Burr Type X Distribution: Revisited

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Abstract

In this paper, we consider the two-parameter Burr-Type X distribution. We observe several interesting properties of this distribution. This particular skewed distribution can be used quite effectively in analyzing lifetime data. It has some interesting relations with the well studied gamma, Weibull distributions and with the recently proposed exponentiated exponential and exponentiated Weibull distributions. Statistical inferences about the shape and scale parameters have been discussed. Analysis of a real data set has been performed.

Keywords: Generalized Weibull distribution; generalized exponential distribution; hazard function; Fisher Information matrix; order statistics.

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1 INTRODUCTION

Burr (1942) introduced twelve different forms of cumulative distribution functions for modelling data. Among those twelve distribution functions, Burr-Type X and Burr-Type XII received the maximum attention. There is a thorough analysis of Burr-Type XII distribution in Rodriguez (1977), see also Wingo (1993) for a nice account of it.

In this paper, we consider the two-parameter Burr-Type X distribution. Two-parameter Burr-Type X distribution has the following cumulative distribution function (CDF);

$$F(x;\alpha,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha}; \quad x > 0, \alpha > 0, \lambda > 0, \tag{1.1}$$

where α and λ are shape and scale parameters, respectively. Several aspects of the one-parameter ($\lambda = 1$) Burr-Type X distribution were studied by Sartawi and Abu-Salih (1991), Jaheen (1995, 1996), Ahmad *et al.* (1997), Raqab (1998) and Surles and Padgett (1998). Recently Surles and Padgett (2001) proposed (1.1) and observed that the Burr-Type X distribution can be used quite effectively in modelling strength data and also modelling general lifetime data.

The main aim of this paper is to consider different aspects of two-parameter Burr-Type X distribution and its relation with other well studied distributions like gamma distribution, Weibull distribution and with not so well studied distributions like generalized exponential distribution and generalized Weibull distribution. The generalized Weibull or exponentiated Weibull (EW) was originally proposed by Mudholkar and Srivastava (1993) [see also Mudholkar, Srivastava and Freimer (1995)]. The EW family has the distribution function;

$$F_{EW}(x;\alpha,\beta,\lambda) = \left(1 - e^{-(\lambda x)^{\beta}}\right)^{\alpha}. \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.2}$$

Here α and β are two shape parameters and λ is a scale parameter. It is observed that the EW family is a very flexible family and it can used for modelling several types of skewedly distributed lifetime data. Several well known distributions like exponential ($\alpha = 1$, $\beta = 1$), Weibull ($\alpha = 1$), Rayleigh ($\alpha = 1$, $\beta = 2$) are particular cases of the EW distribution. Recently Gupta and Kundu (1999) discussed various interesting properties of the generalized exponential (GE) distribution, a particular member of the EW distribution, obtained by substituting $\beta = 1$ in (1.2). It is observed that the GE distribution can be used quite effectively to analyze several lifetime data in place of gamma distribution or Weibull distribution. It can have increasing or decreasing hazard function, like the gamma distribution or the Weibull distribution.

Along the same line of the GE distribution, Burr-Type X distribution can be thought of as the exponentiated Rayleigh or generalized Rayleigh (GR) distribution as correctly mentioned by Surles and Padgett (2001). We also prefer to call the Burr-Type X distribution a GR distribution in this paper for brevity. It is observed that the two-parameter GR distribution has several properties which are quite common to the two-parameter gamma, Weibull and GE distributions. The distribution function and the density function of a GR distribution have closed form. As a consequence of that, it can be used very conveniently even for censored data. Unlike, gamma, Weibull and GE distributions it can have non-monotone hazard function, which can be very useful in many practical applications.

The rest of the paper is organized as follows. In Section 2, we discuss different properties of the GR distribution. Moments and order statistics are discussed in Section 3. Maximum likelihood estimators and the Fisher Information matrix are presented in Section 4. Statistical inferences are discussed in Section 5. A real data set is analyzed and discussed in Section 6.

2 Distributional Properties

If the random variable X has a GR distribution as defined in (1.1), then it has the density function;

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1}; \qquad x>0, \alpha>0, \lambda>0. \tag{2.1}$$

Here α is the shape parameter and λ is the scale parameter. We denote the GR distribution with shape parameter α and scale parameter λ as GR(α,λ). If $\alpha = 1$, GR distribution coincides with the well known Rayleigh distribution. For $\alpha \neq 1$, it plays the role of a shape parameter. If $\alpha \leq \frac{1}{2}$, the density function is a decreasing function and for $\alpha > \frac{1}{2}$, it is a right skewed unimodal function. The mode of the density function can be written as $\frac{x_0}{\lambda}$, where x_0 is the solution of the non-linear equation

$$1 - 2x^2 - e^{-x^2}(1-2\alpha x^2) = 0. \tag{2.2}$$

Clearly the mode is a decreasing function of λ as expected and it is an increasing function of α. Different forms of the density functions are presented in Figure 1. It is clear from Figure 1 that the GR density functions resemble the gamma and Weibull density functions. The median of a GR random variable occurs at $\left[-\frac{1}{\lambda}\ln\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{2}}$ and clearly it is also a decreasing function of λ but an increasing function of α.

