non-informative priors. We assume that $a_1 = a_2 = b_1 = b_2 = 0$ and α has a gamma prior, namely GA(0, 1). Note that the priors on θ_1, θ_2 and α are all non-proper.

All computations are performed at the Indian Institute of Technology Kanpur using Pentium IV processor. All the programs are written in FORTRAN-77 and we used the random deviate generator RAN2, described in Press *et al.* [25]. We consider the following sample sizes; (m, n) = (5,5), (10,10), (15,15), (20,20), (25,25), (50,50) and the following parameter values; $\theta_1 = 1.0$ and $\theta_2 = 1.0$, 1.5, 2.0, 2.5 and 3.0. Since the asymptotic distributions are independent of α, we kept it constant at $\alpha = 1.5$. All the results are based on 1000 replications. In our simulations experiments for both the bootstrap methods, when m or n is less that 20, we computed the confidence intervals based on 250 resampling and for m or n greater than 20 (namely for 25 or 50), the results are based on 200 resampling. The Bayes estimates and the corresponding credible intervals are based on 10,000 sampling, namely M = 10,000. The nominal level for the confidence intervals or the credible intervals is 0.95 in each case.

From the sample, we estimate α using the iterative algorithm (12). We start the iterative process with the initial estimate 1 and the iterative process stops when the difference between the two consecutive iterates are less than 10^{-6}. Once we estimate α, we obtain the MLE of R using (13). We obtain the AMLE of R using the method described in section 3. We compute the Bayes estimate of R as suggested in section 5. We report the average biases and MSEs of the MLEs, AMLEs and Bayes estimators over 1000 replications. The results are reported in Table 1.

We compute the 95% confidence intervals based on the asymptotic distributions of the MLEs and replacing the true parameter values by the MLEs and AMLEs respectively. We further compute Boot-p, Boot-t confidence intervals and the HPD credible intervals. We obtain the average confidence/ credible lengths and the corresponding coverage percentages. The results are reported in Table 2

Some of the points are quite clear from this experiment. Even for small sample sizes, the performance of the MLEs and AMLEs are quite satisfactory in terms of biases and MSEs. Interestingly the MSEs and biases of the MLEs and AMLEs are very close for small sizes and for large sample sizes they coincide. The performance of the Bayes estimators also

Table 1: Biases and MSEs of the MLEs, AMLEs and Bayes estimators of R, when $\theta_1 = 1.0$ and for different values of θ_2. *

S.S.	$\theta_2 = 1.0$	$\theta_2 = 1.5$	$\theta_2 = 2.0$	$\theta_2 = 2.5$	$\theta_2 = 3.0$
(5,5)	-0.0009(0.0328) -0.0010(0.0314) -0.0384(0.0268)	-0.0107(0.0296) -0.0079(0.0284) 0.0836(0.0269)	-0.0162(0.0256) -0.0117(0.0247) 0.1101(0.0215)	-0.0190(0.0221) -0.0135(0.0214) 0.0149(0.0206)	-0.0203(0.0191) -0.0142(0.0187) -0.0123(0.0167)
(10,10)	-0.0071(0.0144) -0.0069(0.0142) 0.0817(0.0145)	-0.0127(0.0134) -0.0111(0.0132) 0.2014(0.0137)	-0.0153(0.0118) -0.0129(0.0117) -0.1402(0.0109)	-0.0164(0.0103) -0.0135(0.0102) -0.0269(0.0091)	-0.0167(0.0090) -0.0136(0.0089) 0.0124(0.0085)
(15,15)	-0.0060(0.0095) -0.0058(0.0094) -0.0331(0.0093)	-0.0095(0.0089) -0.0086(0.0088) -0.1280(0.0082)	-0.0112(0.0079) -0.0098(0.0078) 0.1402(0.0074)	-0.0119(0.0069) -0.0102(0.0068) 0.0489(0.0065)	-0.0120(0.0060) -0.0102(0.0060) -0.0785(0.0056)
(20,20)	-0.0012(0.0067) -0.0011(0.0066) 0.0141(0.0069)	-0.0043(0.0063) -0.0037(0.0063) -0.0495(0.0065)	-0.0060(0.0056) -0.0051(0.0056) 0.0810(0.0054)	-0.0068(0.0049) -0.0057(0.0049) 0.0237(0.0047)	-0.0072(0.0043) -0.0060(0.0043) 0.0630(0.0043)
(25,25)	-0.0006(0.0053) -0.0005(0.0053) -0.0220(0.0050)	-0.0034(0.0049) -0.0029(0.0049) -0.0496(0.0045)	-0.0049(0.0044) -0.0042(0.0044) 0.0365(0.0042)	-0.0057(0.0038) -0.0048(0.0039) -0.0870(0.0041)	-0.0061(0.0034) -0.0051(0.0034) 0.0155(0.0030)
(50,50)	0.0003(0.0029) 0.0003(0.0029) 0.0559(0.0026)	-0.0011(0.0027) -0.0011(0.0027) 0.0062(0.0023)	-0.0019(0.0024) -0.0019(0.0024) 0.0093(0.0020)	-0.0024(0.0021) -0.0024(0.0021) 0.0015(0.0019)	-0.0026(0.0019) -0.0026(0.0019) -0.0478(0.0017)

