Estimation of $P[Y < X]$ for Weibull Distribution

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Abstract

This paper deals with the estimation of $R = P[Y < X]$ when $X$ and $Y$ are two independent Weibull distributions with different scale parameters but having the same shape parameter. The maximum likelihood estimator and the approximate maximum likelihood estimator of $R$ are proposed. We obtain the asymptotic distribution of the maximum likelihood estimator of $R$. Based on the asymptotic distribution, the confidence interval of $R$ can be obtained. We also propose two bootstrap confidence intervals. We consider the Bayesian estimate of $R$ and propose the corresponding credible interval for $R$. Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a real data set has also been presented for illustrative purposes.

Key Words and Phrases: Approximate maximum likelihood estimator; Bayes estimator; Bootstrap confidence intervals; Credible intervals; Maximum likelihood estimator; Stress-Strength model.

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1 Introduction

In this paper we consider the problem of estimating reliability in the stress strength setting when the strength of a unit or a system, $X$, has a cumulative distribution function (CDF) $F_1(x)$ and the ultimate stress to which it is subjected, $Y$ has CDF $F_2(y)$. The main aim of this paper is to focus on the inference on $R = P(X > Y)$, where $X$ and $Y$ are independent Weibull distributions with the same shape parameter $\alpha$ but different scale parameters $\theta_1$ and $\theta_2$ respectively.

Note that the estimation of $R$ is very common in the statistical literature. For example, if $X$ is the strength of a system which is subjected to a stress $Y$, then $R$ is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. This particular problem was considered by McCool [23], but the treatment was not complete. He mainly obtained the MLE of a particular transformation of $R$ and obtained its confidence interval. It may be mentioned that related problems have been widely used in the statistical literature. The maximum likelihood estimator (MLE) of $R$ when $X$ and $Y$ have bivariate exponential distribution has been considered by Awad et al. [3]. Church and Harris [7], Downtown [12], Govidarajulu [17], Woodward and Kelley [32] and Owen, Cresswell and Hanson [24] considered the estimation of $R$ when $X$ and $Y$ are normally distributed. Similar problem for multivariate normal distribution has been considered by Gupta and Gupta [18]. Kelley, Kelley and Schucany [20], Sathe and Shah [27], Tong [30, 31] considered the problem of estimating $R$ when $X$ and $Y$ are independent exponential random variables. Constantine and Kerson [9] considered the estimation of $R$ when $X$ and $Y$ are independent gamma random variables. Ahmad, Fakhry and Jaheen [2] and Surles and Padgett [29, 28] considered the estimation of $R$ when $X$ and $Y$ are Burr type X random variables. Recently, Kundu and Gupta [22] and Raqab and Kundu [26] considered this problem when $X$ and
Y are generalized exponential distributions and Burr type X distributions respectively. A comprehensive treatment of the different stress strength models can be found in the recent monograph of Kotz, Lumelskii and Pensky [21]

In this paper we obtain the MLE of $R$ and it is observed that the MLE can be obtained by solving a non-linear equation. We propose a simple iterative scheme to find the MLE of $R$. Since the MLE can be obtained by using an iterative procedure, it needs an initial guess value to start the iterative process. To avoid this, we propose an approximate maximum likelihood estimator (AMLE) of $R$, which can be obtained without any iterative process. We obtain the asymptotic distribution of the MLE of $R$ and based on the asymptotic distribution, we construct the asymptotic confidence interval of $R$. We also recommend two bootstrap confidence intervals of $R$. Bayes estimator of $R$ and the corresponding credible interval using Gibbs sampling technique have been proposed. Different methods have been compared using Monte Carlo simulations and one data set has been used for illustrative purposes.

We use the following notation. Weibull distribution with the shape parameter $\alpha$ and scale parameter $\theta$ will be denoted by $WE(\alpha, \theta)$ and the corresponding density function $f(x; \alpha, \theta)$, for $\alpha > 0$ and $\theta > 0$, is as follows;

$$f(x; \alpha, \theta) = \frac{\alpha}{\theta} x^{\alpha-1} e^{-x^{\alpha}/\theta}; \quad x > 0. \quad (1)$$

Moreover, the gamma density function with the shape and scale parameters $a$ and $b$ respectively will be denoted by $GA(a, b)$ and the corresponding density function, $f_{GA}(x; a, b)$, for $a, b > 0$ is

$$f_{GA}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \quad x > 0. \quad (2)$$

If $X$ follows $GA(a, b)$, then $1/X$ follows inverse gamma and it will be denoted as $IG(a, b)$.

The rest of the paper is organized as follows. In section 2, we derive the MLE of $R$. It is observed that the MLE can be obtained using an iterative procedure. In section 3,
we propose an AMLE of $R$, which can be obtained explicitly. Different confidence intervals are presented in section 4. Bayesian solutions are presented in section 5. Different proposed methods are compared using Monte Carlo simulation and the results are presented in section 6. One data analysis has been carried out in section 7 and finally the conclusion appears in section 8.

2 Maximum Likelihood Estimator of $R$

Suppose $X$ and $Y$ follow $WE(\alpha, \theta_1)$ and $WE(\alpha, \theta_2)$ respectively and they are independent. Then it can be easily seen (see McCool [23]) that

$$R = P(Y < X) = \frac{\theta_1}{\theta_1 + \theta_2}. \quad (3)$$

Now to compute the MLE of $R$, we need to compute the MLE of $\theta_1$ and $\theta_2$. We will see later that to compute the MLEs of $\theta_1$ and $\theta_2$, we need to compute the MLE of $\alpha$ also. Suppose $X_1, \ldots, X_n$ is a random sample from $WE(\alpha, \theta_1)$ and $Y_1, \ldots, Y_m$ is a random sample from $WE(\alpha, \theta_2)$, then the log-likelihood function of the observed sample is

$$L(\alpha, \theta_1, \theta_2) = (m + n) \ln \alpha - n \ln \theta_1 - m \ln \theta_2 + (\alpha - 1) \left[ \frac{1}{n} \sum_{i=1}^{n} \ln x_i + \frac{1}{m} \sum_{j=1}^{m} \ln y_j \right] - \frac{1}{\theta_1} \sum_{i=1}^{n} x_i^\alpha - \frac{1}{\theta_2} \sum_{j=1}^{m} y_j^\alpha. \quad (4)$$

