On Hybrid Censored Weibull Distribution

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Abstract

A hybrid censoring is a mixture of Type-I and Type-II censoring schemes. This article presents the statistical inferences on Weibull parameters when the data are hybrid censored. The maximum likelihood estimators and the approximate maximum likelihood estimators are developed for estimating the unknown parameters. Asymptotic distributions of the maximum likelihood estimators are used to construct approximate confidence intervals. Bayes estimates and the corresponding highest posterior density credible intervals of the unknown parameters are obtained under suitable priors on the unknown parameters and using the Gibbs sampling procedure. The method of obtaining the optimum censoring scheme based on the maximum information measure is also developed. Monte Carlo simulations are performed to compare the performances of the different methods and one data set is analyzed for illustrative purposes.

Keywords: Maximum likelihood estimators; approximate maximum likelihood estimators; asymptotic distribution; Type-I censoring; Type-II censoring; Gibbs sampling, Optimum censoring scheme.

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1 Introduction

A hybrid censoring sampling scheme can be described as follows. Each unit in a randomly selected sample of n units is subjected to a life test under identical environmental conditions. The lifetimes of the sample units are independent and identically distributed (*i.i.d.*) random variables. The test is terminated when a pre-chosen number R, out of n items have failed or when a pre-determined time, T, on test has been reached. It is also assumed that the failed items are not replaced. A schematic illustration is depicted in Figure 1, when $y_{1:n} < \ldots < y_{R:n}$ denote the observed failure times if $y_{R:n} < T$ and $y_{1:n} < \ldots < y_{d:n}$ denote the observed failure times if $y_{d:n} < T$, $d < R$ and the $(d+1)^{th}$ failure does not take place before time point T.

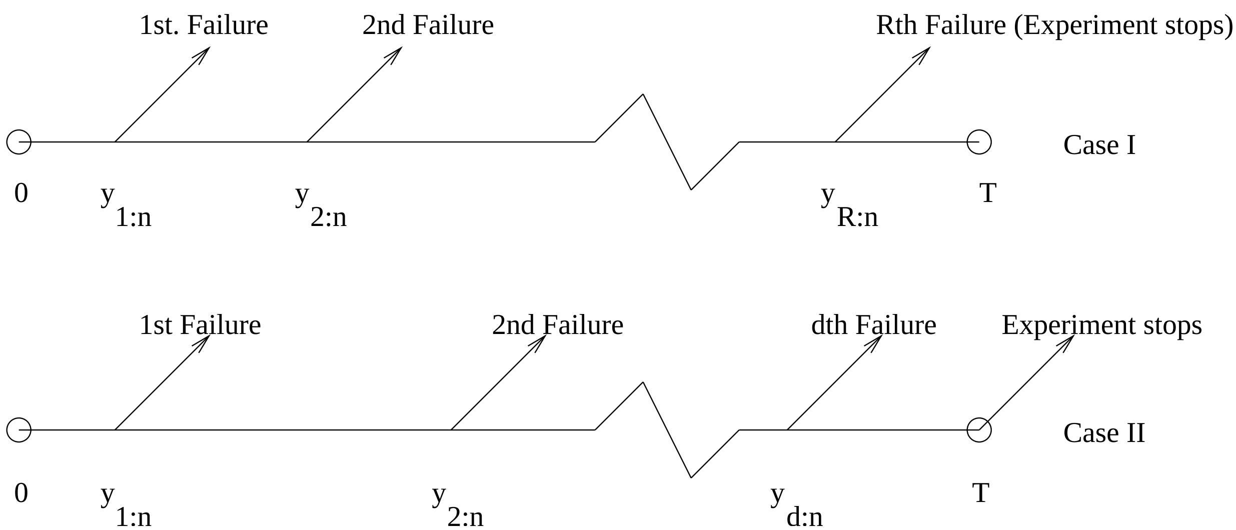

Figure 1: Schematic illustration of the hybrid censoring scheme

The mixture of Type-I and Type-II censoring schemes is known as hybrid censoring scheme and it is quite important in reliability acceptance test in MIL-STD-781C [20]. It may be noted that the familiar complete, Type-I and Type-II right censored samples are special cases of this hybrid scheme.

Epstein [12] introduced this sampling scheme and considered life testing situations where the lifetime X follows the exponential distribution with mean life θ. Epstein [12] proposed two sided confidence intervals for θ without any formal proof. Fairbanks *et al.* [13] modified slightly the proposition of Epstein [12] and suggested a simple set of confidence intervals. Chen and Bhattacharya [5] obtained the exact distribution of the conditional maximum likelihood estimator of θ and proposed a one-sided confidence interval. Draper and Guttman [9] considered this problem from the Bayesian point of view and obtained the two sided credible intervals of the mean lifetime using inverted gamma prior. Comparison of the different methods can be found in Gupta and Kundu [15]. For some related work, one may refer to Ebrahimi [10, 11], Jeong, Park and Yum [18] and Childs *et al.* [7].

In spite of the applicability of the hybrid censoring scheme, it is somewhat surprising to observe that limited attention has been paid in analyzing hybrid censored lifetime data when the lifetimes are not exponential. The main concern is, the analysis becomes too difficult and may not be tractable. Weibull distribution is one of the most common distributions which is used to analyze several lifetime data. The density function of the Weibull distribution can take different shapes and also its hazard function can be increasing, decreasing and constant depending on the shape parameter. Because of the shape parameter, it has lots of flexibility compared to exponential distribution.

In this article we consider the hybrid censored lifetime data, when the lifetime follows two-parameter Weibull distribution. The main aim of this paper is two fold. First we provide different methods to compute the point and interval estimations of the shape and scale parameters. We provide the maximum likelihood estimators of the unknown parameters. It is observed that the maximum likelihood estimators can be obtained by solving a non-linear equation and we propose a simple iterative scheme to solve the non-linear equation. We also suggest approximate maximum likelihood estimators, which have explicit expressions.

It is not possible to compute the exact distributions of the maximum likelihood estimators, and we use the asymptotic distribution to construct approximate confidence intervals of the unknown parameters.

We provide the Bayes estimates under the assumptions of independent gamma priors on the scale and shape parameters. It is observed that the Bayes estimates can not be computed explicitly, and we use the Gibbs sampling procedure to compute the Bayes estimates and also to compute the highest posterior density credible intervals. We compare the performances of the different methods by Monte Carlo simulations. One data set is analyzed for illustrative purposes.

The second aim of this paper is to provide a methodology to compare different sampling schemes and that can be used to find the optimum scheme. For a practitioner, it is quite important to design an efficient experiment. Based on the prior information about the unknown parameters, we describe a simple method to design the optimum censoring scheme which provides the maximum information about the unknown parameters.

The rest of the paper is organized as follows. In section 2, we describe the model and the available data. The maximum likelihood estimators and the approximate maximum likelihood estimators are provided in sections 3 and 4 respectively. Bayesian inferences are provided in section 5. Simulation results are presented in section 6. One data set is analyzed and the results are presented in section 7. In section 8 we provide the procedure to obtain the optimum censoring scheme. Finally we conclude the paper in section 9.

2 Model Description

Suppose the lifetime random variable Y has a Weibull distribution with the shape and scale parameters as α and λ respectively, *i.e.*, the probability density function (PDF) of Y is;

$$f_Y(y;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{y}{\lambda}\right)^{\alpha}}; \qquad y > 0, \tag{1}$$

where $\alpha > 0$, $\lambda > 0$ are the natural parameter space. If the random variable Y has the density function (1), then $X = \ln Y$ has the extreme value distribution with PDF;

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma} e^{\left\{\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}\right\}}; \qquad -\infty < x < \infty, \tag{2}$$

where $\mu = \ln\lambda$, $\sigma = \frac{1}{\alpha}$. The density function as described by (2) is known as the density function of an extreme value distribution with location and scale parameters as μ and σ respectively.