* In each cell the first row represents the average bias of the MLE and the corresponding MSE is reported within bracket. Similarly the second and third rows represent the results for AMLE and Bayes estimators respectively.

Table 2: Average confidence/ credible lengths and coverage percentages for different estimators, when $\theta_1 = 1.0$ and for different values of θ_2. *

S.S.	$\theta_2 = 1.0$	$\theta_2 = 1.5$	$\theta_2 = 2.0$	$\theta_2 = 2.5$	$\theta_2 = 3.0$
(5,5)	0.5540(82%)	0.5337(83%)	0.4947(82%)	0.4537(81%)	0.4162(80%)
	0.6269(87%)	0.6072(87%)	0.5700(86%)	0.5314(85%)	0.4955(84%)
	0.6289(86%)	0.6410(93%)	0.6107(94%)	0.5687(94%)	0.5230(94%)
	0.7481(83%)	0.7580(88%)	0.6916(93%)	0.6179(92%)	0.5415(89%)
	0.5391(92%)	0.5194(91%)	0.4965(92%)	0.4748(91%)	0.4488(93%)
(10,10)	0.4211(90%)	0.4058(90%)	0.3802(88%)	0.3541(88%)	0.3299(87%)
	0.4431(92%)	0.4275(92%)	0.4013(91%)	0.3747(90%)	0.3501(89%)
	0.4402(85%)	0.4482(93%)	0.4257(95%)	0.3946(96%)	0.3635(96%)
	0.5182(80%)	0.5413(92%)	0.5140(95%)	0.4684(95&)	0.4221(94%)
	0.4073(92%)	0.3961(92%)	0.0.3773(93%)	0.0.3566(95%)	0.0.3376(95%)
(15,15)	0.3490(92%)	0.3371(90%)	0.3170(90%)	0.2963(89%)	0.2772(89%)
	0.3604(92%)	0.3483(91%)	0.3279(91%)	0.3068(90%)	0.2873(90%)
	0.3864(97%)	0.3686(97%)	0.3407(97%)	0.3133(97%)	0.2887(97%)
	0.4432(97%)	0.4341(96%)	0.4070(96%)	0.3736(95%)	0.3400(94%)
	0.3410(94%)	0.3318(94%)	0.3148(94%)	0.2935(93%)	0.2774(95%)
(20,20)	0.3048(92%)	0.2954(91%)	0.2785(91%)	0.2609(91%)	0.2446(90%)
	0.3120(94%)	0.3024(93%)	0.2854(93%)	0.2676(92%)	0.2509(91%)
	0.3183(94%)	0.3104(95%)	0.2921(96%)	0.2730(97%)	0.2548(97%)
	0.3486(94%)	0.3378(95%)	0.3217(96%)	0.3059(95%)	0.2894(94%)
	0.2989(93%)	0.2907(94%)	0.2747(95%)	0.2579(94%)	0.2403(93%)
(25,25)	0.2737(93%)	0.2655(93%)	0.2506(93%)	0.2350(93%)	0.2203(92%)
	0.2787(93%)	0.2704(94%)	0.2553(94%)	0.2395(93%)	0.2247(93%)
	0.2867(96%)	0.2778(97%)	0.2603(97%)	0.2433(97%)	0.2280(97%)
	0.3073(95%)	0.2997(96%)	0.2865(96%)	0.2733(95%)	0.2605(95%)
	0.2703(95%)	0.2634(95%)	0.2510(95%)	0.2381(95%)	0.2251(96%)
(50,50)	0.1949(93%)	0.1893(93%)	0.1790(92%)	0.1682(92%)	0.1580(92%)
	0.1964(93%)	0.1907(93%)	0.1804(93%)	0.1696(92%)	0.1593(92%)
	0.1983(94%)	0.1932(94%)	0.1842(94%)	0.1746(94%)	0.1657(93%)
	0.1934(95%)	0.1884(95%)	0.1799(96%)	0.1700(95%)	0.1609(95%)
	0.1928(94%)	0.1871(94%)	0.1770(94%)	0.1662(94%)	0.1661(95%)