The MLEs of $\alpha$, $\theta_1$ and $\theta_2$, say $\hat{\alpha}$, $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively can be obtained as the solutions of

$$\frac{\partial L}{\partial \alpha} = \frac{m + n}{\alpha} + \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j - \frac{1}{\theta_1} \sum_{i=1}^{n} x_i^\alpha \ln x_i - \frac{1}{\theta_2} \sum_{j=1}^{m} y_j^\alpha \ln y_j = 0, \quad (5)$$

$$\frac{\partial L}{\partial \theta_1} = - n \frac{\alpha}{\theta_1} + \frac{1}{\theta_1^2} \sum_{i=1}^{n} x_i^\alpha = 0, \quad (6)$$

$$\frac{\partial L}{\partial \theta_2} = - m \frac{\alpha}{\theta_2} + \frac{1}{\theta_2^2} \sum_{j=1}^{m} y_j^\alpha = 0. \quad (7)$$

From (6) and (7) we obtain

$$\hat{\theta}_1(\alpha) = \frac{1}{n} \sum_{i=1}^{n} x_i^\alpha \quad \text{and} \quad \hat{\theta}_2(\alpha) = \frac{1}{m} \sum_{j=1}^{m} y_j^\alpha. \quad (8)$$
Putting the values of $\hat{\theta}_1(\alpha)$ and $\hat{\theta}_2(\alpha)$ in (5), we obtain

$$m + n + \frac{1}{\alpha} \left[ \sum_{i=1}^{n} \ln x_i^\alpha + \sum_{j=1}^{m} \ln y_j^\alpha \right] - \frac{\sum_{i=1}^{n} x_i^\alpha \ln x_i}{n} - \frac{\sum_{j=1}^{m} y_j^\alpha \ln y_j}{m} = 0. \quad (9)$$

Therefore, $\hat{\alpha}$ can be obtained as a solution of the non-linear equation of the form

$$h(\alpha) = \alpha, \quad (10)$$

where

$$h(\alpha) = \frac{m + n + \sum_{i=1}^{n} \ln x_i^\alpha + \sum_{j=1}^{m} \ln y_j^\alpha}{\sum_{i=1}^{n} x_i^\alpha \ln x_i + \sum_{j=1}^{m} y_j^\alpha \ln y_j}. \quad (11)$$

Since $\hat{\alpha}$ is a fixed point solution of the non-linear equation (10), therefore, it can be obtained by using a simple iterative procedure as follows;

$$h(\alpha_{(j)}) = \alpha_{(j+1)}, \quad (12)$$

where $\alpha_{(j)}$ is the j-th iterate of $\hat{\alpha}$. The iterative procedure should be stopped when the absolute difference between $\alpha_{(j)}$ and $\alpha_{(j+1)}$ is sufficiently small. Note that the iterative process (12) is not the only iterative process for estimating $\alpha$. Since the profile log-likelihood function of $\alpha$, i.e., $L(\alpha, \hat{\theta}_1(\alpha), \hat{\theta}_2(\alpha))$ is a unimodal concave function of $\alpha$, therefore finding $\hat{\alpha}$ is not a very difficult problem. Most of the iterative process should work well.

Once we obtain $\hat{\alpha}$, the scale parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ can be obtained from (8) as $\hat{\theta}_1(\hat{\alpha})$ and $\hat{\theta}_2(\hat{\alpha})$ respectively. The MLE of $R$ becomes

$$\hat{R} = \frac{\frac{1}{n} \sum_{i=1}^{n} x_i^\hat{\alpha}}{\frac{1}{n} \sum_{i=1}^{n} x_i^\hat{\alpha} + \frac{1}{m} \sum_{j=1}^{m} y_j^\hat{\alpha}}. \quad (13)$$

### 3 Approximate Maximum Likelihood Estimator of $R$

In the previous section, we propose the MLE of $R$ and it is observed that the MLE of $R$ can be obtained by an iterative procedure. In this section, we propose an estimator of $R$ which can be obtained with out any iterative technique.
It is known that if the random variable \( U \) has a Weibull distribution, with the shape and scale parameters as \( \alpha \) and \( \theta \) respectively, i.e. the PDF of \( U \) is given by (1), then \( V = \ln U \), has the extreme value distribution with PDF as

\[
f_V(x; \mu, \sigma) = \frac{1}{\sigma} e^{\{\frac{x-\mu}{\sigma}\} - e^{\frac{x-\mu}{\sigma}}}, \quad -\infty < x < \infty,
\]

where \( \mu = \frac{1}{\alpha} \ln \theta \) and \( \sigma = \frac{1}{\alpha} \). The density function as described by (14) is known as the density function of an extreme value distribution, with location and scale parameters as \( \mu \) and \( \sigma \) respectively.

Note that (1) and (14) are equivalent models in the sense, the procedures developed under one model can be easily used for the other model also. Although they are equivalent models, sometimes it is easier to work with the model (14) than (1), because in the model (14), the two parameters \( \mu \) and \( \sigma \) appear as the location and scale parameters respectively. For \( \mu = 0 \) and \( \sigma = 1 \), the model (14) is known as the standard extreme value distribution and it has the following probability density function;

\[
f_W(w; 0, 1) = e^{w-e^w}; \quad -\infty < w < \infty.
\]

We use the following notation for this subsection. \( X_{1:n} < \ldots < X_{n:n} \) and \( Y_{1:m} < \ldots < Y_{m:m} \) are the ordered \( X_i \)'s and \( Y_i \)'s respectively. \( U_{i:n} = \ln X_{i:n}, V_{i:m} = \ln Y_{i:m}, \mu_1 = \frac{1}{\alpha} \ln \theta_1, \mu_2 = \frac{1}{\alpha} \ln \theta_2, \sigma = \frac{1}{\alpha}, a_{i:n} = \frac{U_{i:n} - \mu_1}{\sigma}, b_{i:m} = \frac{V_{i:m} - \mu_2}{\sigma} \).