Models (1) and (2) are equivalent models in the sense, the procedure developed under one model can be easily used for the other model. Although, they are equivalent models, sometimes it is easier to work with the model (2) than (1), because in the model (2), the two parameters μ and σ appear as location and scale parameters. In fact, it is observed that the approximate maximum likelihood estimators can be obtained quite easily using model (2) than model (1). For $\mu = 0$ and $\sigma = 1$, the model (2) is known as the standard extreme value distribution and it has the following PDF;

$$f_Z(z,0,1) = e^{z-e^{z}}; \qquad -\infty < z < \infty. \tag{3}$$

Now we describe the data available under the hybrid censoring scheme. Note that, under the hybrid censoring scheme, it is assumed that R and T are known in advance. Therefore, under this censoring scheme we have one of the two following types of observations:

Case I: $\{y_{1:n} < \ldots < y_{R:n}\}$ if $y_{R:n} < T$.

Case II: $\{y_{1:n} < \ldots < y_{d:n}\}$ if $d < R$ and $y_{d:n} < T < y_{d+1:n}$.

It may be mentioned that although we do not observe $y_{d+1:n}$, but $y_{d:n} < T < y_{d+1:n}$ means that the $d-th$ failure took place before T and no failure took place between $y_{d:n}$ and T.

3 Maximum Likelihood Estimators

In this section we provide the maximum likelihood estimators (MLEs) of the unknown parameters. Based on the observed data, the likelihood function for Case I is

$$l(\alpha,\lambda) = \left(\frac{\alpha}{\lambda}\right)^R \prod_{i=1}^{R} \left(\frac{y_{i:n}}{\lambda}\right)^{\alpha-1} e^{-\left\{\sum_{i=1}^{R}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha}+(n-R)\times\left(\frac{y_{R:n}}{\lambda}\right)^{\alpha}\right\}}, \quad (4)$$

and for Case II, it is

$$\begin{aligned} l(\alpha,\lambda) &= \left(\frac{\alpha}{\lambda}\right)^d \prod_{i=1}^{d} \left(\frac{y_{i:n}}{\lambda}\right)^{\alpha-1} e^{-\left\{\sum_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha}+(n-d)\times\left(\frac{T}{\lambda}\right)^{\alpha}\right\}} \quad \text{if} \quad d>0, & (5)\\ &= e^{-n\left(\frac{T}{\lambda}\right)^{\alpha}} \quad \text{if} \quad d=0. & (6)\end{aligned}$$

The logarithm of (4) and (5), can be written as

$$L(\alpha,\lambda) = R\ln\alpha - R\ln\lambda + (\alpha-1)\sum_{i=1}^{R}\ln y_{i:n} - (\alpha-1)R\ln\lambda - \sum_{i=1}^{R}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha} - (n-R)\left(\frac{y_{R:n}}{\lambda}\right)^{\alpha}, \quad (7)$$

and

$$L(\alpha,\lambda) = d\ln\alpha - d\ln\lambda + (\alpha-1)\sum_{i=1}^{d}\ln y_{i:n} - d(\alpha-1)\ln\lambda - \sum_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha} - (n-d)\left(\frac{T}{\lambda}\right)^{\alpha}, \quad (8)$$

respectively. Taking derivatives with respect to α and λ of (7) and putting then equal to zero we obtain

$$\frac{\partial L}{\partial\lambda} = -\frac{\alpha R}{\lambda} + \frac{\alpha}{\lambda^{\alpha+1}}\left[\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R)y_{R:n}^{\alpha}\right] = 0, \quad (9)$$

$$\frac{\partial L}{\partial\alpha} = \frac{R}{\alpha} - R\ln\lambda + \sum_{i=1}^{R}\ln y_{i:n} - \sum_{i=1}^{R}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha}(\ln y_{i:n} - \ln\lambda) - (n-R)\left(\frac{y_{R:n}}{\lambda}\right)^{\alpha}(\ln y_{R:n} - \ln\lambda) = 0. \quad (10)$$

From (9), we obtain

$$\lambda^{\alpha} = \frac{\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R)y_{R:n}^{\alpha}}{R} = u(\alpha) \quad \text{(say)}. \quad (11)$$

Using (11), (10) can be re-written as

$$\frac{R}{\alpha} = \frac{R}{\alpha}\ln u(\alpha) - \sum_{i=1}^{R}\ln y_{i:n} + \sum_{i=1}^{R}\frac{y_{i:n}^{\alpha}}{u(\alpha)} \times \left(\ln y_{i:n} - \frac{1}{\alpha}\ln u(\alpha)\right)$$
$$+(n-R)\frac{y_{R:n}^{\alpha}}{u(\alpha)} \times \left(\ln y_{R:n} - \frac{1}{\alpha}\ln u(\alpha)\right)$$

or

$$\frac{1}{\alpha}\left[R - R\ln u(\alpha) + \sum_{i=1}^{R}\frac{y_{i:n}^{\alpha}}{u(\alpha)} \times \ln u(\alpha) + (n-R)\frac{y_{R:n}^{\alpha}}{u(\alpha)}\ln u(\alpha)\right] \tag{12}$$

$$= -\sum_{i=1}^{R}\ln y_{i:n} + \sum_{i=1}^{R}\frac{y_{i:n}^{\alpha}}{u(\alpha)} \times \ln y_{i:n} + (n-R)\frac{y_{R:n}^{\alpha}}{u(\alpha)} \times \ln y_{R:n}. \tag{13}$$

Note that (13) can be written in the form:

$$\alpha = h(\alpha), \tag{14}$$

where

$$h(\alpha) = \frac{R(1-\ln u(\alpha)) + \frac{\ln u(\alpha)}{u(\alpha)}\left(\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R)y_{R:n}^{\alpha}\right)}{-\sum_{i=1}^{R}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{R} y_{i:n}^{\alpha}\ln y_{i:n} + (n-R)y_{R:n}^{\alpha}\ln y_{R:n}\right]}$$
$$= \frac{R}{-\sum_{i=1}^{R}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{R} y_{i:n}^{\alpha}\ln y_{i:n} + (n-R)y_{R:n}^{\alpha}\ln y_{R:n}\right]}.$$

We propose a simple iterative scheme to solve for α from (14). Start with an initial guess of α, say $\alpha^{(0)}$, obtain $\alpha^{(1)} = h(\alpha^{(0)})$ and proceeding in this way obtain $\alpha^{(n+1)} = h(\alpha^{(n)})$. Stop the iterative procedure, when $|\alpha^{(n+1)} - \alpha^{(n)}| < \epsilon$, some pre-assigned tolerance limit. In case of (8), the likelihood equation can be re-written as

$$\lambda^{\alpha} = \frac{\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha}}{d} = u(\alpha),$$

and $\alpha = h(\alpha)$, where

$$h(\alpha) = \frac{d(1-\ln u(\alpha)) + \frac{\ln u(\alpha)}{u(\alpha)}\left(\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha}\right)}{-\sum_{i=1}^{d}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}\ln y_{i:n} + (n-d)T^{\alpha}\ln T\right]}$$
$$= \frac{d}{-\sum_{i=1}^{d}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}\ln y_{i:n} + (n-d)T^{\alpha}\ln T\right]}.$$

Similar procedure as above can be used to solve for α. Note that when $d = 0$, the maximum likelihood estimators do not exist. Since the MLEs when they exist, are not in compact forms, we propose the following approximate MLEs and they have explicit expressions.