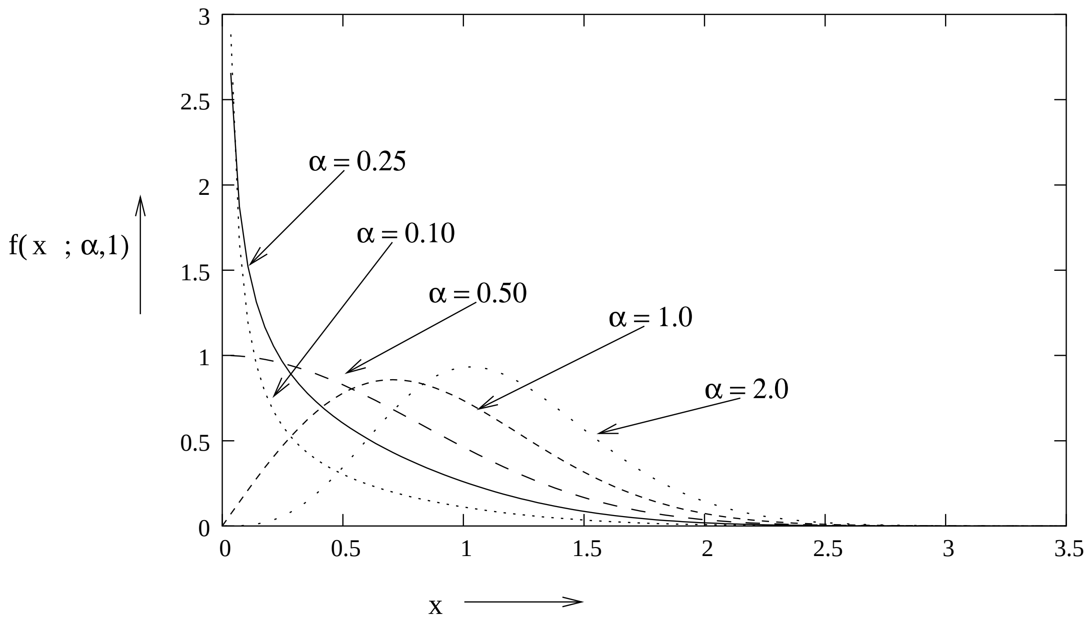

Figure 1: The density functions of the generalized Rayleigh distribution for different shape parameters

Now let us look at the hazard function of the GR distribution. The hazard function of X is given by

$$h(x;\alpha,\lambda) = \frac{2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1}}{1-\left(1-e^{-(\lambda x)^2}\right)^{\alpha}}. \tag{2.3}$$

Clearly if $\alpha = 1$, the hazard function becomes $2\lambda^2 x$, a linear function of x. For general α, we have the following result. From Mudholkar et al. (1995), it follows that if $\alpha \le \frac{1}{2}$, the hazard function of GR(α,λ) is bathtub type and for $\alpha > \frac{1}{2}$, it has an increasing hazard function.

The hazard functions for different values of α are plotted in Figure 2. For $\alpha \le \frac{1}{2}$, it decreases from ∞ to a positive constant and then it increases to ∞. For $\alpha > \frac{1}{2}$, it is an increasing function and it increases from 0 to ∞. It is known that for shape parameter greater than 1, the hazard functions of gamma, Weibull and GE are all increasing functions. The hazard functions of gamma and GE distributions increase from 0 to 1. While for Weibull distribution it increases from 0 to ∞. For $\alpha > \frac{1}{2}$, the hazard function of a GR distribution behaves like the hazard function of a Weibull distribution, whose shape parameter is greater than 1. In this respect the GR distribution behaves more like a Weibull distribution than gamma distribution or GE distribution. Therefore, if the data are coming from an environment where the failure rate is gradually increasing without any bound, the GR distribution can also be used instead of a