The log-likelihood function of the observed data \( U_{1:n} < \ldots < U_{n:n} \) and \( V_{1:m} < \ldots < V_{m:m} \) without the constant is

\[
L(\mu_1, \mu_2, \sigma) = -(m + n) \ln \sigma + \sum_{i=1}^{n} \ln g(a_{i:n}) + \sum_{i=1}^{m} \ln g(b_{i:m}),
\]

where \( g(x) = e^{x-e^x} \). Taking derivatives of (16) with respect to \( \mu_1, \mu_2 \) and \( \sigma \) we obtain the normal equations as;

\[
\frac{\partial L}{\partial \mu_1} = -\frac{1}{\sigma} \sum_{i=1}^{n} \frac{g'(a_{i:n})}{g(a_{i:n})} = 0,
\]
\[ \frac{\partial L}{\partial \mu_2} = -\frac{1}{\sigma} \sum_{i=1}^{m} \frac{g'(b_{i;m})}{g(b_{i;m})} = 0, \]  
(18)

\[ \frac{\partial L}{\partial \sigma} = -m + n - \frac{1}{\sigma} \sum_{i=1}^{n} \frac{g'(a_{i;n})}{g(a_{i;n})} a_{i;n} - \frac{1}{\sigma} \sum_{i=1}^{m} \frac{g'(b_{i;m})}{g(b_{i;m})} b_{i;m} = 0. \]  
(19)

Expanding \( \frac{g'(a_{i;n})}{g(a_{i;n})} \) and \( \frac{g'(b_{i;m})}{g(b_{i;m})} \) in Taylor series around the points \( G^{-1}(p_i) = \ln(-\ln q_i) = c_i \) (say) and \( G^{-1}(\tilde{p}_i) = \ln(-\ln \tilde{q}_i) = d_i \) (say) respectively, where \( G^{-1}(x) = 1 - e^{-e^x}, \_p_i = \frac{i}{n+1}, \_q_i = 1 - p_i, \_\tilde{p}_i = \frac{i}{m+1}, \_\tilde{q}_i = 1 - \tilde{p}_i \). See for reasoning David [10] and Arnold and Balakrishnan [1].

Note that

\[ \frac{g'(a_{i;n})}{g(a_{i;n})} \approx \alpha_i - \beta_i a_{i;n}, \quad i = 1, \ldots, n, \]
\[ \frac{g'(b_{i;m})}{g(b_{i;m})} \approx \bar{\alpha}_i - \bar{\beta}_i b_{i;m}, \quad i = 1, \ldots, m, \]

where

\[ \alpha_i = \frac{g'(c_i)}{g(c_i)} - c_i \left[ \frac{g''(c_i)}{g(c_i)} - \left( \frac{g'(c_i)}{g(c_i)} \right)^2 \right] = 1 + \ln q_i (1 - \ln(-\ln q_i)), \]
\[ \beta_i = \left[ \left( \frac{g'(c_i)}{g(c_i)} \right)^2 - \frac{g''(c_i)}{g(c_i)} \right] = -\ln q_i, \]
\[ \bar{\alpha}_i = \frac{g'(d_i)}{g(d_i)} - d_i \left[ \frac{g''(d_i)}{g(d_i)} - \left( \frac{g'(d_i)}{g(d_i)} \right)^2 \right] = 1 + \ln \tilde{q}_i (1 - \ln(-\ln \tilde{q}_i)), \]
\[ \bar{\beta}_i = \left[ \left( \frac{g'(d_i)}{g(d_i)} \right)^2 - \frac{g''(d_i)}{g(d_i)} \right] = -\ln \tilde{q}_i. \]

Therefore, (17), (18) and (19) can be approximated as

\[ \frac{\partial L}{\partial \mu_1} \approx \frac{\partial L^*}{\partial \mu_1} = -\frac{1}{\sigma} \sum_{i=1}^{n} (\alpha_i - \beta_i a_{i;n}) = 0, \]  
(20)

\[ \frac{\partial L}{\partial \mu_2} \approx \frac{\partial L^*}{\partial \mu_2} = -\frac{1}{\sigma} \sum_{i=1}^{m} (\bar{\alpha}_i - \bar{\beta}_i b_{i;m}) = 0, \]  
(21)

\[ \frac{\partial L}{\partial \sigma} \approx \frac{\partial L^*}{\partial \sigma} = -\frac{1}{\sigma} \left[ (m + n) + \sum_{i=1}^{n} (\alpha_i - \beta_i a_{i;n}) a_{i;n} + \sum_{i=1}^{m} (\bar{\alpha}_i - \bar{\beta}_i b_{i;m}) b_{i;m} \right] = 0. \]  
(22)

If we denote \( \bar{\mu}_1, \bar{\mu}_2 \) and \( \bar{\sigma} > 0 \) as the solutions of (20), (21) and (22), then observe that

\[ \bar{\mu}_1 = A_1 - B_1 \bar{\sigma}, \]  
(23)
\[ \tilde{\mu}_2 = A_2 - B_2 \tilde{\sigma}, \quad (24) \]

where
\[
A_1 = \frac{\sum_{i=1}^{n} \beta_i u_{i:n}}{\sum_{i=1}^{n} \beta_i}, \quad A_2 = \frac{\sum_{i=1}^{m} \beta_i v_{i:m}}{\sum_{i=1}^{m} \beta_i}, \quad B_1 = \frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} \beta_i}, \quad B_2 = \frac{\sum_{i=1}^{m} \tilde{\alpha}_i}{\sum_{i=1}^{m} \beta_i}.
\]

Moreover, \( \tilde{\sigma} > 0 \) can be obtained as the unique solution of the quadratic equation
\[
C\tilde{\sigma}^2 + D\tilde{\sigma} - E = 0,
\]

here
\[
C = (m + n) + B_1 \sum_{i=1}^{n} \alpha_i + B_2 \sum_{i=1}^{m} \tilde{\alpha}_i - B_1^2 \sum_{i=1}^{n} \beta_i - B_2^2 \sum_{i=1}^{m} \beta_i = m + n,
\]
\[
D = \sum_{i=1}^{n} \alpha_i (u_{i:n} - A_1) - 2B_1 \sum_{i=1}^{n} \beta_i (u_{i:n} - A_1) + \sum_{i=1}^{m} \tilde{\alpha}_i (v_{i:m} - A_2) - 2B_2 \sum_{i=1}^{m} \tilde{\beta}_i (v_{i:m} - A_2),
\]
\[
E = \sum_{i=1}^{n} \beta_i (u_{i:n} - A_1)^2 + \sum_{i=1}^{m} \tilde{\beta}_i (v_{i:m} - A_2)^2 > 0.
\]

Therefore,
\[
\tilde{\sigma} = \frac{-D + \sqrt{D^2 + 4E(m + n)}}{2(m + n)}. \quad (25)
\]

Once \( \tilde{\sigma} \) is obtained, \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) can be obtained using (23) and (24). Therefore, we use the approximate MLEs of \( \alpha, \theta_1, \theta_2 \) and \( R \) as
\[
\tilde{\alpha} = \frac{1}{\sigma}, \quad \tilde{\theta}_1 = e^{\frac{1}{\tilde{\sigma}}(A_1 - B_1 \tilde{\sigma})}, \quad \tilde{\theta}_2 = e^{\frac{1}{\tilde{\sigma}}(A_2 - B_2 \tilde{\sigma})}, \quad \text{and} \quad \tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1 + \tilde{\theta}_2}.
\]