4 Approximate Maximum Likelihood Estimators

Let us use the following notations; $x_{i:n} = \ln y_{i:n}$ and $S = \ln T$. Therefore, the likelihood equation of the observed data $x_{i:n}$ for Case I is

$$l(\mu, \sigma) = \frac{c}{\sigma^R} \prod_{i=1}^{R} g(z_{i:n}) \left(\bar{G}(z_{R:n})\right)^{n-R}, \tag{15}$$

and for Case II, it is

$$l(\mu, \sigma) = \frac{c}{\sigma^d} \prod_{i=1}^{d} g(z_{i:n}) \left(\bar{G}(V)\right)^{n-d}, \tag{16}$$

where

$$g(x) = e^{x-e^x}, \quad \bar{G}(x) = e^{-e^x}, \quad z_{i:n} = \frac{x_{i:n} - \mu}{\sigma}, \quad V = \frac{S-\mu}{\sigma}, \quad \mu = \ln \lambda, \quad \sigma = \frac{1}{\alpha}.$$

First consider (15), ignoring the constant, taking the log-likelihood we obtain,

$$L(\mu, \sigma) = \ln(l(\mu, \sigma)) = -R \ln \sigma + \sum_{i=1}^{R} \ln(g(z_{i:n})) + (n-R) \ln(\bar{G}(z_{R:n})). \tag{17}$$

Taking derivatives with respect to μ and σ of $L(\mu, \sigma)$, gives,

$$\frac{\partial L}{\partial \mu} = -\left(\frac{1}{\sigma}\right) \sum_{i=1}^{R} \frac{g'(z_{i:n})}{g(z_{i:n})} + (n-R) \times \left(\frac{1}{\sigma}\right) \times \frac{g(z_{R:n})}{\bar{G}(z_{R:n})} = 0, \tag{18}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{R}{\sigma} - \sum_{i=1}^{R} \frac{g'(z_{i:n})}{g(z_{i:n})} \times \frac{z_{i:n}}{\sigma} + (n-R) \frac{g(z_{R:n})}{\bar{G}(z_{R:n})} \times \frac{z_{R:n}}{\sigma} = 0. \tag{19}$$

Note that (18) and (19) can be written equivalently

$$-\sum_{i=1}^{R} \frac{g'(z_{i:n})}{g(z_{i:n})} + (n-R) \frac{g(z_{R:n})}{\bar{G}(z_{R:n})} = 0, \tag{20}$$

$$-R - \sum_{i=1}^{R} \frac{g'(z_{i:n})}{g(z_{i:n})} \times z_{i:n} + (n-R) \times \frac{g(z_{R:n}) z_{R:n}}{\bar{G}(z_{R:n})} = 0. \tag{21}$$

Clearly, (20) and (21) do not have explicit solutions. We expand the function $\frac{g'(z_{i:n})}{g(z_{i:n})}$ and $\frac{g(z_{R:n})}{\bar{G}(z_{R:n})}$ in Taylor series around the points $G^{-1}(p_i) = \ln(-\ln q_i) = \mu_i$ (say) and $G^{-1}(p_R) =$

$\ln(-\ln q_R) = \mu_R$ respectively, where $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$ for $i = 1, \ldots, R.$, similarly as Balasooriya and Balakrishnan [3], see for reasoning Arnold *et al.* [1]. Note that

$$\frac{g'(z_{i:n})}{g(z_{i:n})} \approx \alpha_i - \beta_i z_{i:n}, \tag{22}$$

$$\frac{g(z_{R:n})}{\bar{G}(z_{R:n})} \approx 1 - \alpha_R + \beta_R z_{R:n}, \tag{23}$$

where for $i = 1, \ldots, R$

$$\begin{aligned} \alpha_i &= \frac{g'(\mu_i)}{g(\mu_i)} - \mu_i \left[\frac{g''(\mu_i)}{g(\mu_i)} - \left(\frac{g'(\mu_i)}{g(\mu_i)} \right)^2 \right] \\ &= 1 + \ln q_i (1 - \ln(-\ln q_i)), \end{aligned} \tag{24}$$

$$\beta_i = \left[\left(\frac{g'(\mu_i)}{g(\mu_i)} \right)^2 - \frac{g''(\mu_i)}{g(\mu_i)} \right] = -\ln q_i. \tag{25}$$

Using the approximation (22) and (23) in (20) and (21), we obtain

$$-\sum_{i=1}^{R} (\alpha_i - \beta_i z_{i:n}) + (n-R)\,(1 - \alpha_R + \beta_R z_{R:n}) = 0, \tag{26}$$

$$-R - \sum_{i=1}^{R} (\alpha_i - \beta_i z_{i:n}) z_{i:n} + (n-R)(1 - \alpha_R + \beta_R z_{R:n}) z_{R:n} = 0. \tag{27}$$

From (26) we obtain solution of $\hat{\mu}_I$ as

$$\hat{\mu}_I = A_I - B_I \hat{\sigma}_I, \tag{28}$$

where

$$A_I = \frac{\sum_{i=1}^{R} \beta_i x_{i:n} + \beta_R (n-R) x_{R:n}}{\sum_{i=1}^{R} \beta_i + \beta_R (n-R)}, \tag{29}$$

$$B_I = \frac{\sum_{i=1}^{R} \alpha_i - (n-R)(1-\alpha_R)}{\sum_{i=1}^{R} \beta_i + \beta_R (n-R)}. \tag{30}$$

From (27), we obtain $\hat{\sigma}_I$ as a solution of the quadratic equation

$$C_I \sigma^2 + D_I \sigma - E_I = 0, \tag{31}$$

where

$$C_I = R + B_I \sum_{i=1}^{R} \alpha_I - B_I (n-R)(1-\alpha_R) - B_I^2 \sum_{i=1}^{R} \beta_i - B_I^2 (n-R) \beta_R = R$$

$$D_I = \sum_{i=1}^{R} \alpha_i (x_{i:n} - A_I) - (n-R)(1-\alpha_R)(x_{R:n} - A_I) - 2B_I \sum_{i=1}^{R} \beta_i (x_{i:n} - A_I) -$$
$$2(n-R)\beta_R B_I (x_{R:n} - A_I)$$
$$E_I = \sum_{i=1}^{R} \beta_i (x_{i:n} - A_I)^2 + (n-R)\beta_R (x_{R:n} - A_I)^2 > 0.$$

Therefore,

$$\hat{\sigma}_I = \frac{-D_I + \sqrt{D_I^2 + 4RE_I}}{2R}$$

is the only positive root.

Similarly for case II, the estimate of μ and σ can be obtained by solving

$$-\sum_{i=1}^{d} \frac{g'(z_{i:n})}{g(z_{i:n})} + (n-d)\frac{g(V)}{\bar{G}(V)} = 0, \quad (32)$$

$$-d - \sum_{i=1}^{d} \frac{g'(z_{i:n})}{g(z_{i:n})} \times z_{i:n} + (n-d) \times \frac{g(V)V}{\bar{G}(V)} = 0. \quad (33)$$

In this case, we expand $\frac{g'(z_{i:n})}{g(z_{i:n})}$ and $\frac{g(V)}{\bar{G}(V)}$ in Taylor series around the points μ_i and $G^{-1}\left(\frac{p_d+p_{d+1}}{2}\right) = \ln\left(-\ln\left(1-\frac{p_d+p_{d+1}}{2}\right)\right)$ respectively. Let us denote $p_{d^*} = \frac{p_d+p_{d+1}}{2}$, $q_{d^*} = 1-p_{d^*}$ and $\ln(-\ln q_{d^*}) = \mu_{d^*}$, $\beta_{d^*} = -\ln q_{d^*}$ and $\alpha_{d^*} = 1 + \ln q_{d^*} \times (1 - \ln(-\ln q_{d^*}))$.