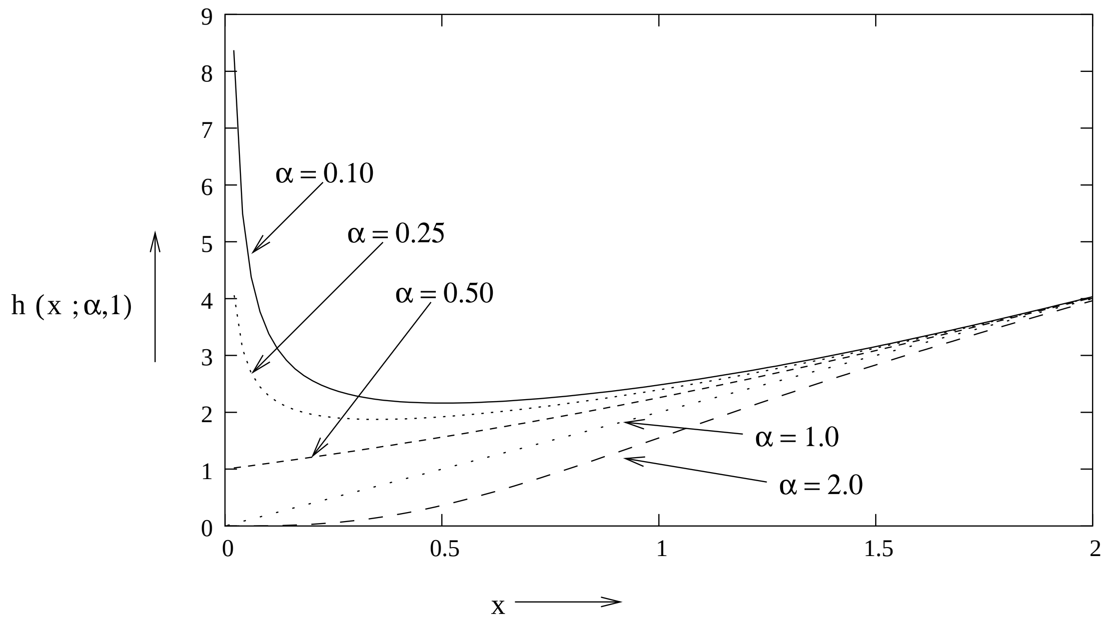

Figure 2: The hazard functions of the generalized Rayleigh distribution for different shape parameters

Weibull distribution.

Now we discuss the reverse hazard rate of the GR distribution. The reverse hazard rate of any distribution function $F(x)$, can be defined as $r(x) = f(x)/F(x)$. Consequently, the reversed hazard rate of GR(α,1), is given by

$$r(x;\alpha) = \frac{2x\alpha}{e^{x^2} - 1}. \tag{2.4}$$

The reverse hazard rate has recently become quite popular in the statistical literature. It can be interpreted as follows. Suppose, the lifetime of a unit has reverse hazard rate $r(x)$, then $r(x)dx$ provides the probability of failing in $(x - dx, x)$, when a unit is found failed at x. In general, the reverse hazard rate is useful in constructing the information matrix and in estimating the survival function for censored data, see Block, Savits and Singh (1998). The reverse hazard rate of the GR distribution with unit scale parameter is a linear function of α like GE distribution.

Ordering of distributions, particularly among the lifetime distributions plays an important role in statistical literature. Johnson, Kotz and Balakrishnan (1995) have a major section on the ordering of various lifetime distributions. Pecaric, Proschan and Tong (1992) also provide a detailed treatment of stochastic ordering, highlighting their growing importance and illustrating their usefulness in several practical applications. It is known that the gamma family has a likelihood ratio ordering ($<_{LR}$), which implies that it has ordering in the hazard rate ($<_{HAZ}$) and also in distribution ($<_{ST}$). The gamma family has the likelihood ratio ordering, and then it has the monotone likelihood ratio property. This implies there exists uniformly most powerful (UMP) test for any one sided hypothesis and uniformly most powerful unbiased test (UMPU) for any two sided hypothesis on the shape parameter, when the scale parameter is known. The gamma family has the dispersive ordering ($<_{DISP}$) (see Johnson, Kotz and Balakrishnan; 1995), and then it has tail ordering in the sense of Lehmann (1966). The gamma family also has the convex ordering ($<_C$) and star shaped ($<_*$) ordering. Although the gamma and GE families enjoy several ordering properties, the same thing is not true for the Weibull family. The Weibull family does not have the stochastic ordering and therefore it does not have the hazard rate or likelihood ratio ordering. The Weibull family has convex ordering and then it has the star shaped ordering. In this context, the GR family has the ordering properties similar to those of the gamma and GE families. Table 1 highlights the ordering relations of the different families (without detailed proof). 'Y' means ordering exists and and 'N' means ordering does not exist.

From Table 1, it is quite clear that the family GR distributions is quite similar to gamma and GE families than the Weibull family. The GR family has the likelihood ratio ordering. When the scale parameter is known, it can be concluded that there exists UMP test and UMPU test for one sided and two sided hypotheses, respectively, on the shape parameter.

Table 1

Ordering relations within the different families

	$<_{LR}$	$<_{HAZ}$	$<_{ST}$	$<_C$	$<_*$	$<_{DISP}$
gamma	Y	Y	Y	Y	Y	Y
Weibull	N	N	N	Y	Y	N
GE	Y	Y	Y	Y	Y	Y
GR	Y	Y	Y	Y	Y	Y

3 Order Statistics and Their Moments

The moment generating function (MGF) of $X \sim GR(\alpha, 1) \stackrel{def}{\longrightarrow} GR(\alpha)$ exists for $-\infty < t < \infty$ and it is given by

$$M(t) = Ee^{tX} = 2\alpha \int_0^\infty e^{tx} x e^{-x^2} (1 - e^{-x^2})^{\alpha-1} dx. \tag{3.1}$$

Using the binomial series expansion, we immediately obtain from (3.1)

$$M(t) = \alpha \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \begin{pmatrix} \alpha - 1 \\ j \end{pmatrix} \left[1 + e^{\frac{t^2}{4(j+1)}} \frac{t\sqrt{\pi}}{\sqrt{j+1}} \Phi\left(\frac{t}{\sqrt{2(j+1)}} \right) \right]. \tag{3.2}$$

It follows that the k^{th} moment of X exists and it is given by ;

$$\mu_k = \alpha \Gamma\left(\frac{k}{2}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)^{\frac{k}{2}+1}} \begin{pmatrix} \alpha - 1 \\ j \end{pmatrix} \tag{3.3}$$

From (3.3), it can be seen that if $X \sim \text{GR}(\alpha)$, then $Y = X^2$ has GE distribution with the shape parameter α. The even moments of X can be written as;

$$\mu_{2k} = \alpha k! \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)^{k+1}} \begin{pmatrix} \alpha - 1 \\ j \end{pmatrix}$$

From Gupta and Kundu (1999), we know that

$$\mu_2 = \psi(\alpha+1) - \psi(1) \quad \text{and} \quad \mu_4 - \mu_2^2 = \psi'(1) - \psi'(\alpha+1), \tag{3.4}$$

where $\psi(.)$ and $\psi'(.)$ are the digamma and poly-gamma functions.

