Analysis of Hybrid Life-tests in Presence of Competing Risks

Debasis Kundu[†] & Rameshwar D. Gupta[‡]

Abstract

The mixture of Type-I and Type-II censoring schemes, called the hybrid censoring scheme is quite common in life-testing or reliability experiments. In this paper, we consider the competing risks model in presence of hybrid censored data. Under this set up, it is assumed that the item may fail due to various causes and the corresponding lifetime distributions are independent and exponentially distributed with different scale parameters. We obtain the maximum likelihood estimators of the mean life of the different causes and derive their exact distributions. Using the exact distributions, all the moments can be obtained. Asymptotic confidence intervals and two bootstrap confidence intervals are also proposed. Bayes estimates and credible intervals of the unknown parameters are obtained under the assumptions of independent inverted gamma priors of the mean life of the different causes. Different methods have been compared using Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

Keywords: Maximum likelihood estimators; Type-I and Type-II censoring; Fisher Information matrix; Asymptotic distribution; Bayesian Inference; Inverted Gamma Distribution.

[†] Corresponding Author: Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, Pin 208016, INDIA. Phone: 91-512-2597141, Fax: 91-512-2597500, e-mail: kundu@iitk.ac.in

[‡] Department of Computer Science and Statistics, The University of New Brunswick at Saint John, New Brunswick, Canada E2L 4L5. Part of the work was supported by a grant from the Natural Sciences and Engineering Research Council.

1 Introduction

Consider the following lifetime experiment in which n units are put on a test. Each unit is exposed to some risks. It is assumed that each unit may fail due to different causes and the corresponding lifetime distributions are independent and identically distributed. The test is terminated when a pre chosen number R out of n items have failed or when a pre determined time, T, on the test has been reached. It is also assumed that the failed items are not replaced.

This particular censoring scheme is known as hybrid censoring scheme. It was first introduced by Epstein [3] without the presence of competing risks and under the assumptions that the underlying lifetime distribution is exponential. The hybrid censoring scheme is quite useful in reliability acceptance test. See for example Childs *et al.* [1] and Jeong, Park and Yum [5] for some recent development on hybrid censored sampling plan for the exponential life time distributions.

In medical studies or in reliability analysis, it is quite common that more than one risk factor may be present at the same time. An investigator is often interested in the assessment of a specific risk in presence of other risk factors. Usually, it is observed that the data consists of a failure time and an indicator denoting the cause of failure. In the statistical literature, it is known as the competing risks model. For the general introduction of the competing risks problem, the readers are referred to the recent monograph by Crowder [2]. Without the presence of covariates, it is usually assumed that the competing causes are independently distributed. For the parametric setup, it is assumed that the different lifetime distributions follow some specific parametric distributions, namely exponential, Weibull, log-normal or gamma distributions. Several estimation procedures are proposed for estimating the unknown parameters in presence of type-I or type-II censoring scheme, see Crowder [2].

In this paper, we assume that the lifetime distribution of the different causes are independent and exponentially distributed. We consider the estimation of the unknown parameters in presence of hybrid censoring scheme. It is observed that the maximum likelihood estimators (MLEs) of the mean lifetimes of the different causes do not exist always. We propose the conditional MLEs and obtain the exact distributions of the conditional MLEs. Based on the exact distributions of the MLEs we can obtain all the moments of the MLEs. Using the exact distributions of the conditional MLEs it is possible to construct the approximate confidence intervals of the unknown parameters. Because of the very complicated nature of the distribution functions, it is not pursued here. Instead, we propose, the asymptotic confidence intervals and two bootstrap confidence intervals. We also compute the Bayes estimates and the credible intervals of the unknown parameters using inverted gamma priors. Different methods are compared using Monte Carlo simulations and for illustrative purposes, we analyze one real data set.

2 Model Description, Notation and MLEs

2.1 Model Description and Notation

For notational simplicity, we assume that the number of causes is two. Let T_i be the lifetime distribution of cause i, for $i = 1$ or 2. It is assumed in this paper that T_1 and T_2 are independent and exponentially distributed with mean θ_1 and θ_2 respectively. Therefore, the density function of T_i is;

$$f_{T_i}(x, \theta) = \frac{1}{\theta_i} e^{-\frac{x}{\theta_i}}; \qquad x > 0. \tag{1}$$

Let $Z = \min\{T_1, T_2\}$, therefore, Z, has the density function

$$f_Z(z) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) e^{-z\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}.$$