* In each cell the first, second, third and fourth rows represent the average confidence lengths based on the asymptotic distributions of the MLEs, and replacing the true parameter values by MLEs and AMLEs, Boot-p method and Boot-t method respectively. The corresponding coverage percentages are reported within brackets. Similarly the fifth rows represent the average credible lengths and the corresponding coverage percentages in brackets for the Bayes estimators.

are quite satisfactory. Interestingly in most of the cases considered here, the MSEs of the Bayes estimators are smaller than the MSEs of the MLEs or AMLEs, although the biases are significantly higher. It is observed that when m, n increase then MSEs of all the estimators decrease.

The confidence intervals based on the asymptotic distributions of the MLEs do not work very well when m and n are very small. The coverage percentages are smaller than the nominal level. For m and n greater then 25, the asymptotic results work quite well. It is quite interesting that in constructing the confidence interval based on the asymptotic distribution of the MLEs if we replace the MLEs by the AMLEs, the performances are marginally better in terms of the coverage percentages. Non-parametric bootstrap methods work quite well even for small sizes. In our experimental results, it is observed that Boot-p confidence intervals perform better than the Boot-t confidence intervals at least for small sizes. One point we should mention that for bootstrap methods the performances depend on the number of resampling. The performances change when the number of resampling changes. The performances of the Bayes estimators also quite satisfactory even for small sample sizes. The Bayes credible intervals also maintain the nominal coverage percentages and the average credible lengths are also smaller than the average confidence intervals in general. Based on all these, we recommend to use Bayes credible intervals or the Boot-p confidence intervals for constructing confidence intervals for small m and n and for m and n greater than 25, asymptotic results can be used.

7 Data Analysis

In this section we present a data analysis of the strength data reported by Badar and Priest [4]. The authors are thankful to Professor J.G. Surles for providing the data The data represent the strength data measured in GPA, for single carbon fibers and impregnated

1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of $1, 10, 20$, and 50mm. Impregnated tows of 1000 fibers were tested at gauge lengths of $20, 50, 150$ and 300 mm. For illustrative purpose, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes $n = 69$ and $m = 63$, respectively. We are presenting the data below for convenience.

Data Set 1

1.312,1.314,1.479,1.552,1.700,1.803,1.861,1.865,1.944,1.958,1.966,1.997,2.006,2.021,2.027,2.055,
2.063,2.098,2.140,2.179,2.224,2.240,2.253,2.270,2.272,2.274,2.301,2.301,2.359,2.382,2.382,2.426,
2.434,2.435,2.478,2.490,2.511,2.514,2.535,2.554,2.566,2.570,2.586,2.629,2.633,2.642,2.648,2.684,
2.697,2.726,2.770,2.773,2.800,2.809,2.818,2.821,2.848,2.880,2.954,3.012,3.067,3.084,3.090,3.096,
3.128,3.233,3.433,3.585,3.585.

Data Set 2

1.901,2.132,2.203,2.228,2.257,2.350,2.361,2.396,2.397,2.445,2.454,2.474,2.518,2.522,2.525,2.532,
2.575,2.614,2.616,2.618,2.624,2.659,2.675,2.738,2.740,2.856,2.917,2.928,2.937,2.937,2.977,2.996,
3.030,3.125,3.139,3.145,3.220,3.223,3.235,3.243,3.264,3.272,3.294,3.332,3.346,3.377,3.408,3.435,
3.493,3.501,3.537,3.554,3.562,3.628,3.852,3.871,3.886,3.971,4.024,4.027,4.225,4.395,5.020.