4 Asymptotic Distribution of \( \hat{R} \) and Different Confidence Intervals

In this section, first we obtain the asymptotic distribution of \( \hat{\theta} = (\hat{\alpha}, \hat{\theta}_1, \hat{\theta}_2) \) and then derive the asymptotic distribution of \( \hat{R} \). Based on the asymptotic distribution of \( \hat{R} \), we obtain
the asymptotic confidence interval of $R$. Let us denote the Fisher information matrix of $\theta = (\alpha, \theta_1, \theta_2)$ as $I(\theta) = ((I_{ij}(\theta)))$ for $i, j = 1, 2, 3$. Therefore,

$$I(\theta) = - \begin{bmatrix} E \left( \frac{\partial^2 L}{\partial \alpha^2} \right) & E \left( \frac{\partial^2 L}{\partial \alpha \partial \theta_1} \right) & E \left( \frac{\partial^2 L}{\partial \alpha \partial \theta_2} \right) \\ E \left( \frac{\partial^2 L}{\partial \theta_1 ^2} \right) & E \left( \frac{\partial^2 L}{\partial \theta_1 \partial \theta_1} \right) & E \left( \frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} \right) \\ E \left( \frac{\partial^2 L}{\partial \theta_2 ^2} \right) & E \left( \frac{\partial^2 L}{\partial \theta_2 \partial \theta_1} \right) & E \left( \frac{\partial^2 L}{\partial \theta_2 \partial \theta_2} \right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad \text{(say)}.$$ 

Moreover

$$E \left( \frac{\partial^2 L}{\partial \alpha^2} \right) = -I_{11} = -\frac{m+n}{\alpha^2} - \frac{1}{\theta_1} E \sum_{i=1}^{n} x_i^\alpha (\ln x_i)^2 - \frac{1}{\theta_2} E \sum_{j=1}^{m} y_j^\alpha (\ln y_j)^2$$

$$= -\frac{1}{\alpha^2} [(m+n)(1+\Gamma''(2)) + 2\Gamma'(2)(n \ln \theta_1 + m \ln \theta_2) + n(\ln \theta_1)^2 + m(\ln \theta_2)^2]$$

$$E \left( \frac{\partial^2 L}{\partial \theta_1 ^2} \right) = -\frac{n}{\theta_1^2} = -I_{22}, \quad E \left( \frac{\partial^2 L}{\partial \theta_1 \partial \theta_1} \right) = \frac{1}{\theta_1^2} E \sum_{i=1}^{n} x_i^\alpha (\ln x_i) = \frac{n}{\alpha \theta_1} [\ln \theta_1 + \Gamma'(2)] = -I_{12} = -I_{21}$$

$$E \left( \frac{\partial^2 L}{\partial \theta_2 ^2} \right) = E \left( \frac{\partial^2 L}{\partial \theta_2 \partial \theta_2} \right) = \frac{1}{\theta_2^2} E \sum_{j=1}^{m} y_j^\alpha (\ln y_j) = \frac{m}{\alpha \theta_2} [\ln \theta_2 + \Gamma'(2)] = -I_{13} = -I_{31}.$$ 

Since the two-parameter Weibull family satisfies all the regularity conditions, therefore we have the following result.

**Theorem 1:** As $n \to \infty$ and $m \to \infty$ and $\frac{n}{m} \to p$, then

$$\sqrt{n}(\hat{\alpha} - \alpha), \sqrt{n}(\hat{\theta}_1 - \theta_1), \sqrt{n}(\hat{\theta}_2 - \theta_2) \quad \overset{d}{\rightarrow} \quad N_3 \left( 0, A^{-1}(\alpha, \theta_1, \theta_2) \right),$$

where

$$A(\alpha, \theta_1, \theta_2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$$

and

$$a_{11} = \frac{1}{\alpha^2} \left[ (1+p)(1+\Gamma''(2)) + 2\Gamma'(2)\left(p \ln \theta_1 + \ln \theta_2 \right) + p(\ln \theta_1)^2 + (\ln \theta_2)^2 \right] = \lim_{n,m \to \infty} \frac{I_{11}}{m},$$

$$a_{22} = \frac{1}{\theta_1^2} = \lim_{n,m \to \infty} \frac{1}{n} I_{22}, \quad a_{12} = -\frac{\sqrt{p}}{\alpha \theta_1} [\ln \theta_1 + \Gamma'(2)] = a_{21} = \lim_{n,m \to \infty} \frac{\sqrt{p}}{n} I_{12},$$

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\[
\begin{align*}
    a_{33} &= \frac{1}{p\theta_2^2} = -\lim_{n,m \to \infty} \frac{1}{n} I_{33}, \quad a_{13} = -\frac{1}{\sqrt{p}\alpha\theta_2} [\ln \theta_2 + \Gamma'(2)] = a_{31} = \lim_{n,m \to \infty} \frac{1}{m\sqrt{p}} I_{13}.
\end{align*}
\]

\textbf{Proof:} The proof follows by expanding the derivative of the log-likelihood function using Taylor series and using the Central limit theorem.

Now we have the main result:

\textbf{Theorem 2:} As \( m \to \infty \) and \( n \to \infty \), so that \( \frac{n}{m} \to p \), then

\[
    \sqrt{m} (\hat{R} - R) \to N(0, B),
\]

where \( B = \frac{2}{u} b^T G b \), and

\[
    u = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}, \quad c = \left(1 + \frac{\theta_2}{\theta_1}\right)^{-4} \left(\frac{\theta_2}{\theta_1}\right)^2 = \frac{\theta_1^2\theta_2^2}{(\theta_1 + \theta_2)^4},
\]

\[
    b^T = \left(0, \frac{1}{\theta_1}, -\frac{1}{\theta_2}\right),
\]

\[
    G = \begin{bmatrix}
    a_{22}a_{33} & -a_{21}a_{33} & -a_{22}a_{31} \\
    -a_{21}a_{13} & a_{11}a_{33} - a_{13}^2 & a_{12}a_{31} \\
    -a_{22}a_{31} & a_{12}a_{31} & a_{11}a_{22} - a_{13}^2
    \end{bmatrix}.
\]

\textbf{Proof:} It follows from Theorem 1.

\textbf{Remark 1:} The asymptotic distribution of \( \hat{R} \) is independent of \( \alpha \).