Let $Z_1, \ldots, Z_n$ be n independent and identically distributed (*i.i.d.*) sample of size n from Z. Suppose, $Z_{1:n} < \ldots < Z_{n:n}$ denote, the ordered $Z_1, \ldots, Z_n$. Moreover, we denote T^* $= \min\{Z_{R:n}, T\}$, where R and T are some prefixed numbers as mentioned in the previous section. If δ_i denotes the cause of failure of the $i-th$ ordered unit, then in this particular case δ_i can take only two values. In presence of hybrid censoring, we have the following observations;

$$\text{Case I:} \qquad \{(Z_{1:n}, \delta_1), \ldots, (Z_{R:n}, \delta_R)\}; \quad \text{if} \quad Z_{R:n} < T, \quad \text{or} \tag{2}$$

$$\text{Case II:} \qquad \{(Z_{1:n}, \delta_1), \ldots, (Z_{J:n}, \delta_J)\}; \quad \text{if} \quad Z_{J:n} < T < Z_{J+1:n}. \tag{3}$$

Here $J =$ total number of observed failures up to time point T for Case II. For Case I, the experiment stops at $Z_{R:n}$ and for Case II, the experiment stops at T. For Case II, it is known that $Z_{J:n} < T < Z_{J+1:n} < \ldots < Z_{R:n}$ and $Z_{J+1:n} < \ldots < Z_{R:n}$ are not observed. We also denote D_1 and D_2 as the number of failures due to cause I and cause II respectively. So $J = D_1 + D_2$. Suppose D denotes the total number of failures up to time point T. Therefore, for Case I, $D \geq R$ and for Case II, $D = J$.

We also use the following notation in this paper. $G(\alpha, \lambda)$ and $IG(\alpha, \lambda)$ for $\alpha, \lambda > 0$, denote the gamma and inverted gamma distributions when $x > 0$, with density functions

$$f_G(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad \text{and} \quad f_{IG}(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\frac{\lambda}{x}} x^{-\alpha-1},$$

respectively.

2.2 Maximum Likelihood Estimators

Based on the observations (2) or (3), the log-likelihood function, $L(\theta_1, \theta_2)$, of the observed data can be written as (ignoring the constant)

$$L(\theta_1, \theta_2) = -D_1 \ln \theta_1 - D_2 \ln \theta_2 - W \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right), \tag{4}$$

where $D_2 = R - D_1$ or $J - D_1$ for Case I and Case II respectively. Moreover, $W = \sum_{i=1}^{R} Z_{i:n} + (n-R) Z_{R:n}$ or $\sum_{i=1}^{J} Z_{i:n} + (n-J)T$ represents the total time on tests for Case I and Case II respectively. It is immediate that for Case I, (W, D_1) is a joint minimal sufficient statistic and for Case II, (W, D_1, J) is a joint minimal sufficient statistic for (θ_1, θ_2). From (4), it is clear that the MLE of θ_1 (θ_2) exists only when $D_1(D_2) > 0$ and they are as follows.

$$\hat{\theta}_1 = \frac{W}{D_1} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{W}{D_2}. \tag{5}$$

3 Conditional Distributions of the MLEs

In this section, we obtain the conditional distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$, namely

$$F_{\hat{\theta}_1}(x) = P[\hat{\theta}_1 \le x | D_1 > 0] \qquad \text{and} \qquad F_{\hat{\theta}_2}(x) = P[\hat{\theta}_2 \le x | D_2 > 0].$$

In this section, we denote $\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$. We compute $F_{\hat{\theta}_1}(x)$, $F_{\hat{\theta}_2}(x)$ can be obtained along the same line. Now

$$\begin{aligned} F_{\hat{\theta}_1}(x) &= P[\hat{\theta}_1 \le x | D_1 > 0] = P[\hat{\theta}_1 \le x, Z_{R:n} \le T | D_1 > 0] + P[\hat{\theta}_1 \le x, Z_{R:n} > T | D_1 > 0] \\ &= \sum_{i=1}^{R} P[\hat{\theta}_1 \le x, Z_{R:n} \le T, D_1 = i | D_1 > 0] + \sum_{j=1}^{R-1} P[\hat{\theta}_1 \le x, Z_{R:n} > T, J = j | D_1 > 0] \\ &= \sum_{i=1}^{R} G_i(x) q_i + \sum_{j=1}^{R-1} \sum_{i=1}^{j} G_{ij}(x) q_{ij}, \end{aligned}$$

where

$$\begin{aligned} G_i(x) &= P[\hat{\theta}_1 \le x | Z_{R:n} \le T, D_1 = i, D_1 > 0], \qquad q_i = P[Z_{R:n} \le T, D_1 = i | D_1 > 0], \\ G_{ij}(x) &= P[\hat{\theta}_1 \le x | Z_{R:n} > T, J = j, D_1 = i, D_1 > 0], \qquad q_{ij}(x) = P[Z_{R:n} > T, J = j, D_1 = i | D_1 > 0]. \end{aligned}$$

We will provide the expressions for $G_i(x)$, q_i, $G_{ij}(x)$ and q_{ij}. To compute $G_i(x)$, we use the conditional moment generating function of $\hat{\theta}_1$ conditioning on $D_1 = i$ and $Z_{R:n} \le T$. We have the following results.