Let $X_1, \ldots X_n$ be a random sample of size n from the GR distribution, with $\lambda = 1$ and let $X_{1:n} < \ldots < X_{n:n}$ denote the order statistics obtained from this sample. The r^{th} order statistic $X_{r:n}$ represents the life length of a $(n-r+1)$-out-of n system made up of n identical components with independent life lengths. It is important to point out here that when α is a positive integer, the GR distribution is the distribution of the maximum of a random sample of size α from the standard Rayleigh distribution GR(1,1). In this respect, the following two results provide the distributional properties of $X_{n:n}$.

Lemma 3.1: *Let $X_1, ..., X_n$ be independent random variables (r.v.'s) from GR(α_i, λ), for $1 \leq i \leq n$. Then $X_{n:n}$ is distributed as GR($\sum_{i=1}^{n} \alpha_i$, λ).*

Lemma 3.2: *Let $X_1, ..., X_n$ be iid r.v.'s from GR distribution. Then $X_{n:n}$ has a GR distribution iff X_i has GR distribution.*

The density function of $X_{r:n}$ can be written as (Arnold, Balakrishnan and Nagaraja; 1992);

$$f_{r:n}(x) = c_{r:n} \sum_{i=0}^{\infty} \frac{(-1)^i}{(r+i)} \begin{pmatrix} n - r \\ i \end{pmatrix} f(x; \alpha(r+i)), \tag{3.5}$$

where $c_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and $f(x;.) = f(x;.,1)$ as defined in (2.1). As special case of (3.5), we have the density function of $X_{n:n}$ as;

$$f_{n:n}(x) = f(x; \alpha n). \tag{3.6}$$

Using $f_{r:n}$, the k^{th} moment of $X_{r:n}$ can be obtained as

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = c_{r:n} \sum_{i=0}^{n-r} \frac{(-1)^i}{(r+i)} \binom{n-r}{i} \mu^k(\alpha(r+i)), \tag{3.7}$$

where $\mu^k(\alpha(r+i))$ denote the k^{th} moment of the GR$(\alpha(r+i))$ distribution.

The joint probability density function of any two order statistics $U = X_{r:n}$ and $V = X_{s:n}$ $(1 \le r < s \le n)$ is given by

$$f_{r,s:n}(u,v) = 4\alpha^2 c_{r,s:n} uve^{-u^2} e^{-v^2} \left(1-e^{-u^2}\right)^{\alpha(r-1)} \left[\left(1-e^{-v^2}\right)^{\alpha} - \left(1-e^{-u^2}\right)^{\alpha}\right]^{s-r-1}$$
$$\times \left[1-\left(1-e^{-u^2}\right)^{\alpha}\right]^{n-s} \left(1-e^{-u^2}\right)^{\alpha-1} \left(1-e^{-v^2}\right)^{\alpha-1}, \quad 0 < u < v < \infty, \tag{3.8}$$

where $c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Therefore, we obtain the product moment of order statistics as

$$\mu_{r,s:n} = E(X_{r:n}X_{s:n}) = \alpha c_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{n-s}{i} \binom{s-r-1}{j} \phi(i,j), \tag{3.9}$$

where

$$\phi(i,j) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \binom{(r+j)\alpha - 1}{k} (k+1)^l}{l!(l+\frac{3}{2})(s-r-j+i)} \mu^{2l+4}((s-r-j+i)\alpha) \tag{3.10}$$

The detailed proof of (3.9) is provided in the appendix.

4 MLE's and Information Matrix

For completeness purposes, in this section, we briefly discuss the maximum likelihood estimators (MLE's) of the two-parameter GR distribution and discuss their asymptotic properties. Detailed comparisons of the performances of the MLE's with the other estimates by extensive simulations, have been performed by the authors in Kundu and Raqab (2005).