Therefore, in this case following the same steps as before, we obtain the solutions of $\hat{\mu}_{II}$ and $\hat{\sigma}_{II}$ as

$$\hat{\mu}_{II} = A_{II} - B_{II}\hat{\sigma}_{II}, \quad (34)$$

where

$$A_{II} = \frac{\sum_{i=1}^{d} \beta_i x_{i:n} + \beta_{d^*}(n-d)S}{\sum_{i=1}^{d} \beta_i + \beta_{d^*}(n-d)},$$
$$B_{II} = \frac{\sum_{i=1}^{d} \alpha_i - (n-d)(1-\alpha_{d^*})}{\sum_{i=1}^{d} \beta_i + \beta_{d^*}(n-d)}$$

and

$$\hat{\sigma}_{II} = \frac{-D_{II} + \sqrt{D_{II}^2 + 4dE_{II}}}{2d},$$

where

$$D_{II} = \sum_{i=1}^{d} \alpha_i (x_{i:n} - A_{II}) - (n-d)(1-\alpha_{d^*})(S - A_{II}) - 2B_{II} \sum_{i=1}^{d} \beta_i (x_{i:n} - A_{II}) -$$
$$2(n-d)\beta_R B_{II}(S - A_{II}),$$
$$E_{II} = \sum_{i=1}^{d} \beta_i (x_{i:n} - A_{II})^2 + (n-d)\beta_{d^*}(S - A_{II})^2 > 0.$$

5 Bayes Estimators and Credible Intervals

In this section we consider the Bayes estimations of the unknown parameters and also constructions of the credible intervals. We re-parametrize the model as follows $\theta = \frac{1}{\lambda^\alpha}$. Based on the new parametrization, we consider the Bayes estimates of α and θ.

5.1 Prior and Posterior Distributions

Following the approach of Berger and Sun [4] it is assumed that θ has a gamma prior, $Gamma(a, b)$, for $a, b > 0$, *i.e.*

$$\pi_1(\theta) \propto \theta^{a-1} e^{-b\theta}; \qquad \theta > 0. \tag{35}$$

No specific form of priors $\pi_2(\alpha)$ on α is assumed here. It is only assumed that the support of $\pi_2(\alpha)$ is $(0, \infty)$ and it is independent of θ. Based on the above prior assumptions, the joint density functions of the *data*, α and θ for the two different cases become;

Case 1:

$$l(data, \alpha, \theta) \propto \alpha^R \theta^{a+R-1} \prod_{i=1}^{R} y_{i:n}^{\alpha-1} e^{-\theta\left[\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R) y_{R:n}^{\alpha} + b\right]} \pi_2(\alpha). \tag{36}$$

Case 2:

$$l(data, \alpha, \theta) \propto \alpha^d \theta^{a+d-1} \prod_{i=1}^{d} y_{i:n}^{\alpha-1} e^{-\theta\left[\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d) T^{\alpha} + b\right]} \pi_2(\alpha); \qquad \text{if } d > 0 \tag{37}$$
$$l(data, \alpha, \theta) \propto \theta^{a-1} e^{-\theta(nT^\alpha + b)} \pi_2(\alpha); \qquad \text{if } d = 0. \tag{38}$$

Based on $l(data, \alpha, \theta)$, we obtain the joint posterior density function of α and θ given the $data$ as

$$l(\alpha, \theta | data) = \frac{l(data, \alpha, \theta)}{\int_0^\infty \int_0^\infty l(data, \alpha, \theta) d\alpha d\theta}. \qquad (39)$$

Since (39) can not be computed analytically, we adopt the Gibbs sampling procedures to compute Bayes estimates of α and λ. To perform the Gibbs sampling procedure, we further assume that $\pi_2(\alpha)$ is log-concave. It may be mentioned that the well known distributions like Weibull and gamma have log-concave density functions if the corresponding shape parameters are greater than or equal to one, whereas normal and log-normal have always log-concave density functions.

Now we consider the posterior density functions of α and θ. We need the following results;

THEOREM 1: The conditional density function of θ given α and $data$ for Case 1 is

$$\pi_1(\theta | \alpha, data) = Gamma(a + R, \sum_{i=1}^{R} y_{i:n}^{\alpha} + (n - R) y_{R:n}^{\alpha} + b).$$

For Case 2;

$$\pi_1(\theta | \alpha, data) = Gamma(a + d, \sum_{i=1}^{d} y_{i:n}^{\alpha} + (n - d) T^{\alpha} + b)$$

if $d > 0$ and for $d = 0$,

$$\pi_1(\theta | \alpha, data) = Gamma(a, n T^{\alpha} + b).$$

PROOF: It simply follows from the joint density function.

THEOREM 2: The conditional density function of α given the $data$ is log-concave.

PROOF: See in the appendix A.

Comments: It easily follows from the proof of theorem 2 that if the prior distribution of α is gamma (the shape parameter can be less than one or can be zero also) then the posterior density function of α is log-concave if $R \geq 1$.

We use the method proposed by Devroye [8] to generate sample from a log-concave density function for Gibbs sampling purposes. The details are provided below.

5.2 Bayes Estimate and Credible Interval

We use the idea of Geman and Geman [14] to generate samples from the conditional posterior density function. We adopt the following scheme:

- Step 1: Generate α_1, from the log-concave density $l(.|data)$, as given in (43), (45) or (46) depending on the situation.
- Step 2: Generate θ_1 from $\pi_1(.|\alpha, data)$ as provided in Theorem 1.
- Step 3: Repeat Steps 1 and 2, M times and obtain α_i, θ_i for $i = 1, \ldots, M$.

Now the approximate posterior means and posterior variances of α and θ can be easily obtained. Based on the generated M α and θ values and using the method proposed by Chen and Shao [6] the approximate highest posterior density (HPD) credible intervals of α and θ can be easily constructed.

6 Numerical Experiments

In this section we present some simulation results to compare the performances of the different methods proposed in the previous sections. We mainly compare the performances of the MLEs, AMLEs and Bayes estimators of the unknown parameters, in terms of their biases and mean squared errors (MSEs) for different censoring schemes. We also compare the average lengths of the asymptotic confidence intervals and credible intervals and their coverage percentages.

All the computations are performed at IIT Kanpur in Pentium IV processor using FORTRAN-77 program. In all cases we use the random deviate generator RAN2 proposed in Press *et al.* [21]. Since λ is the scale parameter, we have taken in all cases $\lambda = 1$ with out loss of generality. For simulation purposes, we present the results when T is of the form $T^{\frac{1}{\alpha}}$. The reason to choose T in that form is the following; if $\hat{\alpha}$ represents the MLE or AMLE of α, then the distribution of $\frac{\hat{\alpha}}{\alpha}$ becomes independent of α in that case for $\lambda = 1$. For that purpose we report the result only for $\alpha = 1$ without loss of generality. But these results can be used for any other α also and it will be explained later in details.