Lemma 1: The conditional density function (PDF) of $\hat{\theta}_1$ given that $D_1 = i$ and $Z_{R:n} \le T$, is

$$\begin{aligned} f_{\hat{\theta}_1 | D_1 = i, Z_{R:n} \le T}(x) &= \frac{1}{P[Z_{R:n} \le T]} \times \left[f_G\left(x; R, \frac{i}{\theta}\right) + R\binom{n}{R} \sum_{k=1}^{R} \binom{R-1}{k-1} \frac{(-1)^k}{n-R+k} e^{-\frac{T}{\theta}(n-R+k)} \times \right. \\ & \left. f_G\left(x; \frac{T}{i}(n-R+k), R, \frac{i}{\theta}\right)\right]. \end{aligned}$$

Proof of Lemma 1: The proof mainly follows by writing the conditional moment generating function of $\hat{\theta}_1$ given $D_1 = i, Z_{R:n} \leq T$ and then inverting it. The details can be obtained from the authors.

Note that for $i \geq 1$,

$$\begin{aligned} q_i &= P[D_1 = i, Z_{R:n} \leq T | D_1 > 0] = \frac{P[D_1 = i, Z_{R:n} \leq T, D_1 > 0]}{P[D_1 > 0]} \\ &= \binom{R}{i} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{R-i} \times \frac{P[Z_{R:n} \leq T]}{P[D_1 > 0]} \end{aligned}$$

and

$$P[D_1 > 0] = 1 - \sum_{i=0}^{R-1} \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^i \binom{n}{i} \left(1 - e^{-\frac{T}{\theta}}\right)^i e^{-(n-i)\frac{T}{\theta}} - \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^R \sum_{i=R}^{n} \binom{n}{i} \left(1 - e^{-\frac{T}{\theta}}\right)^i e^{-(n-i)\frac{T}{\theta}}.$$

Now we would like to compute $G_{ij}(x)$ and for that we need the conditional moment generating function of $\hat{\theta}_1$ conditioning on $D_1 = i, J = j$ and $Z_{R:n} > T$. We have the following result whose proof can be obtained along the same line as the proof of lemma 1.

Lemma 2: The conditional PDF of $\hat{\theta}_1$ given $D_1 = i$, $J = j$ and $Z_{R:n} > T$ is given by

$$f_{\hat{\theta}_1 | D_1 = i, J = j, Z_{R:n} > T}(x) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} e^{\frac{T}{\theta}(j-k)} f_G(x; \frac{T}{i}(k+n-j), j, \frac{i}{\theta}).$$

Also for $j < R$,

$$\begin{aligned} q_{ij} &= P[D_1 = i, J = j, Z_{R:n} > T | D_1 > 0] = P[D_1 = i, J = j | D_1 > 0] = \frac{P[D_1 = i, J = j]}{P[D_1 > 0]} \\ &= \frac{P[D_1 = i | J = j] P[J = j]}{P[D_1 > 0]} = \binom{j}{i} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{j-i} \frac{P[J = j]}{P[D_1 > 0]}. \end{aligned}$$

If we denote

$$p_j = P[J = j] = \binom{n}{j} \left(1 - e^{-\frac{T}{\theta}}\right)^j e^{-(n-j)\frac{T}{\theta}},$$

then

$$q_{ij} = \binom{j}{i} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{j-i} \frac{p_j}{\left(1 - \sum_{i=0}^{R-1} \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^i p_i - \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^R \sum_{i=R}^{n} p_i\right)}.$$

Therefore, we have the final result.