\textbf{Remark 2:} In Theorem 1, the normalizing constants \( \sqrt{m} \) and \( \sqrt{n} \) can be interchanged and the necessary changes are required in the corresponding dispersion matrix.

\textbf{Remark 3:} Theorem 2 can be used to construct asymptotic confidence interval of \( R \). To compute the confidence interval of \( R \), the variance \( B \) needs to be estimated. We recommend to use the empirical Fisher information matrix and the MLE estimates of \( \alpha, \theta_1 \) and \( \theta_2 \) to estimate \( B \), which is very convenient.

\textbf{Remark 4:} Theoretically it is very difficult to compute the asymptotic distributions of the AMLEs of the unknown parameters. In our simulation experiments we observe that for large sample sizes the MLEs and AMLEs match very well. Although we could not prove it
theoretically but we believe that the MLEs and AMLEs are asymptotically equivalent. It needs further investigation.

It is observed that the asymptotic confidence intervals do not perform very well for small sample sizes. We propose the following two bootstrap confidence intervals mainly for small sample sizes, which might be computationally very demanding for large samples.

4.1 Bootstrap Confidence Intervals

In this subsection, we propose to use two confidence intervals based on the non-parametric bootstrap methods; (i) percentile bootstrap method (we call it from now on as Boot-p) based on the idea of Efron (1982), (ii) bootstrap-t method (we refer it as Boot-t from now on) based on the idea of Hall (1988). We illustrate briefly how to estimate confidence intervals of $R$ using both methods.

**Boot-p Method:**

Step 1: Generate a bootstrap sample of size $n$, $\{x_1^*, \ldots, x_n^*\}$ from $\{x_1, \ldots, x_n\}$ and generate a bootstrap sample of size $m$, $\{y_1^*, \ldots, y_m^*\}$ from $\{y_1, \ldots, y_m\}$. Based on $\{x_1^*, \ldots, x_n^*\}$ and $\{y_1^*, \ldots, y_m^*\}$ compute the bootstrap estimate of $R$, say $\hat{R}^*$, using (13).

Step 2: Repeat step 1, NBOOT times.

Step 3: Let $G(x) = P(\hat{R}^* \leq x)$, be the cumulative distribution function of $\hat{R}^*$. Define $\hat{R}_{Boot-p}(x) = G^{-1}(x)$ for a given $x$. The approximate $100(1 - \gamma)$% confidence interval of $R$ is given by

$$\left(\hat{R}_{Boot-p}\left(\frac{\gamma}{2}\right), \hat{R}_{Boot-p}\left(1 - \frac{\gamma}{2}\right)\right).$$  

(26)
Boot-t Method:

Step 1: From the sample \(\{x_1, \ldots, x_n\}\) and \(\{y_1, \ldots, y_m\}\) compute \(\hat{R}\).

Step 2: Same as in Boot-p method first generate bootstrap sample \(\{x_1^*, \ldots, x_n^*\}\), \(\{y_1^*, \ldots, y_m^*\}\) and then compute \(\hat{R}^*\), the bootstrap estimate of \(R\). Also compute the following statistic:

\[
T^* = \frac{\sqrt{m}(\hat{R}^* - \hat{R})}{\sqrt{V(\hat{R}^*)}}.
\]

Compute \(V(\hat{R}^*)\) using Remark 2.

Step 3: Repeat step 2, NBOOT times.

Step 4: From the NBOOT \(T^*\) values obtained, determine the upper and lower bound of the 100(1 - \(\gamma\))% confidence interval of \(R\) as follows: Let \(H(x) = P(T^* \leq x)\) be the cumulative distribution function of \(T^*\). For a given \(x\), define

\[
\hat{R}_{\text{Boot-}t} = \hat{R} + m^{-\frac{1}{2}} \sqrt{V(\hat{R})} H^{-1}(x).
\]

Here also, \(V(\hat{R})\) can be computed as mentioned in Remark 2. The approximate 100(1 - \(\gamma\))% confidence interval of \(R\) is given by

\[
\left(\hat{R}_{\text{Boot-}t}(\frac{\gamma}{2}), \hat{R}_{\text{Boot-}t}(1 - \frac{\gamma}{2})\right).
\]

5 Bayesian Inference on \(R\)

In this section, we consider Bayesian inference on \(R\). We mainly obtain the Bayes estimate of \(R\) under the squared error loss and the corresponding credible interval by Gibbs sampling technique.
5.1 The Prior and the Posterior Distributions

Following the approach of Berger and Sun [5], it is assumed that the prior density of \( \theta_j \) is inverted \( IG(a_j, b_j) \). Therefore, the prior density function of \( \theta_j \), for \( j = 1, 2 \), becomes

\[
\pi_{1j}(\theta_j) = \pi_{1j}(\theta_j | a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{-(1+a_j)} e^{-b_j \theta_j},
\]

where \( a_j > 0 \) and \( b_j > 0 \). Moreover, it is assumed that \( \theta_1 \) and \( \theta_2 \) are independent. The joint density function of \( \theta_1 \) and \( \theta_2 \) is

\[
\pi_1(\theta_1, \theta_2) = \prod_{j=1}^{2} \pi_{1j}(\theta_j).
\]

No specific form of prior, \( \pi_2(\alpha) \) on \( \alpha \) is assumed here. It is only assumed that the support of \( \pi_2(\alpha) \) is \((0, \infty)\) and it is independent of \( \theta_1 \) and \( \theta_2 \). Based on the above assumptions, we have the likelihood function of the observed data as

\[
l(data|\alpha, \theta_1, \theta_2) = \alpha^{m+n} \theta_1^{-n} \theta_2^{-m} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{j=1}^{m} y_j^{\alpha-1} e^{-\frac{1}{\theta_1} \sum_{i=1}^{n} x_i^\alpha} e^{-\frac{1}{\theta_2} \sum_{j=1}^{m} y_j^\alpha}.
\]

Therefore, the joint density of the \( data, \alpha, \theta_1 \) and \( \theta_2 \) can be obtained as

\[
l(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1, \theta_2) \times \pi_2(\alpha).
\]

Based on (29), we obtain the joint posterior density of \( \alpha, \theta_1 \) and \( \theta_2 \) given the data as

\[
l(\alpha, \theta_1, \theta_2|data) = \frac{\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} l(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1, \theta_2) \times \pi_2(\alpha) d\alpha d\theta_1 d\theta_2}{\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} l(data|\alpha, \theta_1, \theta_2) d\alpha d\theta_1 d\theta_2}.
\]

Since (30) can not be obtained analytically, we adopt the Gibbs sampling technique to compute the Bayes estimate of \( R \) and the corresponding credible interval of \( R \). To perform the Gibbs sampling, in this case we further assume that \( \pi_2(\alpha) \) is log-concave. It may be mentioned that most common densities are log-concave, for example gamma, Weibull, normal, log-normal all have log-concave densities. Another natural concern is how to choose \((a_j, b_j)\) for the conditional prior distribution of \( \theta_j \). Two methods as suggested by Berger and Sun
[5], (a) specify the first two marginal moments for \( \theta_j \); (b) specify the two marginal quantiles of \( \theta_j \), and solve for \( (a_j, b_j) \), can be adopted in this set up also. It is not pursued here. Based on the above priors, we provide different posterior densities.