Let $X_1, \ldots, X_n$ be a random sample from GR, then the log-likelihood function can be written as;

$$L(\alpha,\lambda) = C + n\ln\alpha + 2n\ln\lambda + \sum_{i=1}^{n} \ln x_i - \lambda^2 \sum_{i=1}^{n} x_i^2 + (\alpha-1)\sum_{i=1}^{n} \ln\left(1-e^{-(\lambda x_i)^2}\right), \tag{4.1}$$

where C is constant. Therefore, to obtain the MLE's of α and λ, we can maximize (4.1) directly with respect to α and λ or we can solve the following two non-linear equations:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1-e^{-(\lambda x_i)^2}\right) = 0, \tag{4.2}$$

$$\frac{\partial L}{\partial \lambda} = \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 + 2\lambda(\alpha - 1) \sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{1 - e^{-(\lambda x_i)^2}} = 0 \tag{4.3}$$

From (4.2), we obtain the MLE of α as a function of λ, say $\hat{\alpha}(\lambda)$, as

$$\hat{\alpha}(\lambda) = -\frac{n}{\sum_{i=1}^{n} \ln\left(1 - e^{-(\lambda x_i)^2}\right)}. \tag{4.4}$$

If the scale parameter is known, the MLE of α can be obtained from (4.4). If both parameters are unknown, we can first estimate the scale parameter by maximizing the profile likelihood of λ. Ignoring the constant, the profile log-likelihood of λ can be written as;

$$g(\lambda) = L(\hat{\alpha}(\lambda), \lambda) = -n \ln\left(-\sum_{i=1}^{n} \ln\left(1 - e^{-(\lambda x_i)^2}\right)\right) + 2n \ln \lambda$$

$$-\lambda^2 \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \ln\left(1 - e^{-(\lambda x_i)^2}\right). \tag{4.5}$$

It can be easily shown that $g(\lambda)$ is a uni-modal function of λ. Therefore, most of the maximization routine will work well from a reasonable starting value of λ. To maximize $g(\lambda)$, we differentiate $g(\lambda)$ with respect to λ and equate it to 0 as follows;

$$g'(\lambda) = -2n\lambda \frac{\sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{(1-e^{-(\lambda x_i)^2})}}{\sum_{i=1}^{n} \ln(1 - e^{-(\lambda x_i)^2})} + \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{(1 - e^{-(\lambda x_i)^2})} = 0 \tag{4.6}$$

Setting $\lambda^2 = \mu$, (4.6) can be written as

$$h(\mu) = \mu, \tag{4.7}$$

where

$$h(\mu) = \left[\frac{\sum_{i=1}^{n} \frac{x_i^2 e^{-\mu x_i^2}}{(1-e^{-\mu x_i^2})}}{\sum_{i=1}^{n} \ln(1 - e^{-\mu x_i^2})} + \frac{1}{n} \sum_{i=1}^{n} x_i^2 + \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^2 e^{-\mu x_i^2}}{(1 - e^{-\mu x_i^2})} \right]^{-1}. \tag{4.8}$$

Therefore, a fixed point solution of of the non-linear equation (4.7) $\hat{\mu}$ can be easily obtained using a simple iterative procedure as follows;

$$\mu_{j+1} = h(\mu_j) \tag{4.9}$$

where the $(j+1)^{th}$ iterate μ_{j+1}, can be obtained from the j^{th} iterate μ_j using (4.9). The iteration can be stopped when the absolute difference between the $(j+1)^{th}$ and j^{th} is less than a prespecified value, say ϵ. Once $\hat{\mu}$ is obtained the MLE of λ, $\hat{\lambda}$, can be obtained as positive square root of $\hat{\mu}$. Once $\hat{\lambda}$ is obtained, $\hat{\alpha}$ can be obtained from (4.4) as $\hat{\alpha}(\hat{\lambda})$.

The asymptotic normality results of the MLE's of $\theta = (\alpha, \lambda)$ can be stated as follows:

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow N_2(0, \mathbf{I^{-1}}(\theta)), \tag{4.10}$$

where $\mathbf{I}(\theta)$ is the Fisher Information matrix, *i.e.*,

$$\mathbf{I}(\theta) = -\frac{1}{n} \begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) \\ \\ E\left(\frac{\partial^2 L}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) \end{bmatrix} = - \begin{bmatrix} I_{11} & I_{12} \\ \\ I_{21} & I_{22} \end{bmatrix}.$$

Surles and Padgett (2001) showed that the GR family satisfies all the regularity conditions for $\alpha > 0$ and $\lambda > 0$. Further, they provided the elements of the Fisher Information matrix in terms of infinite series. Here, we provide alternative expressions in terms of digamma and polygamma functions, which may be easier to use in practice as they are readily available nowadays in most of the standard packages like MAPLE, MATHEMATIKA etc. It can be shown that for $\alpha > 2$

$$\begin{aligned} I_{11} &= -\frac{1}{\alpha^2} \\ I_{12} &= I_{21} = \frac{2}{\lambda^2(\alpha-1)}\left((\psi(\alpha) - \psi(1)) - \frac{\alpha-1}{\alpha}\right) \\ I_{22} &= -\frac{2}{\lambda^2} - \frac{2}{\lambda^2}\left[\psi(\alpha+1) - \psi(1)\right] - \frac{2\alpha}{\lambda^3}\left[\psi(1) - \psi(\alpha)\right] - \frac{2(\alpha-1)}{\lambda^3} \\ &\quad -\frac{4\alpha}{\lambda^3(\alpha-2)}\left[(\psi(2) - \psi(\alpha))^2 + \psi'(2) - \psi'(\alpha)\right] \end{aligned}$$

and for $0 < \alpha \leq 2$, we have

$$\begin{aligned} I_{11} &= -\frac{1}{\alpha^2} \\ I_{12} &= I_{21} = \frac{2\alpha}{\lambda^2}\int_0^\infty ye^{-2y}\left(1 - e^{-y}\right)^{\alpha-2} dy \\ I_{22} &= -\frac{2}{\lambda^2}\Bigg[1 + (\psi(\alpha+1) - \psi(1)) - \frac{\alpha(\alpha-1)}{\lambda} \times \\ &\quad \int_0^\infty ye^{-2y}(1 - 2y - e^{-y})(1 - e^{-y})^{\alpha-3} dy\Bigg] \end{aligned}$$