THEOREM 1: The conditional PDF of $\hat{\theta}_1$, conditioning on $D_1 > 0$, is given by

$$\begin{aligned} f_{\hat{\theta}_1}(x) &= \frac{1}{P[D_1>0]}\left\{\sum_{i=1}^{R}\binom{R}{i}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{i}\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{R-i}\times\right. \\ & \left[f_G(x;R,\frac{i}{\theta})+R\binom{n}{R}\sum_{k=1}^{R}\binom{R-1}{k-1}\frac{(-1)^k}{n-R+k}\times e^{-\frac{T}{\theta}(n-R+k)}f_G\left(x;\frac{T}{i}(n-R+k),R,\frac{i}{\theta}\right)\right] \\ & \left.+\sum_{j=1}^{R-1}\sum_{i=1}^{j}\left[\sum_{k=0}^{j}(-1)^k\binom{j}{k}e^{\frac{T}{\theta}(j-k)}f_G(x;\frac{T}{i}(k+n-j),j,\frac{i}{\theta})\right]\binom{j}{i}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{i}\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{j-i}p_j\right\}. \end{aligned}$$

Similarly, we can obtain the conditional PDF of $\hat{\theta}_2$, conditioning on $D_2 > 0$, by interchanging the role of θ_1 and θ_2. Note that Theorem 1, can be used to derive different moments of $\hat{\theta}_1$. For example,

$$\begin{aligned} E(\hat{\theta}_1) &= \frac{1}{P[D_1>0]}\left\{\sum_{i=1}^{R}\binom{R}{i}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{i}\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{R-i}\times\right. \\ & \left[\frac{R\theta}{i}+R\binom{n}{R}\sum_{k=1}^{R}\binom{R-1}{k-1}\frac{(-1)^k}{n-R+k}\times e^{-\frac{T}{\theta}(n-R+k)}\times\left(\frac{T}{i}(n-R+k)+\frac{R\theta}{i}\right)\right]+ \\ & \left.\sum_{j=1}^{R-1}\sum_{i=1}^{j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}e^{\frac{T}{\theta}(j-k)}\left(\frac{T}{i}(k+n-j)+\frac{j\theta}{i}\right)\times\binom{j}{i}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{i}\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{j}p_j\right\}. \end{aligned}$$

Other moments also can be obtained similarly. Note that using the approach of Kundu and Basu [6], it is possible to construct the approximate confidence interval of $\hat{\theta}_1$, using the PDF of $\hat{\theta}_1$. Since it is computationally quite involved particularly for large R, we recommend the following confidence intervals for large R and n.

4 CONFIDENCE INTERVALS

First we propose to use the asymptotic confidence intervals. Using the asymptotic normality of the MLEs, we obtain $100(1-\alpha)\%$ confidence intervals of θ_1 and θ_2 as

$$\hat{\theta}_1 \pm z_{\frac{\alpha}{2}}\frac{W}{D_1^{\frac{3}{2}}}, \quad \text{and} \quad \hat{\theta}_2 \pm z_{\frac{\alpha}{2}}\frac{W}{D_2^{\frac{3}{2}}}.$$

Here $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ -th percentile point of $N(0,1)$.

We propose the following two parametric Bootstrap confidence intervals, namely percentile Bootstrap (Boot-p) and Bootstrap-t (Boot-t) confidence intervals conditioning on the number of failures within the time interval T.

[1] Determine $\hat{\theta}_1$ and $\hat{\theta}_2$ from the sample and compute $\hat{\theta}$ by $\frac{1}{\hat{\theta}} = \frac{1}{\hat{\theta}_1} + \frac{1}{\hat{\theta}_2}$.

[2] CASE I

(i) First we need to generate a bootstrap sample of J, say J^*, and it is generated from the conditional probability mass function;

$$\frac{\binom{n}{i}\left(1-e^{-\frac{T}{\hat{\theta}}}\right)^{i} e^{-(n-i)\frac{T}{\hat{\theta}}}}{\sum_{j=R}^{n}\binom{n}{j}\left(1-e^{-\frac{T}{\hat{\theta}}}\right)^{j} e^{-(n-j)\frac{T}{\hat{\theta}}}}; \qquad i = R, \ldots, n. \tag{6}$$

(ii) Generate a sample of size J^* from the truncated distribution, which has the non-zero PDF between $(0, T)$ as

$$\frac{\frac{1}{\hat{\theta}} e^{-\frac{x}{\hat{\theta}}}}{1-e^{-\frac{T}{\hat{\theta}}}}; \qquad 0 < x < T. \tag{7}$$

(iii) Take the first R order statistics from J^* and assign Cause I or Cause II to each failure with probability $\frac{\hat{\theta}_2}{\hat{\theta}_1+\hat{\theta}_2}$ and $\frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2}$ respectively.

[2′] CASE II

(i) Generate a sample of size D from the truncated distribution function given in (7).

(ii) Assign Cause I or Cause II to each failure with probability $\frac{\hat{\theta}_2}{\hat{\theta}_1+\hat{\theta}_2}$ and $\frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2}$ respectively.