**Theorem 3:** The conditional density of \( \theta_1 \), given \( \alpha \) and \( \text{data} \) is

\[
\pi_1(\theta_1|\alpha, \text{data}) = \frac{(b_1 + \sum_{i=1}^{n} x_i^\alpha)^{a_1+n}}{\Gamma(a_1+n)} \theta_1^{-(1+a_1+n)} e^{-\frac{1}{\theta_1}(b_1 + \sum_{i=1}^{n} x_i^\alpha)},
\]

(31)

similarly, the conditional density of \( \theta_2 \), given \( \alpha \) and \( \text{data} \) is

\[
\pi_2(\theta_2|\alpha, \text{data}) = \frac{(b_2 + \sum_{j=1}^{m} y_j^\alpha)^{a_2+m}}{\Gamma(a_2+m)} \theta_2^{-(1+a_2+m)} e^{-\frac{1}{\theta_2}(b_2 + \sum_{j=1}^{m} y_j^\alpha)},
\]

(32)

and they are independent.

**Proof:** See in the appendix.

**Theorem 4:** The conditional density of \( \alpha \) given the \( \text{data} \) is log-concave.

**Proof:** See in the appendix.

We use the method proposed by Devroye [11] to generate sample from a log-concave density function for Gibbs sampling purposes. The details are provided below.

5.2 Bayes Estimate and Credible Interval

We use the idea of Geman and Geman [16] to generate sample from the conditional posterior densities. We adopt the following scheme:

- **Step 1:** Generate \( \alpha_1 \), from the log-concave density \( l(.|\text{data}) \), as given in (35).
- **Step 2:** Generate \( \theta_{1,1} \) and \( \theta_{2,1} \) from \( \pi_1(.|\alpha_1, \text{data}) \) and \( \pi_2(.|\alpha_1, \text{data}) \) as given in (31) and (32) respectively. Calculate \( R \).
- **Step 3:** Repeat Steps 1 and 2, \( M \) times.
Now the approximate posterior mean and posterior variance of $R$ becomes

$$\hat{E}(R|data) = \frac{1}{M} \sum_{k=1}^{M} \frac{\theta_{1,k}}{\theta_{1,k} + \theta_{2,k}}$$

and

$$\hat{V}(R|data) = \frac{1}{M} \sum_{k=1}^{M} \left( \frac{\theta_{1,k}}{\theta_{1,k} + \theta_{2,k}} - (\hat{E}(R|data)) \right)^2,$$

respectively. Based on $M$, $R$ values, using the method proposed by Chen and Shao [6] the approximate highest posterior density (HPD) credible of $R$ can be easily constructed.

6 Numerical Experiments and Discussions

In this section we mainly present some Monte Carlo simulation results, which we performed to observe the behavior of the different methods for different sample sizes and for different parameter values. We mainly compare the performances of the MLEs, AMLEs and the Bayes estimates with respect to the squared error loss function in terms of biases and mean squares errors (MSEs). We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLEs and two different bootstrap confidence intervals in terms of the average confidence lengths and coverage percentages. Since it is observed that for large sample sizes the MLEs and AMLEs match very well, we also compute the confidence intervals based on the asymptotic distributions of MLEs but replacing the MLEs by AMLEs. For comparison purposes we also compute the HPD credible intervals of $R$, under the non-informative priors. We assume that $a_1 = a_2 = b_1 = b_2 = 0$ and $\alpha$ has a gamma prior, namely GA(0, 1). Note that the priors on $\theta_1$, $\theta_2$ and $\alpha$ are all non-proper.

All computations are performed at the Indian Institute of Technology Kanpur using Pentium IV processor. All the programs are written in FORTRAN-77 and we used the random deviate generator RAN2, described in Press et al. [25]. We consider the following sample
sizes; \((m, n) = (5,5), (10,10), (15,15), (20,20), (25,25), (50,50)\) and the following parameter values; \(\theta_1 = 1.0\) and \(\theta_2 = 1.0, 1.5, 2.0, 2.5\) and 3.0. Since the asymptotic distributions are independent of \(\alpha\), we kept it constant at \(\alpha = 1.5\). All the results are based on 1000 replications. In our simulations experiments for both the bootstrap methods, when \(m\) or \(n\) is less that 20, we computed the confidence intervals based on 250 resampling and for \(m\) or \(n\) greater than 20 (namely for 25 or 50), the results are based on 200 resampling. The Bayes estimates and the corresponding credible intervals are based on 10,000 sampling, namely \(M = 10,000\). The nominal level for the confidence intervals or the credible intervals is 0.95 in each case.

From the sample, we estimate \(\alpha\) using the iterative algorithm (12). We start the iterative process with the initial estimate 1 and the iterative process stops when the difference between the two consecutive iterates are less than \(10^{-6}\). Once we estimate \(\alpha\), we obtain the MLE of \(R\) using (13). We obtain the AMLE of \(R\) using the method described in section 3. We compute the Bayes estimate of \(R\) as suggested in section 5. We report the average biases and MSEs of the MLEs, AMLEs and Bayes estimators over 1000 replications. The results are reported in Table 1.

We compute the 95% confidence intervals based on the asymptotic distributions of the MLEs and replacing the true parameter values by the MLEs and AMLEs respectively. We further compute Boot-p, Boot-t confidence intervals and the HPD credible intervals. We obtain the average confidence/credible lengths and the corresponding coverage percentages. The results are reported in Table 2.