Surles and Padgett [28], [29] and Raqab and Kundu [26] observed that generalized Rayleigh distribution works quite well for these strength data. We are analyzing the data by subtracting 0.75 from both the data sets. We fit the Weibull models to the two data sets separately. We present the estimated shape and scale parameters, log-likelihood values, Kolmogorov-Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution functions and corresponding p values in Table 3. We also present the observed frequencies and the expected frequencies based on the fitted models in the Tables 4 and 5. For data sets 1 and 2, the chi-square values are 1.220 and 0.861 respectively. Therefore, it is clear that Weibull model fits quite well to both the data sets. Since the two

Table 3: Shape parameter, scale parameter, log-likelihood, K-S and p values of the fitted Weibull models to Data sets 1 and 2.

Data Set	Shape Parameter	Scale Parameter	Log-Likelihood	K-S	p value
1	3.8428	11.3142	-48.8703	0.0461	0.9985
2	3.9090	38.5449	-60.1524	0.0800	0.8154

Table 4: Observed frequencies and expected frequencies for modified data set 1 when fitting the Weibull model

Intervals	Observed Frequencies	Expected Frequencies
0.00 - 1.78	5	6.06
1.78 - 2.22	15	15.97
2.22 - 2.68	27	23.85
2.68 - 3.13	18	17.36
3.13 - ∞	4	5.76

shape parameters are not very different, assuming the two shape parameters are equal we estimated the parameters and the results are reported in Table 6.

Based on the log-likelihood values it is clear we can not reject the null hypothesis that the two shape parameters are equal. Therefore, the assumption that the two shape parameters are equal is justified for these data sets. The K-S values and the corresponding p values indicate that the Weibull models with equal shape parameters fit reasonably well to the transformed data sets.

The MLE and AMLE of R become 0.7624, 0.7608 and the corresponding 95% confidence intervals become (0.6968,0.8279) and (0.6946,0.8270) respectively. We also obtain the 95% Boot-p and Boot-t confidence intervals as (0.6983, 0.8285) and (0.7001, 0.8415) respectively.

For Bayesian estimate of R, we use non-proper priors on θ_1 and θ_2, *i.e.*, $a_1 = a_2 = b_1 =$

$b_2 = 0$. Moreover for α we assume non-proper gamma priors, namely $GA(0,1)$, see (2). Based on the above priors we obtain 0.7636 as the Bayes estimate of R, under the squared error loss function. We also compute 95% highest posterior density (HPD) credible interval of R, based on the idea of Chen and Shao [6] and it is (0.7014, 0.8244).

8 Conclusions

In this paper we consider estimation of $R = P(Y < X)$ by different methods when X and Y are independent Weibull random variables with equal shape parameters. Note that our method can be easily extended for estimating $P(X_1 < \ldots < X_k)$ where X_i's are independent $WE(\alpha, \theta_i)$ random variables, and also estimating $P(Y < cX)$ when c is known. One important question we have not addressed in this paper, namely the estimation of R when the shape parameters are different. The problem becomes quite complicated because in that case R does have any explicit form. We might need some approximation to provide the estimate of R in that case. Work is in progress, it will be reported elsewhere.

Table 5: Observed frequencies and expected frequencies for modified data set 2 when fitting the Weibull model

Intervals	Observed Frequencies	Expected Frequencies
0.00 - 2.53	15	13.99
2.53 - 3.15	21	20.71
3.15 - 3.77	18	19.32
3.77 - 4.39	7	7.82
4.39 - ∞	2	1.16

Table 6: Shape parameter, scale parameter, log-likelihood, K-S and p values of the fitted Weibull models to Data sets 1 and 2 assuming that the two shape parameters are equal.

Data Set	Shape Parameter	Scale Parameter	Log-Likelihood	K-S	p value
1	3.8770	11.6064	-48.8747	0.0464	0.9984
2	3.8770	37.2333	-60.1566	0.0767	0.8528

Appendix

Proof of Theorem 3: Note that (29) can be written as

$$\begin{aligned} l(data, \alpha, \theta_1, \theta_2) \;&\propto\; \alpha^{m+n}\theta_1^{-(1+a_1+n)}\theta_2^{-(1+a_2+m)}\prod_{i=1}^{n} x_i^{\alpha-1}\prod_{j=1}^{m} y_j^{\alpha-1} \\ &\times e^{-\frac{1}{\theta_1}\left(b_1+\sum_{i=1}^{n} x_i^{\alpha}\right)} \times e^{-\frac{1}{\theta_2}\left(b_2+\sum_{j=1}^{m} y_j^{\alpha}\right)}\pi_2(\alpha). \end{aligned} \tag{33}$$