[3] From the bootstrap sample compute $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ and repeat the process NBOOT times.

[4] Let $\widehat{CDF}(x) = P(\hat{\theta}_1 \le x)$ be the cumulative distribution function of $\hat{\theta}_1$. Define for a given x, $\hat{\theta}_{1,boot}(x) = \widehat{CDF}^{-1}(x)$. Then, approximate $100(1-\alpha)\%$ confidence interval of θ_1 is given by

$$\left(\hat{\theta}_{1,boot}(\alpha/2), \hat{\theta}_{1,boot}(1-\alpha/2)\right). \tag{8}$$

Similarly, we can obtain the confidence interval for $\hat{\theta}_2$ also.

The following algorithm is proposed for constructing Boot-t confidence intervals of θ_1 and θ_2.

[1]-[3] Same as Boot-p.

[4] Determine the statistics T_1^* and T_2^* as follows;

$$T_1^* = \frac{(\hat{\theta}_1^* - \hat{\theta}_1)}{\sqrt{V(\hat{\theta}_1^*)}} \qquad \text{and} \qquad T_2^* = \frac{(\hat{\theta}_2^* - \hat{\theta}_2)}{\sqrt{V(\hat{\theta}_2^*)}},$$

where $V(\hat{\theta}_1^*)$ and $V(\hat{\theta}_2^*)$ are the asymptotic variances of $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ respectively, and they can be obtained using the Fisher information matrix.

[5] Repeat [1] - [4] NBOOT times and determine the upper and lower bounds of θ_1 as follows. Let $\widehat{CDF}_1(x) = P(T_1 \leq x)$ be the cumulative distribution function of T_1. For a given x, define

$$\hat{\theta}_{1,Boot-t}(x) = \hat{\theta}_1 + \sqrt{V(\hat{\theta}_1)}\widehat{CDF}_1^{-1}(x).$$

The approximate $100(1-\alpha)\%$ Boot-t confidence interval for θ_1 is given by

$$\left(\hat{\theta}_{1,Boot-t}(\alpha/2), \hat{\theta}_{1,Boot-t}(1-\alpha/2)\right). \tag{9}$$

Similarly the approximate $100(1-\alpha)\%$ Boot-t confidence interval for θ_2 can also be obtained.

5 Bayesian Analysis

In this section, we approach the problem from the Bayesian point of view. In the context of exponential lifetimes, $\hat{\theta}_1$ and $\hat{\theta}_2$ may be reasonably modeled using the inverted gamma priors. We assume that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independently distributed with $IG(a_1, b_1)$ and $IG(a_2, b_2)$ respectively. The parameters a_1, b_1, a_2, b_2 are all assumed to be positive. Note that when $a_1 = a_2 = b_1 = b_2 = 0$, they are the non-informative priors of θ_1 and θ_2 respectively.

The joint posterior density function of θ_1 and θ_2, given the data can be written as

$$l(\theta_1, \theta_2|Data) \propto \frac{1}{\theta_1^{D_1+a_1+1}} e^{-\frac{W+b_1}{\theta_1}} \times \frac{1}{\theta_2^{D_2+a_2+1}} e^{-\frac{W+b_2}{\theta_2}}. \qquad (10)$$

From (10), it is clear that the posterior density function of θ_1 and θ_2 are independent. Moreover, the posterior density function of θ_1 given the data, $l(\theta_1|data)$, is $IG(D_1 + a_1, W + b_1)$. Similarly, the posterior density function of θ_2 given the data, $l(\theta_2|data)$, is $IG(D_2 + a_2, W + b_2)$. The Bayes estimators of θ_1 and θ_2, under squared error loss functions are;

$$\hat{\theta}_{1,Bayes} = \frac{W + b_1}{D_1 + a_1}, \qquad \text{and} \qquad \hat{\theta}_{2,Bayes} = \frac{W + b_2}{D_2 + a_2}. \qquad (11)$$

Interestingly, when $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimators coincide with the corresponding MLEs.

The credible intervals for θ_1 and θ_2 are obtained easily from the joint posterior distribution function. We observe that *a posteriori*;

$$Z_1 = \frac{2(W + b_1)}{\theta_1} \qquad \text{and} \qquad Z_2 = \frac{2(W + b_2)}{\theta_2},$$

follow $\chi^2_{2(D_1+a_1)}$ and $\chi^2_{2(D_2+a_2)}$ respectively provided $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are positive integers. Therefore, $100(1 - \alpha)\%$ credible intervals for θ_1 and θ_2 are

$$\left[\frac{2(W + b_1)}{\chi^2_{2(D_1+a_1),1-\alpha/2}}, \frac{2(W + b_1)}{\chi^2_{2(D_1+a_1),\alpha/2}}\right] \qquad \text{and} \qquad \left[\frac{2(W + b_2)}{\chi^2_{2(D_2+a_2),1-\alpha/2}}, \frac{2(W + b_2)}{\chi^2_{2(D_2+a_2),\alpha/2}}\right] \qquad (12)$$

for $D_1 + a_1 > 0$ and $D_2 + a_2 > 0$ respectively.