We have considered different N, R and T values. In each case we compute the MLEs, AMLEs and also the Bayes estimates of the unknown parameters. For computing the Bayes estimators, it is assumed that α and θ have $Gamma(a, b)$ and $Gamma(\gamma, \delta)$ priors respectively. Moreover we use the non-informative priors of both α and θ, by considering $a = b = \gamma = \delta = 0$. The Bayes estimators are computed under the squared error loss function and with respect to the above non-proper priors.

We compute the 95% asymptotic confidence intervals based on MLEs and replacing the MLEs by AMLEs. For comparison purposes, we also compute the 95% HPD credible intervals from the Gibbs samples. We replicate the process 1000 times and report the average estimates, the MSEs, the average confidence/ credible lengths and coverage percentages. The results are reported in the following Tables 1 - 4 [1].

From the Tables 1 - 4 the following general observations can be made. For all the methods, (i) for fixed N and R when T increases from 1 to 2, the MSEs decrease, (ii) for fixed R and T as N increases from 30 to 40 the MSEs decrease, (iii) for fixed N and T as R increases the

[1] Correspond to each method, the first figure represents the average estimates of α and the corresponding MSEs are reported within brackets. The second figure represents the average confidence/ credible length and the corresponding coverage percentages are reported within bracket. Similarly, the second line represents the results corresponding to λ

Table 1: The average biases, the mean squared errors, average confidence/ credible lengths and coverage percentages for $N = 30$ & $T = 1.0$

	$R = 20$	$R = 25$	$R = 30$
MLE	1.078 (0.068), 0.794(93) 1.024 (0.067), 0.875(92)	1.057 (0.059), 0.687(91) 1.031 (0.063), 0.758(90)	1.057 (0.059), 0.616 (88) 1.032 (0.062), 0.663 (87)
AMLE	1.704 (0.066), 0.790(93) 1.000 (0.075), 0.833(90)	1.050 (0.057), 0.684(91) 1.017 (0.065), 0.739(89)	1.050 (0.057), 0.610 (88) 1.017 (0.065), 0.645 (85)
BAYES	1.074 (0.075), 0.932(96) 1.019 (0.094), 0.927(91)	1.021 (0.054), 0.861(95) 1.032 (0.079), 0.929(92)	1.028 (0.051), 0.900 (98) 1.025 (0.062), 0.923 (96)

MSEs decrease. The performances of the MLEs and AMLEs are very similar in all aspects. The MSEs of the Bayes estimators are marginally larger than the MLEs or AMLEs for small R but for large R they are the other way. The average credible lengths are larger than the average confidence lengths in all the cases considered, but the coverage percentages of the credible intervals are usually larger than the confidence intervals in most cases considered. Finally, it should be mentioned that Bayes estimates are most computationally expensive followed by MLEs and AMLEs.

Now, we explain how we can use the results in Tables 1 - 4 for any other α values also. For example when $\alpha = 2$, then for $N = 30$, $R = 20$ and $T = 1$ (Table 1), the average MLEs of α will be 2×1.078, the MSE will be 4×0.068, the average 95% confidence length will be 2×0.794 and the coverage percentage will be 0.93. Similarly, when $\alpha = 2$, then for $N = 30$, $R = 20$ and $T = 2^{\frac{1}{2}}$ (Table 3), the average MLEs of α will be 2×1.093, the MSEs will be 4×0.064, the average 95% average confidence length will be 2×0.802 and the coverage percentage will be 0.95.

Table 2: The average biases, the mean squared errors, average confidence/ credible lengths and coverage percentages for $N = 40$ and $T = 1.0$

	$R = 25$	$R = 30$	$R = 40$
MLE	1.059 (0.048), 0.702(94) 1.021 (0.053), 0.791(94)	1.039 (0.040), 0.625(92) 1.029 (0.050), 0.707(93)	1.037 (0.040), 0.527 (88) 1.030 (0.049), 0.564 (87)
AMLE	1.052 (0.047), 0.699(94) 1.000 (0.060), 0.755(90)	1.033 (0.040), 0.622(92) 1.020 (0.052), 0.693(91)	1.032 (0.039), 0.527 (89) 1.021 (0.051), 0.556 (86)
BAYES	1.047 (0.051), 0.755(93) 1.388 (0.071), 0.876(92)	1.043 (0.043), 0.779(97) 1.036 (0.059), 0.793(91)	1.041 (0.035), 0.774 (96) 1.020 (0.042), 0.810 (93)

7 Data Analysis

In this section, we present the data analysis of the strength data reported by Badar and Priest [2] for illustrative purposes. The data represent the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbons fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purposes, we will be considering the single fibers of 20 mm in gauge length with sample size $n = 69$. We subtract 0.75 from the uncensored data and consider the following two sampling schemes.

Scheme 1: $R = 50$ and $T = 2.5$

Scheme 2: $R = 25$ and $T = 1.5$.

To compute the MLE of α in all cases in this section, we have used the iterative process described in section 3. We have used the initial estimate of $\alpha = 1$ and stopped the iterative process when the difference between two consecutive iterates is less than 10^{-6}. Based on the uncensored sample the maximum likelihood estimators of α and $\theta = \dfrac{1}{\lambda^{\alpha}}$ are 3.8436

Table 3: The average biases, the mean squared errors, average confidence/ credible lengths and coverage percentages for $N = 30$ and $T = 2.0$

	$R = 20$	$R = 25$	$R = 30$
MLE	1.093 (0.064), 0.802(95) 1.004 (0.052), 0.881(93)	1.062 (0.050), 0.676(95) 1.009 (0.045), 0.785(93)	1.042 (0.034), 0.599 (93) 1.014 (0.043), 0.715 (92)
AMLE	1.084 (0.063), 0.801(95) 0.977 (0.061), 0.835(89)	1.054 (0.049), 0.676(95) 1.001 (0.049), 0.773(91)	1.036 (0.034), 0.596 (93) 1.017 (0.046), 0.712 (92)
BAYES	1.041 (0.060), 0.827(93) 1.017 (0.067), 0.918(94)	1.011 (0.051), 0.712(96) 1.035 (0.052), 0.793(97)	1.036 (0.038), 0.682 (95) 0.977 (0.044), 0.759 (95)

and 0.0883 respectively. The approximate maximum likelihood estimators of α and θ are 3.8508 and 0.0884. For computing the Bayes estimators, we mainly consider squared error loss functions and gamma priors on both α and θ same as the previous section. Based on the above assumptions we obtain the Bayes estimators of α and θ as 3.8438 and 0.0874 respectively. We plotted the three density functions in Figure 2 and they are almost identical.