Therefore, the conditional distribution of θ_1 and θ_2 given α and $data$ is

$$l(\theta_1, \theta_2|\alpha, data) \propto \theta_1^{-(1+a_1+n)}\theta_2^{-(1+a_2+m)} \times e^{-\frac{1}{\theta_1}\left(b_1+\sum_{i=1}^{n} x_i^{\alpha}\right)} \times e^{-\frac{1}{\theta_2}\left(b_2+\sum_{j=1}^{m} y_j^{\alpha}\right)}. \tag{34}$$

Therefore, the result follows.

Proof of Theorem 4: Note that from (33), the conditional density of α given the $data$ is

$$l(\alpha|data) \propto \pi_2(\alpha)\alpha^{m+n}\prod_{i=1}^{n} x_i^{\alpha-1}\prod_{j=1}^{m} y_j^{\alpha-1} \times \frac{1}{(b_1+\sum_{i=1}^{n} x_i^{\alpha})^{a_1+n}} \times \frac{1}{\left(b_2+\sum_{j=1}^{m} y_j^{\alpha}\right)^{a_2+m}}. \tag{35}$$

Therefore,

$$\begin{aligned} \ln(l(\alpha|data)) \;&=\; k + \ln\pi_2(\alpha) + (m+n)\ln\alpha + (\alpha-1)\left(\sum_{i=1}^{n}\ln x_i + \sum_{j=1}^{m}\ln y_j\right) \\ &\quad -(a_1+n)\ln\left(\sum_{i=1}^{n} x_i^{\alpha} + b_1\right) - (a_2+m)\ln\left(\sum_{j=1}^{m} y_j^{\alpha} + b_2\right). \end{aligned} \tag{36}$$

Now to prove that (36) is log-concave, observe the following inequality. Suppose

$$g(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} + b_1, \qquad \text{therefore,}$$

$$g'(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \ln x_i \quad \text{and} \quad g''(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2.$$

Since

$$\left(\sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 \right) \times \left(\sum_{i=1}^{n} x_i^{\alpha} \right) - \left(\sum_{i=1}^{n} x_i^{\alpha} \ln x_i \right)^2 = \sum_{1 \le i < j \le n} x_i^{\alpha} x_j^{\alpha} \left(\ln x_i - \ln x_j \right)^2 \ge 0,$$

therefore for $b_1 \ge 0$,

$$g''(\alpha) g(\alpha) \ge (g'(\alpha))^2. \tag{37}$$

From (37) it follows that if $\pi_2(\alpha)$ is log-concave, then for $a_1, b_1, a_2, b_2 \ge 0$, the second derivative of the right hand side of (36) with respect to α is negative. Therefore, $l(\alpha|data)$ is log-concave.

References

[1] Arnold, B. and Balakrishnan, N. (1989) *Relations, Bounds and Approximations for Order Statistics*, Lecture Notes in Statistics, No. 53, Springer Verlag.

[2] Ahmad, K.E., Fakhry, M.E. and Jaheen, Z.F. (1997), "Empirical Bayes estimation of $P(Y < X)$ and characterizations of the Burr-type X model", *Journal of Statistical Planning and Inference*, vol. 64, 297 - 308.

[3] Awad, A. M., Azzam, M.M. and Hamadan, M.A. (1981), "Some inference results in $P(Y < X)$ in the bivariate exponential model", *Communications in Statistics - Theory and Methods*, vol. 10, 2515 - 2524.

[4] Badar, M.G. and Priest, A.M. (1982), "Statistical aspects of fiber and bundle strength in hybrid composites", *Progress in Science and Engineering Composites*, Hayashi, T., Kawata, K. and Umekawa, S. (eds.), ICCM-IV, Tokyo, 1129-1136.

[5] Berger, J.O. and Sun, D. (1993), "Bayesian analysis for the Poly-Weibull distribution", *Journal of the American Statistical Association*, vol. 88, 1412-1418.

[6] Chen, M.H. and Shao, Q.M. (1999), "Monte Carlo estimation of Bayesian Credible and HPD intervals", *Journal of Computational and Graphical Statistics*, vol. 8, 69 - 92.

[7] Church, J.D. and Harris, B. (1970), "The estimation of reliability from stress strength relationships", *Technometrics*,vol. 12, 49 - 54.