Therefore, if no prior information is available, the non-informative priors can be used to compute the credible intervals for θ_1 and θ_2 using (12). Note that if $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are not integers then the corresponding credible intervals can be obtained using gamma distributions.

6 Numerical Results and Data Analysis

6.1 Numerical Results

Note that the performances of the different confidence intervals can not be compared theoretically. In order to compare the performances of the different confidence intervals we consider Monte Carlo simulations for different sample sizes, for different parameter values and for different censoring schemes. We consider different sample sizes, namely $n = 25$, 50, 75 and 100 and two different R values, *i.e.* $R = [0.75 \times n]$ (25% censoring) and $R = [0.60 \times n]$ (40% censoring), where $[a]$ means the largest integer less than or equal to a. Without loss of generality, we take $\theta_1 = 1$ and two different values of θ_2, *i.e.* $\theta_2 = 2.0$ and $\theta_2 = 1.75$. We consider two different values of T also, namely, $T = 1.5$ and $T = 2.25$.

The generation of the sample is as follows. For a fixed n, θ_1 and θ_2, first we generate n exponential random variables with mean $\frac{\theta_1\theta_2}{\theta_1+\theta_2}$. From the n exponential random variables we obtain the hybrid censored data based on R and T. To each uncensored observation we assign failure Cause 1 or Cause 2 with probability $\frac{\theta_2}{\theta_1+\theta_2}$ and $\frac{\theta_1}{\theta_1+\theta_2}$ respectively. For each hybrid censored competing risks data we compute the 95% confidence intervals using three different methods, *i.e* asymptotic, Boot-p and Boot-t methods. For comparison purposes, we also compute the 95% Bayes credible intervals using non-informative prior. We replicate the process 1000 times in each case and report the average confidence/ credible lengths and the coverage percentages. The results are reported in Tables 1 - 6.

Some of the points are quite clear from these results. For all the methods as the sample size increases the average confidence/ credible lengths decrease as expected. Interestingly, for all the cases considered here, the asymptotic, Boot-p and Bayes confidence/ credible intervals have the coverage percentages quite close to the nominal level, where as the Boot-t confidence intervals have the coverage percentages far below than the nominal level.

Now we compare the performances of the different confidence intervals for different cen-

Table 1: T = 1.5, R = 0.75 × n, θ_1 = 1.0, θ_2 = 2.0

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.2075 (0.92)	1.2794 (0.95)	1.3761 (0.85)	1.4508 (0.94)
	θ_2	4.3905 (0.92)	5.3413 (0.95)	5.4536 (0.80)	10.8348 (0.95)
n = 50	θ_1	0.8109 (0.93)	0.8135 (0.93)	0.8717 (0.87)	0.9648 (0.95)
	θ_2	2.5011 (0.93)	3.0880 (0.96)	2.7617 (0.86)	3.0611 (0.96)
n = 75	θ_1	0.6578 (0.94)	0.6611 (0.94)	0.6949 (0.90)	0.7336 (0.95)
	θ_2	1.9472 (0.94)	2.0937 (0.95)	2.0342 (0.87)	2.4223 (0.96)
n = 100	θ_1	0.5635 (0.95)	0.5599 (0.94)	0.5849 (0.91)	0.6098 (0.95)
	θ_2	1.6575 (0.94)	1.7743 (0.95)	1.7471 (0.88)	1.9796 (0.95)

Table 2: T = 1.5, R = 0.60 × n, θ_1 = 1.0, θ_2 = 2.0

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.3430 (0.92)	1.4499 (0.96)	1.3761 (0.85)	1.6619 (0.94)
	θ_2	5.1202 (0.91)	6.1737 (0.96)	5.4536 (0.80)	15.6571 (0.94)
n = 50	θ_1	0.9120 (0.92)	0.9184 (0.95)	0.8717 (0.87)	1.1334 (0.96)
	θ_2	2.8390 (0.93)	3.7174 (0.94)	2.7617 (0.86)	3.6010 (0.95)
n = 75	θ_1	0.7372 (0.93)	0.7586 (0.95)	0.6949 (0.90)	0.8475 (0.95)
	θ_2	2.2164 (0.93)	2.4462 (0.94)	2.0342 (0.87)	2.7323 (0.95)
n = 100	θ_1	0.6333 (0.94)	0.6325 (0.94)	0.5849 (0.91)	0.7006 (0.95)
	θ_2	1.8729 (0.93)	2.0634 (0.95)	1.7471 (0.88)	2.3217 (0.95)