Some of the points are quite clear from this experiment. Even for small sample sizes, the performance of the MLEs and AMLEs are quite satisfactory in terms of biases and MSEs. Interestingly the MSEs and biases of the MLEs and AMLEs are very close for small sizes and for large sample sizes they coincide. The performance of the Bayes estimators also
Table 1: Biases and MSEs of the MLEs, AMLEs and Bayes estimators of $R$, when $\theta_1 = 1.0$ and for different values of $\theta_2$. *

<table>
<thead>
<tr>
<th>S.S.</th>
<th>$\theta_2 = 1.0$</th>
<th>$\theta_2 = 1.5$</th>
<th>$\theta_2 = 2.0$</th>
<th>$\theta_2 = 2.5$</th>
<th>$\theta_2 = 3.0$</th>
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<td>(5,5)</td>
<td>-0.0009(0.0328)</td>
<td>-0.0107(0.0296)</td>
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<td>-0.0190(0.0221)</td>
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<td>-0.0123(0.0167)</td>
</tr>
<tr>
<td>(10,10)</td>
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<td>-0.0153(0.0118)</td>
<td>-0.0164(0.0103)</td>
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<td>(25,25)</td>
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<td>-0.0057(0.0038)</td>
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<td>0.0093(0.0020)</td>
<td>0.0015(0.0019)</td>
<td>-0.0478(0.0017)</td>
</tr>
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</table>

* In each cell the first row represents the average bias of the MLE and the corresponding MSE is reported within bracket. Similarly the second and third rows represent the results for AMLE and Bayes estimators respectively.
Table 2: Average confidence/credible lengths and coverage percentages for different estimators, when $\theta_1 = 1.0$ and for different values of $\theta_2$. *

<table>
<thead>
<tr>
<th>S.S.</th>
<th>$\theta_2 = 1.0$</th>
<th>$\theta_2 = 1.5$</th>
<th>$\theta_2 = 2.0$</th>
<th>$\theta_2 = 2.5$</th>
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<td>0.1700(95%)</td>
<td>0.1609(95%)</td>
</tr>
<tr>
<td></td>
<td>0.1928(94%)</td>
<td>0.1871(94%)</td>
<td>0.1770(94%)</td>
<td>0.1662(94%)</td>
<td>0.1661(95%)</td>
</tr>
</tbody>
</table>

* In each cell the first, second, third and fourth rows represent the average confidence lengths based on the asymptotic distributions of the MLEs, and replacing the true parameter values by MLEs and AMLEs, Boot-p method and Boot-t method respectively. The corresponding coverage percentages are reported within brackets. Similarly the fifth rows represent the average credible lengths and the corresponding coverage percentages in brackets for the Bayes estimators.
are quite satisfactory. Interestingly in most of the cases considered here, the MSEs of the
Bayes estimators are smaller than the MSEs of the MLEs or AMLEs, although the biases are
significantly higher. It is observed that when $m, n$ increase then MSEs of all the estimators
decrease.

The confidence intervals based on the asymptotic distributions of the MLEs do not work
very well when $m$ and $n$ are very small. The coverage percentages are smaller than the
nominal level. For $m$ and $n$ greater then 25, the asymptotic results work quite well. It
is quite interesting that in constructing the confidence interval based on the asymptotic
distribution of the MLEs if we replace the MLEs by the AMLEs, the performances are
marginally better in terms of the coverage percentages. Non-parametric bootstrap methods
work quite well even for small sizes. In our experimental results, it is observed that Boot-p
confidence intervals perform better than the Boot-t confidence intervals at least for small
sizes. One point we should mention that for bootstrap methods the performances depend
on the number of resampling. The performances change when the number of resampling
changes. The performances of the Bayes estimators also quite satisfactory even for small
sample sizes. The Bayes credible intervals also maintain the nominal coverage percentages
and the average credible lengths are also smaller than the average confidence intervals in
general. Based on all these, we recommend to use Bayes credible intervals or the Boot-p
confidence intervals for constructing confidence intervals for small $m$ and $n$ and for $m$ and $n$
greater than 25, asymptotic results can be used.

7 Data Analysis

In this section we present a data analysis of the strength data reported by Badar and Priest
[4]. The authors are thankful to Professor J.G. Surles for providing the data The data
represent the strength data measured in GPA, for single carbon fibers and impregnated
1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20, and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purpose, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes $n = 69$ and $m = 63$, respectively. We are presenting the data below for convenience.

**Data Set 1**


**Data Set 2**


Surles and Padgett [28], [29] and Raqab and Kundu [26] observed that generalized Rayleigh distribution works quite well for these strength data. We are analyzing the data by subtracting 0.75 from both the data sets. We fit the Weibull models to the two data sets separately. We present the estimated shape and scale parameters, log-likelihood values, Kolmogorov-Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution functions and corresponding $p$ values in Table 3. We also present the observed frequencies and the expected frequencies based on the fitted models in the Tables 4 and 5. For data sets 1 and 2, the chi-square values are 1.220 and 0.861 respectively. Therefore, it is clear that Weibull model fits quite well to both the data sets. Since the two
Table 3: Shape parameter, scale parameter, log-likelihood, K-S and $p$ values of the fitted Weibull models to Data sets 1 and 2.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Shape Parameter</th>
<th>Scale Parameter</th>
<th>Log-Likelihood</th>
<th>K-S</th>
<th>$p$ value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8428</td>
<td>11.3142</td>
<td>-48.8703</td>
<td>0.0461</td>
<td>0.9985</td>
</tr>
<tr>
<td>2</td>
<td>3.9090</td>
<td>38.5449</td>
<td>-60.1524</td>
<td>0.0800</td>
<td>0.8154</td>
</tr>
</tbody>
</table>

Table 4: Observed frequencies and expected frequencies for modified data set 1 when fitting the Weibull model

<table>
<thead>
<tr>
<th>Intervals</th>
<th>Observed Frequencies</th>
<th>Expected Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00 - 1.78</td>
<td>5</td>
<td>6.06</td>
</tr>
<tr>
<td>1.78 - 2.22</td>
<td>15</td>
<td>15.97</td>
</tr>
<tr>
<td>2.22 - 2.68</td>
<td>27</td>
<td>23.85</td>
</tr>
<tr>
<td>2.68 - 3.13</td>
<td>18</td>
<td>17.36</td>
</tr>
<tr>
<td>3.13 - ∞</td>
<td>4</td>
<td>5.76</td>
</tr>
</tbody>
</table>

shape parameters are not very different, assuming the two shape parameters are equal we estimated the parameters and the results are reported in Table 6.

Based on the log-likelihood values it is clear we can not reject the null hypothesis that the two shape parameters are equal. Therefore, the assumption that the two shape parameters are equal is justified for these data sets. The K-S values and the corresponding $p$ values indicate that the Weibull models with equal shape parameters fit reasonably well to the transformed data sets.