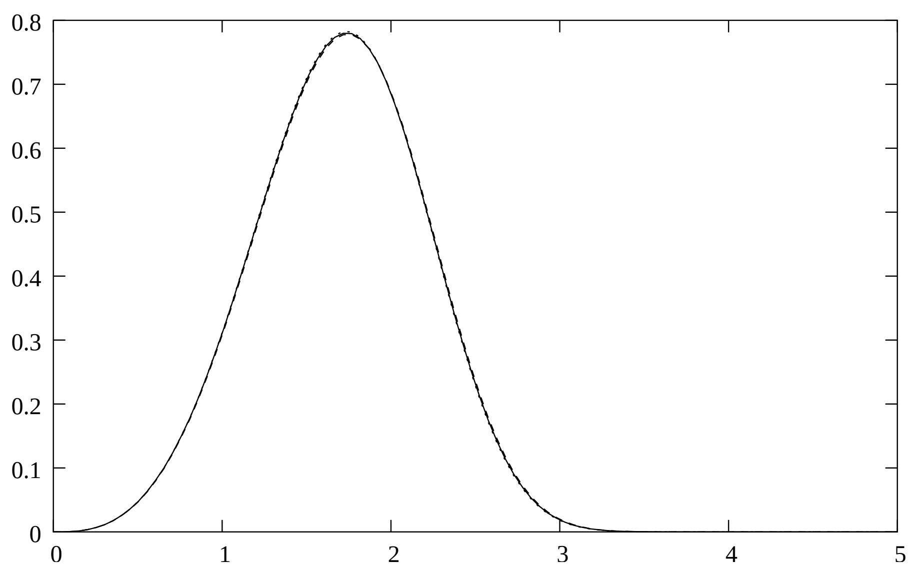

Figure 2: Three fitted density functions for complete data

Table 4: The average biases, the mean squared errors, average confidence/ credible lengths and coverage percentages for $N = 40$ and $T = 2.0$

	$R = 25$	$R = 30$	$R = 40$
MLE	1.070 (0.047), 0.706(95) 1.006 (0.042), 0.783(94)	1.056 (0.034), 0.631(96) 1.005 (0.036), 0.712(94)	1.029 (0.023), 0.517 (94) 1.014 (0.031), 0.617 (94)
AMLE	1.064 (0.045), 0.703(95) 0.983 (0.049), 0.742(90)	1.050 (0.034), 0.629(96) 0.995 (0.039), 0.697(92)	1.025 (0.023), 0.517 (94) 1.013 (0.035), 0.620 (94)
BAYES	1.066 (0.047), 0.781(96) 1.063 (0.061), 0.896(97)	1.029 (0.022), 0.696(97) 1.018 (0.035), 0.743(95)	1.030 (0.026), 0.594 (96) 1.026 (0.028), 0.694 (97)

Therefore, all the three methods produce very similar estimates for the unknown parameters for uncensored samples. To check how good is the Weibull fit, we compute the Kolmogorv-Smirnov distance and the corresponding p value and they are 0.0461 and 0.9985 respectively. Therefore, Weibull provides a very good fit to the uncensored data.

For Scheme 1, it is observed that it is from Case 1 and we obtain the MLEs, AMLEs and Bayes estimates of α and θ as (4.0468,0.0807), (4.0362,0.0813), (4.0848,0.0825) respectively. The 95% confidence intervals of α and θ based on the empirical Fisher information matrix and using the MLEs are (3.0510,5.0426) and (0.0275,0.1338) respectively. Similarly, using the AMLEs the corresponding confidence intervals are (3.0412,5.0311) and (0.0277,0.1349). We also compute the 95% highest posterior density (HPD) credible intervals of α and θ and they are (3.0806,5.2908) and (0.0313,0.1399) respectively. We plotted the three fitted density functions in Figure 3.

For Scheme 2, it is observed that the data come from Case 2 and $d = 19$. Based on the sample we obtain the MLEs, AMLEs and Bayes estimates of α and θ as (3.1933,0.0894), (3.1870,0.0898) and (3.2178,0.0918) respectively. The corresponding 95% confidence intervals

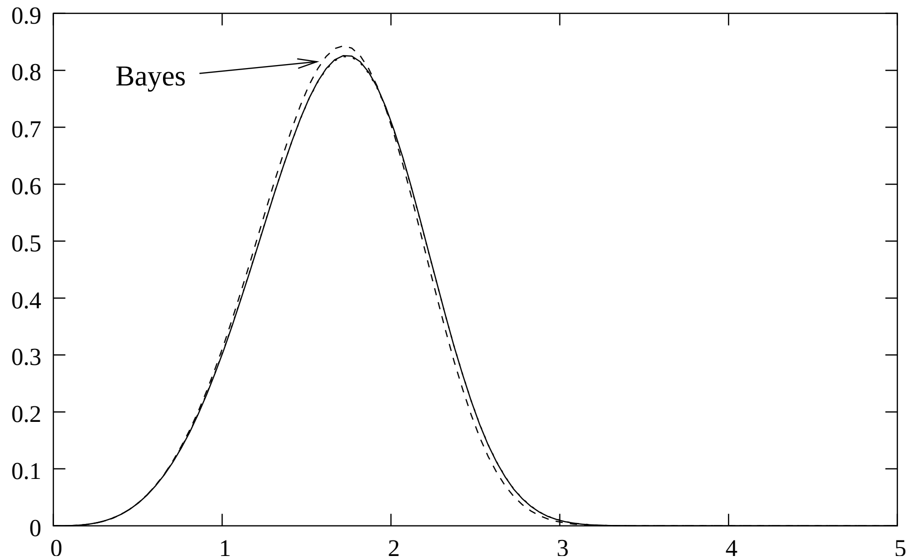

Figure 3: Three fitted density functions for Scheme 1 data

of α and θ based on the MLEs are (1.8234,4.5631) and (0.0388,0.1399). Similarly, based on AMLEs the intervals are (1.8187,4.5553) and (0.0389,0.1406). The 95% HPD credible intervals of α and λ are (1.9173,5.1083) and (0.0378,0.1733) respectively.

From the Figures 2, 3, 4 it is observed that for complete sample, the three fitted density functions are almost identical. For the censored observation although the MLEs and AMLEs produce almost identical fitted density functions but the fitted density function based on Bayes estimators is marginally different than the other two. Comparing the MLEs and AMLEs it is quite clear that for all practical purposes AMLEs can be used to avoid computations/iterative process.

Although for both the schemes, the MLEs, AMLEs and Bayes estimators are almost identical, but the lengths of the confidence intervals are slightly smaller than the corresponding credible intervals. Similar findings were observed in case of exponential distribution also, see Gupta and Kundu [15]. Comparing the two schemes, it is observed that for Scheme 1, all the estimators have smaller standard errors than Scheme 2, as expected.

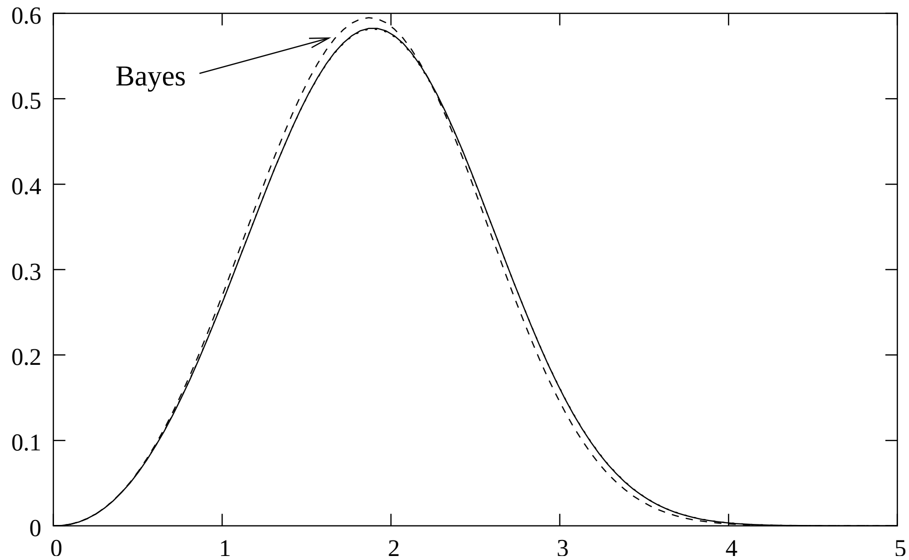

Figure 4: Three fitted density functions for Scheme 2 data

8 Optimum Censoring Scheme

In practice, it is quite important to choose the 'optimum' censoring scheme from a class of possible schemes. Here possible schemes mean, for fixed sample size 'n', different choices of R and T. In this section it is assumed that n is fixed and a particular censoring scheme will be denoted by (R, T). Now, to compare two different schemes, say (R_1, T_1) and (R_2, T_2), the scheme (R_1, T_1) is *better* than (R_2, T_2), if (R_1, T_1) provides more *information* than (R_2, T_2) about the unknown parameters. In this respect comparing the two Fisher information matrices is a very natural choice. If only one parameter is unknown then it boils down to compare two real numbers. But if both the parameters are unknown, then comparing the two Fisher information matrices is not a trivial task. Some of the existing choices in this case are to compare the traces or the determinants of the two Fisher information matrices. Unfortunately in presence of shape and scale parameters, it is observed (Gupta and Kundu [16]) that the trace or the determinant is not scale invariant. It may happen that for a particular scheme, the determinant or the trace of the Fisher information matrix is more than another scheme, but if we multiply the data by some positive constant then the inequality becomes reversed, which is not desirable.