Table 3: T = 2.25, R = 0.60 × n, θ_1 = 1.0, θ_2 = 2.0

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.3430 (0.92)	1.6092 (0.95)	1.5697 (0.84)	1.6619 (0.94)
	θ_2	5.1202 (0.91)	5.9609 (0.96)	6.2160 (0.81)	15.6571 (0.94)
n = 50	θ_1	0.9120 (0.92)	0.9359 (0.94)	1.0166 (0.88)	1.1334 (0.96)
	θ_2	2.8390 (0.93)	3.7793 (0.95)	3.2016 (0.88)	3.6010 (0.95)
n = 75	θ_1	0.7372 (0.93)	0.7444 (0.93)	0.7857 (0.90)	0.8475 (0.95)
	θ_2	2.2164 (0.93)	2.5140 (0.94)	2.3534 (0.86)	2.7323 (0.95)
n = 100	θ_1	0.6333 (0.94)	0.6283 (0.94)	0.6553 (0.90)	0.7006 (0.95)
	θ_2	1.8728 (0.93)	2.1077 (0.94)	2.0300 (0.89)	2.3217 (0.95)

Table 4: T = 2.25, R = 0.75 × n, $\theta_1 = 1.0$, $\theta_2 = 2.0$

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.2070 (0.92)	1.3030 (0.95)	1.4141 (0.82)	1.4502 (0.94)
	θ_2	4.3902 (0.92)	5.8993 (0.96)	5.3139 (0.81)	10.8340 (0.95)
n = 50	θ_1	0.8109 (0.93)	0.8309 (0.93)	0.8852 (0.88)	0.9648 (0.95)
	θ_2	2.5012 (0.93)	2.9463 (0.95)	2.6926 (0.87)	3.0611 (0.95)
n = 75	θ_1	0.6579 (0.94)	0.6606 (0.93)	0.6931 (0.90)	0.7336 (0.95)
	θ_2	1.9472 (0.94)	2.0992 (0.94)	2.0306 (0.86)	2.4223 (0.95)
n = 100	θ_1	0.5636 (0.95)	0.5662 (0.93)	0.5876 (0.91)	0.6098 (0.96)
	θ_2	1.6575 (0.94)	1.8016 (0.94)	1.7651 (0.89)	1.9796 (0.95)

Table 5: T = 1.50, R = 0.75 × n, $\theta_1 = 1.0$, $\theta_2 = 1.75$

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.2388 (0.92)	1.3826 (0.95)	1.4043 (0.82)	1.4974 (0.95)
	θ_2	3.5890 (0.91)	4.4814 (0.95)	4.0705 (0.81)	8.1218 (0.95)
n = 50	θ_1	0.8329 (0.93)	0.8408 (0.93)	0.8853 (0.89)	1.0003 (0.95)
	θ_2	2.0701 (0.94)	2.4439 (0.94)	2.2854 (0.86)	2.5271 (0.96)
n = 75	θ_1	0.6730 (0.94)	0.6826 (0.95)	0.7123 (0.91)	0.7548 (0.95)
	θ_2	1.6274 (0.94)	1.7423 (0.95)	1.7254 (0.89)	2.0146 (0.95)
n = 100	θ_1	0.5772 (0.95)	0.5792 (0.94)	0.6005 (0.92)	0.6272 (0.95)
	θ_2	1.3802 (0.94)	1.4583 (0.95)	1.4486 (0.90)	1.6208 (0.96)

Table 6: T = 1.50, R = 0.60 × n, $\theta_1 = 1.0$, $\theta_2 = 1.75$

S.S.	Parameters	Asymp	Boot-p	Boot-t	Bayes
n = 25	θ_1	1.3815 (0.92)	1.6268 (0.96)	1.5876 (0.83)	1.7307 (0.94)
	θ_2	4.0537 (0.91)	5.0761 (0.95)	5.3880 (0.82)	10.3670 (0.95)
n = 50	θ_1	0.9381 (0.92)	0.9595 (0.93)	1.0573 (0.88)	1.1736 (0.96)
	θ_2	2.3549 (0.93)	3.1240 (0.93)	2.6570 (0.85)	2.9292 (0.95)
n = 75	θ_1	0.7575 (0.94)	0.7556 (0.94)	0.8050 (0.90)	0.8780 (0.95)
	θ_2	1.8285 (0.94)	2.0771 (0.95)	1.9306 (0.88)	2.2681 (0.96)
n = 100	θ_1	0.6505 (0.95)	0.6517 (0.94)	0.6774 (0.91)	0.7238 (0.95)
	θ_2	1.5507 (0.94)	1.6678 (0.94)	1.6465 (0.90)	1.9016 (0.95)