5 Statistical Inferences

It is well known that (Rao; 1973) the testing of hypotheses and confidence intervals problems are equivalent. In this section, we mainly consider these two problems simultaneously for shape and scale parameters. GR family does not belong to the exponential family like gamma family or it does not enjoy some of the nice properties of the extreme value distributions, like Weibull family. From the testing point of view, GR family is very similar to the GE family. We propose to use the likelihood ratio test (LRT) or the asymptotic normality results for testing purposes when both parameters are unknown. If the scale parameter is known there exists UMP test or UMPU test depending on the alternatives.

When both parameters are unknown, one can test the following hypotheses on the shape parameter α

$$H_0 : \alpha = \alpha_0 \quad H_1 : \alpha \neq \alpha_0. \tag{5.1}$$

Unfortunately, it is not possible to obtain the UMP test or UMPU test in this case as expected. We propose the LRT as follows:

$$\Lambda_1 = -2\left[L(\alpha_0, \tilde{\lambda}) - L(\hat{\alpha}, \hat{\lambda})\right], \tag{5.2}$$

where $\hat{\alpha}$, $\hat{\lambda}$ are the MLE's of α, λ respectively. Here, $\tilde{\lambda}$ is the restricted MLE of λ, obtained by maximizing $L(\alpha_0, \lambda)$, as defined in (4.1), with respect to λ. The asymptotic distribution of Λ_1 under H_0 is chi-square with one degree of freedom. Moreover, it can be seen (see Bain and Engelhardt; 1991) that the exact distribution of Λ_1 under H_0 does not depend on the scale parameter λ. It is also possible to use the asymptotic normality result (4.10), to test the hypothesis (5.1). Along the same line the testing on the scale parameter when the shape parameter is unknown, or simultaneous testing of both parameters are also possible.

When the scale parameter λ is known, the following hypotheses can be tested:

$$H_0 : \alpha = \alpha_0 \quad H_1 : \alpha \neq \alpha_0. \tag{5.3}$$

$$H_0 : \alpha = \alpha_0 \quad H_1 : \alpha > \alpha_0. \tag{5.4}$$

$$H_0 : \alpha = \alpha_0 \quad H_1 : \alpha < \alpha_0. \tag{5.5}$$

Without loss of generality, we consider $\lambda = 1$. The LRT for testing (5.5), takes the following form: Reject H_0 if

$$Y = -2\sum_{i=1}^{n} \ln(1 - e^{-X_i^2}) < c \tag{5.6}$$

and accept otherwise. It can be easily seen that under the null hypothesis $\alpha_0 Y$ has a chi-square distribution with $2n$ degrees of freedom. Therefore, the critical value of c can be easily obtained from the chi-square table. Since the GR family has the monotone likelihood ratio property, therefore the above test is a UMP test. Similarly, we can obtain UMP test for (5.4) and UMPU test for (5.3). Moreover, if λ is known, an exact 95% confidence interval for α can be obtained as

$$\left(\frac{\chi^2_{2n,0.025}}{-2\sum_{i=1}^n \ln(1-e^{-X_i^2})}, \frac{\chi^2_{2n,0.975}}{-2\sum_{i=1}^n \ln(1-e^{-X_i^2})} \right),$$

where $\chi^2_{2n,0.025}$ and $\chi^2_{2n,0.975}$ are the 2.5^{th} and 97.5^{th} percentiles of the χ^2 distributions with $2n$ degrees of freedom. It should be mentioned here that, when the shape parameter is known then there does not exist any UMP or UMPU test for testing the scale parameter. We may use the LRT as mentioned before.

6 Data Analysis and Discussions

In this section we present the analysis of a real data set using the GR model and compare it with the other fitted models like Weibull, gamma and GE models.

Example:: The following data represent the number of million revolution before failing for each of the 23 ball bearings in the life test (from Lawless, 1982, p.228).
17.88,28.92, 33.00, 41.52, 42.12, 45.60, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 174.40.

The GR model is used to fit this data set. We plot the profile likelihood function $g(\lambda)$ as defined in (4.5) in the Figure 3.

It is immediate from the Figure 3 that the profile likelihood surface is unimodal and therefore most of the iterative process should work. We have started the iterative procedure (4.9) with an initial guess $\lambda = 0.1$, far away from the solution. We have used $\epsilon = 10^{-6}$ and the iterative process stopped only after 6 iterations. We obtain $\hat{\lambda} = 0.0131$, $\hat{\alpha} = 1.19925$ and the corresponding log-likelihood value = -113.544.

To study the goodness of fit of the GR model, we compute the Kolmogorov-Smirnov statistic and it is 0.1565 with the corresponding p value = 0.6265. Therefore, the high p value clearly indicates that GR model can be used to analyze this data set. We plot the empirical survival function and the fitted survival function in Figure 4 and the P-P plot in Figure 5. Figure 4 indicates reasonable match between the empirical survival function and the fitted survival function. As can be seen from Figure 5 that the data do not deviate dramatically from the line.