An alternative way of comparing the information measures of the two different schemes is to compare the precisions of the $100p-th$ quantile estimators, see for example Zhang and Meeker [23] and the references cited there. Recently, Gupta and Kundu [16] has proposed the following information measure, $I(R,T) = \left[\int_0^1 V((R,T)_p)dp\right]^{-1}$, where $V((R,T)_p)$ denotes the asymptotic variance of the $100p-th$ quantile obtained using the censoring scheme (R,T). Moreover, the information measure $I(R,T)$ is independent of p. It is observed by Gupta and Kundu [16] that the above information measure can be used quite effectively to discriminate between two families. In this paper, we use this measure to choose the best scheme among different schemes.

Suppose, we want to choose the best schemes, *i.e.*, the scheme which provides maximum $I(R,T)$, obviously, one of the solution is $R = n$ and $T = \infty$. Therefore, without restricting the *duration of the experiment*, the problem is not of much use. One way to bring the time into consideration is to introduce a cost factor on time and *i.e.* to maximize

$$I(R,T) - C \times \text{Expected duration of the experiment}, \tag{40}$$

with respect to the different schemes (R,T). Here the known constant $C > 0$ denotes the cost per unit time of the experiment. This approach is used under the Bayesian framework for Type-I exponential distribution by Yeh [22] and Lin *et al.* [19]. In this paper, we do not use the cost factor approach, instead we want to choose that scheme which provides the maximum $I(R,T)$, among the schemes with a given maximum expected duration of the experiment. Therefore, for a given expected duration of the experiment, first we choose different schemes which have the same expected duration and among those schemes, we choose that scheme which has the maximum $I(R,T)$.

Note that to choose the optimum scheme, the experimenter needs to know about the unknown parameters. In practice, if it is completely unknown, the experimenter needs to perform some pilot experiments to get some idea about the unknown parameters and accordingly he chooses the optimum scheme. We are going to discuss a simple method to choose the optimum scheme for a given α and θ. Since θ is the scale parameter, with out loss of generality, we can assume $\theta = 1$, and the method can be easily modified for general θ. We take $\alpha = 2$ and $n = 20$. For illustrative purposes we take only three different censoring schemes, namely $(20, T_1)$, $(18, T_2)$ and $(15, T_3)$ where T_1, T_2 and T_3 are such that all the censoring schemes have the same expected duration of the experiment. We will describe how to choose the best scheme from these three schemes.

When, the sample size $n = 20$, the expected duration of the experiment for $R = 20$ is $E(Z_{(20)})$, when $Z_{(20)}$ is the maximum order statistics of a sample of size 20 from a Weibull distribution with the shape parameter 2 and scale parameter 1. Therefore,

$$E(Z_{(20)}) = 40 \int_0^{\infty} x^2 e^{-x^2} \left(1 - e^{-x^2}\right)^{19} dx = 1.8698 \approx 1.87. \tag{41}$$

The expected duration of the experiment for general (R, T) is provided in the Appendix B. In the Figure 5 we plot the expected duration of the experiment for different R values, namely for $R = 20$, $R = 18$ and $R = 15$. From the Figure 5, it is clear that for fixed R, the expected duration of the experiment is an increasing function of T at the beginning, after a specific value (depending on R) of T, the expected duration of the experiment remains constant.

The experimenter knows before starting the experiment that he needs approximately 1.87 units of time to run the experiment without adopting any censoring procedure. Suppose he can afford only 1.40 units ($\approx 75\%$ of 1.87) of time and he wants to choose the scheme (among the three possible schemes) which provides him the maximum information.

It is observed that the following schemes $\{(20, T), T \leq 1.405\}$, $\{(18, T), T \leq 1.554\}$ and

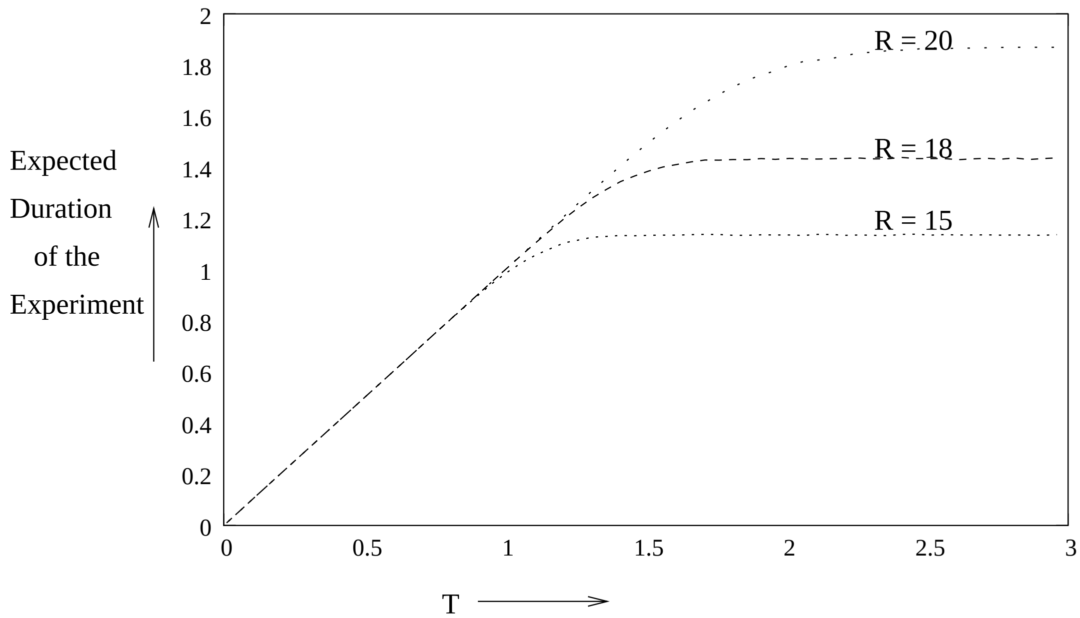

Figure 5: The expected duration of the experiments for three different schemes

$\{(15, T), T < \infty\}$ have the expected duration of the experiments less than or equal to 1.40 units of time. Among these schemes, the experimenter wants to choose the scheme which has the maximum information. In Figure 6 we plot $I(R, T)$ as a function of T for three different R. From the Figure 6 it is clear that for fixed R, the information measure is an increasing function of T at the beginning, and after a fixed time point depending on R, the information measure reaches its maximum.