The MLE and AMLE of $R$ become 0.7624, 0.7608 and the corresponding 95% confidence intervals become (0.6968,0.8279) and (0.6946,0.8270) respectively. We also obtain the 95% Boot-p and Boot-t confidence intervals as (0.6983, 0.8285) and (0.7001, 0.8415) respectively.

For Bayesian estimate of $R$, we use non-proper priors on $\theta_1$ and $\theta_2$, i.e., $a_1 = a_2 = b_1 =$

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\[ b_2 = 0. \] Moreover for \( \alpha \) we assume non-proper gamma priors, namely \( \text{GA}(0,1) \), see (2). Based on the above priors we obtain 0.7636 as the Bayes estimate of \( R \), under the squared error loss function. We also compute 95\% highest posterior density (HPD) credible interval of \( R \), based on the idea of Chen and Shao [6] and it is (0.7014, 0.8244).

**8 CONCLUSIONS**

In this paper we consider estimation of \( R = P(Y < X) \) by different methods when \( X \) and \( Y \) are independent Weibull random variables with equal shape parameters. Note that our method can be easily extended for estimating \( P(X_1 < \ldots < X_k) \) where \( X_i \)'s are independent \( \text{WE}(\alpha, \theta_i) \) random variables, and also estimating \( P(Y < cX) \) when \( c \) is known. One important question we have not addressed in this paper, namely the estimation of \( R \) when the shape parameters are different. The problem becomes quite complicated because in that case \( R \) does have any explicit form. We might need some approximation to provide the estimate of \( R \) in that case. Work is in progress, it will be reported elsewhere.

Table 5: Observed frequencies and expected frequencies for modified data set 2 when fitting the Weibull model

<table>
<thead>
<tr>
<th>Intervals</th>
<th>Observed Frequencies</th>
<th>Expected Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00 - 2.53</td>
<td>15</td>
<td>13.99</td>
</tr>
<tr>
<td>2.53 - 3.15</td>
<td>21</td>
<td>20.71</td>
</tr>
<tr>
<td>3.15 - 3.77</td>
<td>18</td>
<td>19.32</td>
</tr>
<tr>
<td>3.77 - 4.39</td>
<td>7</td>
<td>7.82</td>
</tr>
<tr>
<td>4.39 - \infty</td>
<td>2</td>
<td>1.16</td>
</tr>
</tbody>
</table>
Table 6: Shape parameter, scale parameter, log-likelihood, K-S and \( p \) values of the fitted Weibull models to Data sets 1 and 2 assuming that the two shape parameters are equal.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Shape Parameter</th>
<th>Scale Parameter</th>
<th>Log-Likelihood</th>
<th>K-S</th>
<th>( p ) value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8770</td>
<td>11.6064</td>
<td>-48.8747</td>
<td>0.0464</td>
<td>0.9984</td>
</tr>
<tr>
<td>2</td>
<td>3.8770</td>
<td>37.2333</td>
<td>-60.1566</td>
<td>0.0767</td>
<td>0.8528</td>
</tr>
</tbody>
</table>

**Appendix**

**Proof of Theorem 3:** Note that (29) can be written as

\[
l(data, \alpha, \theta_1, \theta_2) \propto \alpha^{m+n} \theta_1^{-(1+a_1+n)} \theta_2^{-(1+a_2+m)} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{j=1}^{m} y_j^{\alpha-1} \\
\times e^{-\frac{1}{\theta_1}(b_1 + \sum_{i=1}^{n} x_i^\alpha)} \times e^{-\frac{1}{\theta_2}(b_2 + \sum_{j=1}^{m} y_j^\alpha)} \pi_2(\alpha). \tag{33}
\]

Therefore, the conditional distribution of \( \theta_1 \) and \( \theta_2 \) given \( \alpha \) and data is

\[
l(\theta_1, \theta_2|\alpha, data) \propto \theta_1^{-(1+a_1+n)} \theta_2^{-(1+a_2+m)} \times e^{-\frac{1}{\theta_1}(b_1 + \sum_{i=1}^{n} x_i^\alpha)} \times e^{-\frac{1}{\theta_2}(b_2 + \sum_{j=1}^{m} y_j^\alpha)}. \tag{34}
\]

Therefore, the result follows.

**Proof of Theorem 4:** Note that from (33), the conditional density of \( \alpha \) given the data is

\[
l(\alpha|data) \propto \pi_2(\alpha) \alpha^{m+n} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{j=1}^{m} y_j^{\alpha-1} \times \frac{1}{(b_1 + \sum_{i=1}^{n} x_i^\alpha)^{a_1+n}} \times \frac{1}{(b_2 + \sum_{j=1}^{m} y_j^\alpha)^{a_2+m}}. \tag{35}
\]

Therefore,

\[
\ln(l(\alpha|data)) = k + \ln \pi_2(\alpha) + (m+n) \ln \alpha + (\alpha - 1) \left( \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j \right) \\
-(a_1 + n) \ln \left( \sum_{i=1}^{n} x_i^\alpha + b_1 \right) - (a_2 + m) \ln \left( \sum_{j=1}^{m} y_j^\alpha + b_2 \right). \tag{36}
\]

Now to prove that (36) is log-concave, observe the following inequality. Suppose

\[
g(\alpha) = \sum_{i=1}^{n} x_i^\alpha + b_1, \quad \text{therefore,}
\]
\[
g'(\alpha) = \sum_{i=1}^{n} x_i^\alpha \ln x_i \quad \text{and} \quad g''(\alpha) = \sum_{i=1}^{n} x_i^\alpha (\ln x_i)^2.
\]

Since
\[
\left(\sum_{i=1}^{n} x_i^\alpha (\ln x_i)^2\right) \times \left(\sum_{i=1}^{n} x_i^\alpha \right) - \left(\sum_{i=1}^{n} x_i^\alpha \ln x_i \right)^2 = \sum_{1 \leq i < j \leq n} x_i^\alpha x_j^\alpha (\ln x_i - \ln x_j)^2 \geq 0,
\]

therefore for \(b_1 \geq 0\),
\[
g''(\alpha)g(\alpha) \geq (g'(\alpha))^2. \quad (37)
\]

From (37) it follows that if \(\pi_2(\alpha)\) is log-concave, then for \(a_1, b_1, a_2, b_2 \geq 0\), the second derivative of the right hand side of (36) with respect to \(\alpha\) is negative. Therefore, \(l(\alpha|\text{data})\) is log-concave.

References