The non-parametric Pearson goodness-of-fit test is also used. In order to apply the Pearson

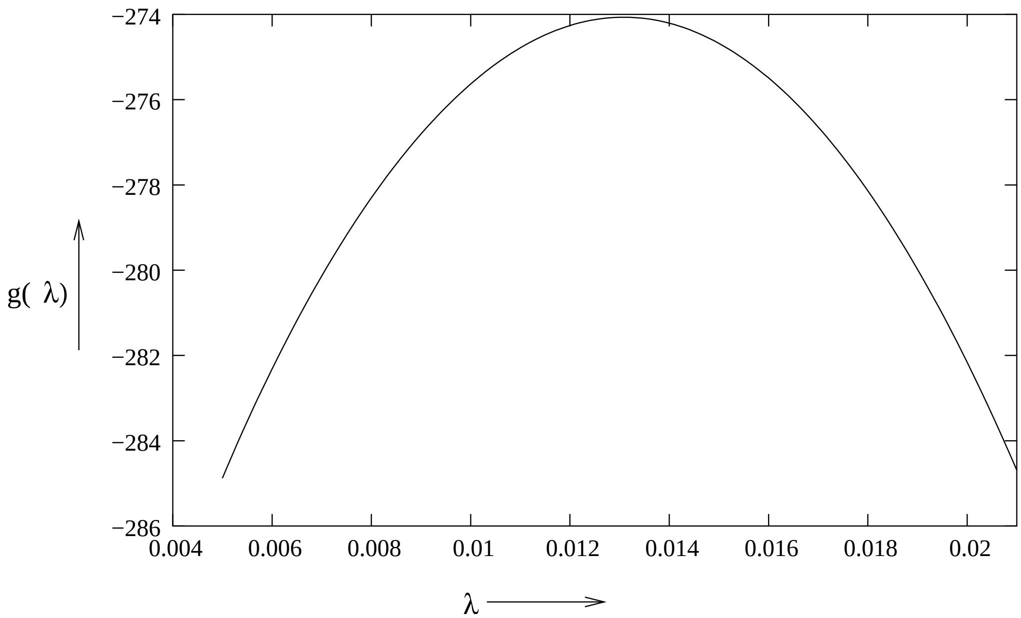

Figure 3: The profile likelihood function $g(\lambda)$ of the data

goodness of fit test to the data set, the support of the cumulative distribution function is divided into the following 5 intervals: [0,35], (35, 55], (55, 80], (80, 100], (100, ∞). Using the MLEs to estimate the GR parameters the test statistic is 1.745. Since the parameters were estimated using the MLEs, using the results of Chernoff and Lehmann (1954), it follows that under the null hypothesis the test statistic converges to a distribution between a chi-square distribution with $k-1$ degrees of freedom and a chi-square distribution with $k-s-1$ degrees of freedom, where k is the number of intervals and s is the number of unknown parameters. Since it is of interest to fail to reject the null hypothesis, based on the test statistic we can not reject the null hypothesis at the 5% level of significance because test statistic is less than the critical value $\chi^2_{2,.095} = 5.99$. Therefore, the null hypothesis that the data are from a GR distribution cannot be rejected.

In this article we consider the two-parameter GR model and study its various properties and develop some new inferential procedures. It is observed that the two-parameter GR family are quite similar in nature to the other two-parameter families like Weibull, gamma or GE family. This distribution has certain desirable properties. Unlike Weibull, gamma and GE model, the GR model can have bathtub shape hazard function. For the censored data, the GR model can be used very conveniently.

We obtained the asymptotic properties of the MLEs of the unknown parameters but we

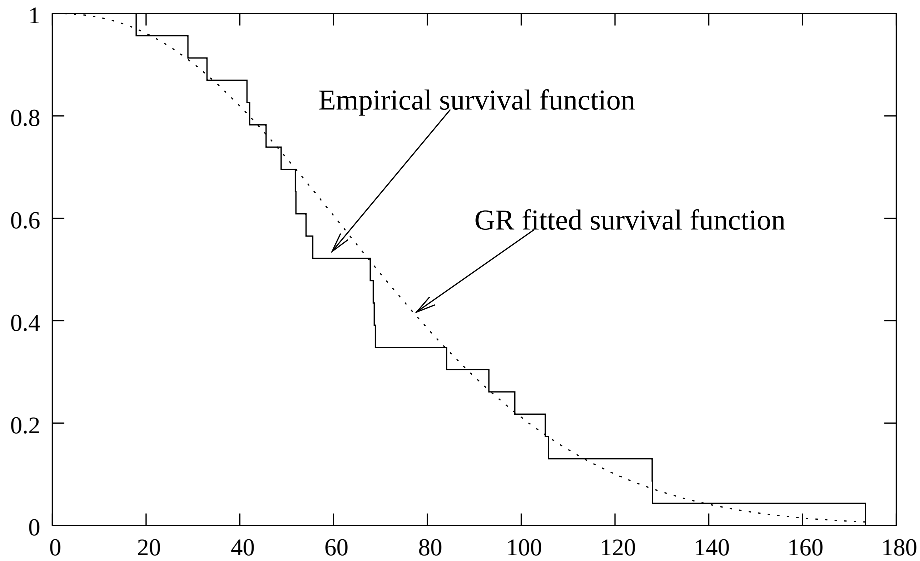

Figure 4: Empirical survival function and the fitted GR survival function of the data