Now for the scheme $\{(20, T), T \leq 1.405\}$, the maximum information occurs at $T = 1.405$ and it is 4.765. Similarly, for the scheme $\{(18, T), T \leq 1.554\}$, the maximum information occurs at $T = 1.554$ and it is 4.913. For the scheme $\{(15, T), T < \infty\}$, the maximum information occurs at $T = \infty$ and it is 3.995. Therefore , among the three different schemes the scheme $(18, 1.554)$ provides the maximum information.

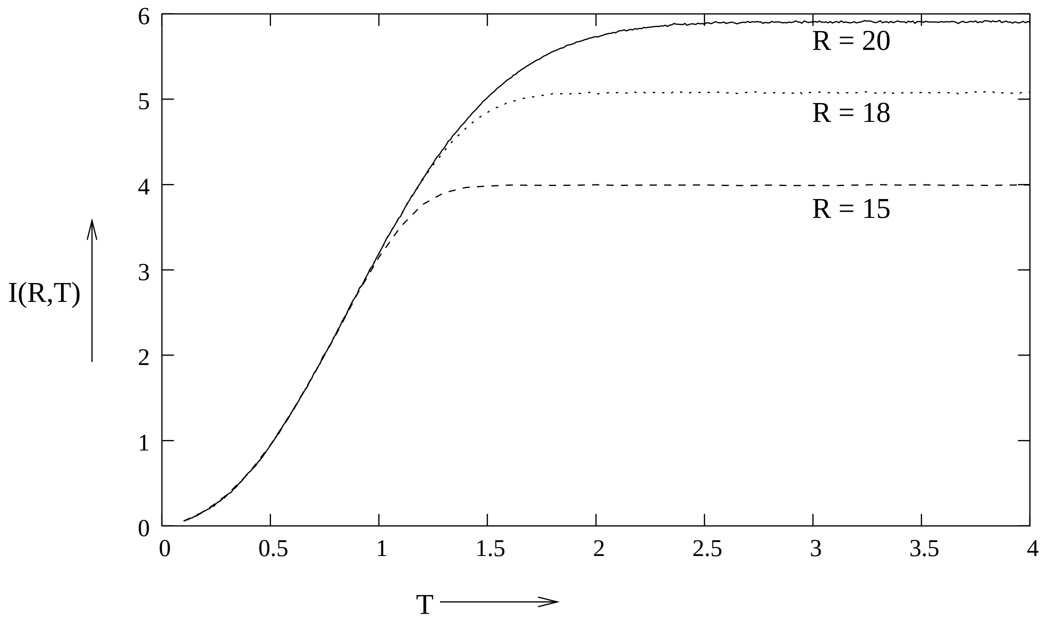

Figure 6: The information measure $I(R, T)$ as a function of T for three different schemes

9 Conclusions

In this paper we consider the classical and Bayesian inference procedures for the hybrid censored Weibull parameters. It is observed that the maximum likelihood estimator of the shape parameter can be obtained by using an iterative procedure. The proposed approximate maximum likelihood estimators of the shape and scale parameters can be obtained in explicit forms. Bayes estimates of the unknown parameters can be obtained using Gibbs sampling procedures and the performances of the Bayes estimators under the assumption of the non-informative priors are quite similar to the MLEs or the AMLEs. We also provide a simple procedure to obtain the optimum censoring scheme. It may be of interest to tabulate different optimum sampling schemes for different values of α and n. Various approximate confidence intervals along the line of Jeng and Meeker [17] can be constructed. Extensive simulations are needed to compare their performances. We have not considered another important aspect, namely goodness-of-fit test for hybrid censored data. It is not a trivial task, because the data are censored. More work is needed in those directions.

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Appendix A

To prove Theorem 2, we need the following lemma.

Lemma 1: For $x_i \geq 0$ and $b \geq 0$, define $g(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} + b$. Then $\frac{d^2}{d\alpha^2} \ln g(\alpha) \geq 0$.

Proof of Lemma 1: Note that,

$$g'(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \ln x_i \quad \text{and} \quad g''(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2.$$

Since

$$\left(\sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 \right) \times \left(\sum_{i=1}^{n} x_i^{\alpha} \right) - \left(\sum_{i=1}^{n} x_i^{\alpha} \ln x_i \right)^2 = \sum_{1 \leq i < j \leq n} x_i^{\alpha} x_j^{\alpha} \left(\ln x_i - \ln x_j \right)^2 \geq 0,$$

therefore for $b \geq 0$,

$$g''(\alpha) g(\alpha) \geq (g'(\alpha))^2. \tag{42}$$

Proof of Theorem 2: We will consider only case I, case II follows exactly in the same manner. The conditional density of α given the *data* is

$$l(\alpha | data) \propto \frac{\alpha^R \pi_2(\alpha) \prod_{i=1}^{R} y_{i:n}^{\alpha-1}}{\left(\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R) y_{R:n}^{\alpha} + b \right)^{a+R}}. \tag{43}$$

Therefore, ignoring the additive constant, the log-likelihood function of the posterior density function of α can be written as

$$\ln(l(\alpha|data)) = \ln \pi_2(\alpha) + R \ln \alpha + (\alpha - 1) \left(\sum_{i=1}^{R} \ln y_{i:n} \right) - (a+R) \ln \left(\sum_{i=1}^{R} y_{i:n}^{\alpha} + (n-R) y_{R:n}^{\alpha} + b \right). \tag{44}$$

Therefore, using Lemma 1 and the assumption on $\pi_2(\alpha)$, it easily follows that $l(\alpha|data)$ is log-concave.

Now we just provide the posterior density function of α for case II. Note that for $d > 0$

$$l(\alpha|data) \propto \frac{\alpha^d \pi_2(\alpha) \prod_{i=1}^d y_{i:n}^{\alpha-1}}{\left(\sum_{i=1}^d y_{i:n}^{\alpha} + (n-d)T^{\alpha} + b\right)^{a+d}}, \tag{45}$$

and for $d = 0$,

$$l(\alpha|data) \propto \frac{\pi_2(\alpha)}{(nT^{\alpha} + b)^a}, \tag{46}$$

Appendix B

In this appendix, we provide the expression for the expected duration of the experiment for a general censoring scheme (R, T), when $\theta = 1$. Suppose DE denotes duration of the experiment and D $(0 \le D \le R)$ denotes the number of death occurs during $[0, T]$. Therefore,

$$DE = \begin{cases} T & \text{if} \quad D < R \\ Y_{R:n} & \text{if} \quad D = R. \end{cases}$$

Note that

$$P(D = i) = \begin{cases} \binom{n}{i} \left(1 - e^{-T^{\alpha}}\right)^i e^{-(n-i)T^{\alpha}} & \text{for} \quad i = 0 \dots R-1 \\ \sum_{i=R}^n \binom{n}{i} \left(1 - e^{-T^{\alpha}}\right)^i e^{-(n-i)T^{\alpha}} & \text{for} \quad i = R. \end{cases}$$

and

$$E(DE) = TP(D < R) + E(Y_{R:n}|D = R)P(D = R).$$

After some calculations, it can be shown that

$$E(Y_{R:n}|D = R) = \sum_{j=R}^n \frac{\alpha j! U(\alpha, j)}{(R-1)!(j-R)!\,(1 - e^{-T^{\alpha}})^j},$$

where

$$U(\alpha, j) = \int_0^T x^{\alpha} e^{-x^{\alpha}} \left(1 - e^{-x^{\alpha}}\right)^{R-1} \left(e^{-x^{\alpha}} - e^{-T^{\alpha}}\right)^{j-R} dx.$$

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