soring schemes, namely for 25% and 40% censoring when the other variables are fixed. Comparing Tables 1 & 2 and Tables 3 & 4 it is quite clear that as the censoring percentages increase then average confidence/ credible lengths increase at it should be. On the other hand comparing Tables 1 & 4 and Tables 2 & 3 it is clear that T does not have much effect on the performances of the different estimators.

Now we compare the performances of the different estimators for the different parameter values. Comparing Tables 1 & 5 and Tables 2 & 6, it is clear that as θ_2 becomes closer to θ_1, the average confidence/ credible lengths increase for θ_1 but the corresponding lengths decrease for θ_2.

Comparing all the methods it is observed that the asymptotic confidence intervals perform quite good even when the sample size is only 25. The confidence lengths of the asymptotic confidence intervals are the smallest although their coverage percentages are slightly lower than the nominal level. The performances of the Boot-p confidence intervals are very good. The average lengths of the Boot-p confidence intervals are quite close to the corresponding asymptotic confidence lengths and the coverage percentages are closer to the nominal level. The performances of the Boot-t confidence intervals are quite poor in terms of the coverage percentages, they are much lower than the nominal level. Interestingly, the Bayes credible intervals maintain the nominal level in all the cases considered, but the lengths of the credible intervals are significantly larger than the other confidence intervals. Computationally, the asymptotic confidence intervals and the Bayes credible intervals are much easier to compute than the bootstrap confidence intervals. Considering all the points we recommend to use the asymptotic confidence intervals, if the computation is not of a major concern Boot-p method also can be used.

6.2 Data Analysis

For illustrative purposes we analyze one data set using the proposed methods. We consider the data set, which was originally analyzed by Hoel [4]. The data arose from a laboratory

Table 7: 95% Confidence and credible intervals for θ_1 and θ_2.

Methods	θ_1	θ_2
Asymptotic	(731.2911, 1785.7997)	(909.3463, 2782.3872)
Bootstrap-p	(810.9471, 1432.7617)	(1019.3655, 2652.1709)
Bootstrap-t	(500.7881, 1410.3297)	(397.6293, 2280.5234)
Bayes	(862.5344, 2008.2250)	(1178.7345, 3297.9966)

experiment in which male mice received a radiation dose of 300 roentgens at 5 to 6 weeks of age. The cause of death of each mouse was determined by autopsy to be thymic lymphoma (Cause 1) or reticulum cell sarcoma (Cause 2). Although, Hoel [4] had the complete data, we created artificially hybrid censored data from the total sample (n = 60) by considering R = 50 and T = 600. We have the following hybrid censored data:

(159 1), (189 1), (191 1), (198 1), (200 1), (207 1), (220 1), (235 1), (245 1), (250 1), (256 1), (261 1), (265 1), (266 1), (280 1), (317 2), (318 2), (343 1), (356 1), (383 1), (399 2), (403 1), (414 1), (428 1), (432 1), (495 2), (525 2), (536 2), (549 2), (552 2), (554 2), (557 2), (558 2), (571 2), (586 2), (594 2), (596 2).

From the above data we obtain the following J = 37, $\sum_{i=1}^{37} Z_i$ = 13888, D_1 = 22, D_2 = 15, and W = 13888 + (60-37) × 600 = 27688. Therefore, $\hat{\theta}_1 = \frac{27688}{22} = 1258.5455$ and $\hat{\theta}_2 = \frac{27688}{15} = 1845.8667$. The asymptotic standard deviations of $\hat{\theta}_1$ and $\hat{\theta}_2$ becomes 268.32 and 476.6 respectively. The different 95% confidence/ credible intervals for θ_1 and θ_2 are presented in Table 7.

It is interesting to observe that for the data example Boot-p confidence interval has the smallest length and the Bayes confidence interval has the maximum length. Moreover, if we want to test the hypothesis; $H_0 : \theta_1 = \theta_2$, vs. $H_1 : \theta_1 \neq \theta_2$, then we can not reject the null hypothesis by any of these methods at the 5% significance levels.

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