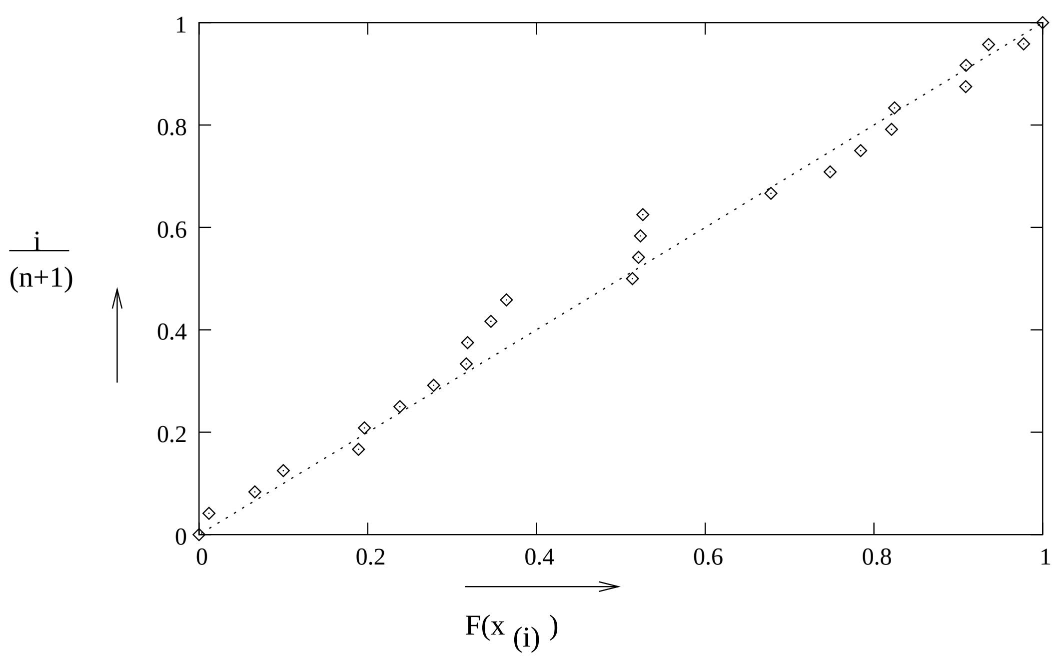

Figure 5: The P-P plot of the data

have not studied the behavior of the MLEs for small samples. Extensive simulation study is required to study the rate of convergence of the MLE's and their various properties. We will report on that in the near future.

Appendix

In the appendix we provide the proof of (3.9). Let

$$I(y) = \int_0^y x^2 (1 - e^{-x^2})^{(r+j)\alpha - 1} e^{-x^2} dx.$$

Using the binomial expansion, we obtain

$$\begin{aligned} I(y) &= \sum_{k=0}^{\infty} (-1)^k \binom{(r+j)\alpha - 1}{k} \int_0^y x^2 e^{-(k+1)x^2} dx \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{(r+j)\alpha - 1}{k}}{(k+1)^{\frac{3}{2}}} \int_0^{(k+1)y^2} u^{\frac{1}{2}} e^{-u} du. \end{aligned}$$

Using Gradshteyn and Ryzhik (2000) (pp 364), we have

$$\int_0^z x^{p-1} e^{-x} dx = \sum_{l=0}^{\infty} \frac{(-1)^l z^{p+l}}{l!(p+l)}.$$

Therefore,

$$I(y) = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \binom{(r+j)\alpha - 1}{k} (k+1)^l}{l!(\frac{3}{2} + l)} y^{3+2l}.$$

Now, for $1 \le r < s \le n$,

$$\begin{aligned} E(X_{r:n} X_{s:n}) &= 4\alpha^2 c_{r,s:n} \int_0^{\infty} \int_0^y x^2 y^2 (1 - e^{-x^2})^{r\alpha - 1} \left[(1 - e^{-y^2})^{\alpha} - (1 - e^{-x^2})^{\alpha} \right]^{s-r-1} \times \\ & \left[1 - (1 - e^{-y^2})^{\alpha} \right]^{n-s} (1 - e^{-y^2})^{\alpha - 1} e^{-x^2} e^{-y^2} dx dy \\ &= 4\alpha^2 c_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{n-s}{i} \binom{s-r-1}{j} \times \\ & \int_0^{\infty} y^2 (1 - e^{-y^2})^{(s-r-j+i)\alpha - 1} I(y) e^{-y^2} dy. \end{aligned}$$

It can be easily seen that

$$\int_0^{\infty} y^{5+2l} (1 - e^{-y^2})^{(s-r-j+i)\alpha - 1} e^{-y^2} dy = \frac{\mu^{2l+4}((s-r-j+i)\alpha)}{2(s-r-j+i)\alpha}.$$

Therefore, the result follows immediately.

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