[8] Congdon, P. (2001), *Bayesian Statistical Modeling*, John Wiley, New York.

[9] Constantine, K. and Karson, M. (1986), "The estimation of $P(Y < X)$ in gamma case", *Communications in Statistics - Computations and Simulations*, vol. 15, 365 - 388.

[10] David, H.A. (1981), *Order Statistics*, John Wiley and Sons, New York.

[11] Devroye, L. (1984), "A simple algorithm for generating random variates with a log-concave density", *Computing*, vol. 33, 247-257.

[12] Downtown, F. (1973), "The estimation of $P(X > Y)$ in the normal case", *Technometrics*, vol. 15, 551 - 558.

[13] Durham, S.D. and Padgett, W.J. (1997), "Cumulative damage models for system strength with applications to carbon fibers and composites", *Technometrics*, vol. 39, 34-44.

[14] Efron, B. (1982), *The Jackknife, the Bootstrap and other Resampling Plans*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 38, SIAM, Philadelphia, PA.

[15] Ferguson, T.S. (1976), *Mathematical Statistics; A Decision Theoretic Approach*, Academic Press, New York.

[16] Geman, S. and Geman, A. (1984), "Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 6, 721-740.

[17] Govidarajulu, Z. (1967), "Two sided confidence limits for $P(X > Y)$ based on normal samples of X and Y", *Sankhya B*, vol. 29, 35 - 40.

[18] Gupta, R.D. and Gupta, R.C. (1990), "Estimation of $P(a'x > b'y)$ in the multivarite normal case", *Statistics*, vol. 1, 91 - 97.

[19] Hall, P. (1988), "Theoretical comparison of bootstrap confidence intervals", *Annals of Statistics*, vol. 16, 927-953.

[20] Kelley, G.D., Kelley, J.A. and Schucany, W.R. (1976), "Efficient estimation of $P(Y < X)$ in the exponential case", *Technometrics*, 359 - 360.

[21] Kotz, S., Lumelskii, Y. and Pensky, M., (2003), *The Stress-Strength Model and its Generalizations*, World Scientific Press, Singapore.

[22] Kundu, D. and Gupta, R.D. (2005), "Estimation of $P(Y < X)$ for the generalized exponential distribution", *Metrika*, (to appear).

[23] McCool, J.I. (1991). Inference on $P(Y < X)$ in the Weibull case. *Communications in Statistics - Simulation and Computations*, 20, 129 - 148.

[24] Owen, D.B., Craswell, K.J. and Hanson, D.L. (1977), "Non-parametric upper confidence bounds for $P(Y < X)$ and confidence limits for $P(Y < X)$ when X and Y are normal", *Journal of the American Statistical Association*, vol. 59, 906 - 924.

[25] Press, W.H., Teukolsky, S.A., Vellerling, W.T., Flannery, B.P. (1992), *Numerical recipes in FORTRAN, the art of scientific computing*, 2nd. Edition, Cambridge University Press, Cambridge.

[26] Raqab, M.Z. and Kundu, D. (2005), "Comparison of Different Estimators of P[$Y < X$] for a Scaled Burr Type X Distribution", *Communications in Statistics - Simulation and Computation* (to appear).

[27] Sathe, Y.S. and Shah, S.P. (1981), "On estimation $P(Y < X)$ for the exponential distribution", *Communications in Statistics - Theory and Methods*, vol. 10, 39 - 47.

[28] Surles, J.G. and Padgett, W.J. (1998), "Inference for $P(Y < X)$ in the Burr type X model", *Journal of Applied Statistical Sciences*, vol. 7, 225 - 238.

[29] Surles, J.G. and Padgett, W.J. (2001), "Inference for reliability and stress-strength for a scaled Burr-type X distribution", *Lifetime Data Analysis*, vol. 7, 187 - 200.

[30] Tong, H. (1975), " A note on the estimation of $P(Y < X)$ in the exponential case. *Technometrics*", vol. 16, 625. (Errata 1975, vol. 17, 395.)

[31] Tong, H. (1977), "On the estimation of $P(Y < X)$ for exponential families", *IEEE Transactions on Reliability*, vol. 26, 54 - 56.

[32] Woodward, W.A. and Kelley, G.D. (1977), "Minimum variance unbiased estimation of $P(Y < X)$ in the normal case", *Technometrics*, vol. 19, 95 - 98.