A Convenient Way of Generating Gamma Random Variables Using Generalized Exponential Distribution

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Abstract

In this paper we propose a very convenient way to generate gamma random variables using generalized exponential distribution, when the shape parameter lies between 0 and 1. The new method is compared with the most popular Ahrens & Dieter method and the method proposed by Best. Like Ahrens & Dieter and Best methods our method also uses the acceptance-rejection principle. But it is observed that our method has greater acceptance proportion than Ahrens & Dieter or Best methods.

Key Words and Phrases: Generalized exponential distribution; Random number generator; Gamma generator; Ahrens & Dieter method; Best method.

Short Running Title: Generating gamma random numbers

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1 Introduction

Generating gamma random numbers is an old and very important problem in the statistical literature. Particularly, in the recent days because of the popularity of MCMC techniques it has gained more importance. Several methods are available in the literature to generate gamma random numbers. The book of Johnson *et al.* [11] provides an extensive list of references of the different methods available today. It is well known that the available algorithms can be divided into two distinct cases. Case 1: Shape parameter < 1; Case 2: Shape parameter > 1. Although several methods are available for case 2, but for case 1, mainly two methods are well known; (a) the most popular and very simple method proposed by Ahrens & Dieter [1], (b) the modified Ahrens & Dieter's method proposed by Best [3], see for example the book by Law and Kelton [12] or Fishman [4]. Both the methods mainly use the majorization functions and the acceptance-rejection principle. It is known that Best's method has lower rejection proportion than Ahrens & Dieter's method. Since now a days particularly for MCMC sampling, often a large number of gamma random numbers needs to be generated, one naturally prefers an acceptance-rejection method which has lower rejection proportion to achieve larger period of the corresponding generator.

Recently generalized exponential (GE) distribution has been proposed and studied quite extensively by Gupta and Kundu [5, 6, 7, 8, 9]. The readers may be referred to some of the related literature on GE distribution by Raqab [13], Raqab and Ahsanullah [14], Raqab and Madi [15] and Zheng [16]. The two-parameter GE for $\alpha, \lambda > 0$, has the following distribution function;

$$F_{GE}(x; \alpha, \lambda) = \begin{cases} 0 & \text{if} \quad x < 0, \\ (1 - e^{-\lambda x})^{\alpha} & \text{if} \quad x > 0. \end{cases} \tag{1}$$

The corresponding density function is;

$$f_{GE}(x; \alpha, \lambda) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x} & \text{if} \quad x > 0. \end{cases} \tag{2}$$

Here α and λ are the shape and scale parameters respectively. When $\alpha = 1$, it coincides with the exponential distribution. If $\alpha \leq 1$, the density function of a GE distribution is a strictly decreasing function and for $\alpha > 1$ it has uni-modal density function. The shape of the density function of the GE distribution for different α can be found in Gupta and Kundu [6].

In a recent paper by Gupta and Kundu [10], it is observed that the distribution functions of the GE and gamma can be very close and it is sometimes very difficult to distinguish between the two distribution functions. In fact using the closeness property, the authors proposed to generate approximate gamma random variables using GE distribution for certain ranges of the shape parameters.

In this paper we mainly consider the generation of gamma random deviates for the shape parameter less than one. We denote the density function of a gamma random variable with scale parameter 1 and shape parameter α as

$$f_{GA}(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}; \qquad x > 0. \tag{3}$$

From now on, we take $0 < \alpha < 1$, unless otherwise mentioned. Note that, it is enough to provide the generation for scale parameter equal to 1. It is observed that when the shape parameter is less than one, then a constant multiplication of the GE density function can be used as a majorization function. Therefore, using the acceptance-rejection principle, gamma random deviates can be easily generated using GE random numbers. We further make some simple modifications to the proposed generator. It is observed that our proposed method has greater acceptance proportion than Ahrens & Dieter and Best generators for all $0 < \alpha < 1$.

The rest of the paper is organized as follows. In section 2 we briefly describe the two most popular methods. The proposed methods are discussed in section 3. The numerical comparisons are provided in section 4 and finally we conclude the paper in section 5.

2 The Two Most Popular Methods

In this section we briefly provide the two most popular methods which have been used to generate gamma random numbers when the shape parameter is less than 1. The first method is proposed by Ahrens & Dieter [1] and the second one is by Best [3]. Both the methods are based on the acceptance-rejection principle with proper choice of the majorization functions.

Ahrens & Dieter [1] used the following majorization function;

$$t_{AD}(x;\alpha) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{if} \quad 0 < x < 1, \\ \\ \frac{1}{\Gamma(\alpha)}e^{-x} & \text{if} \quad \quad x > 1. \end{cases} \tag{4}$$

Since $c_{AD} = \displaystyle\int_0^\infty t_{AD}(x;\alpha)dx = \frac{(e+\alpha)}{e\Gamma(\alpha+1)}$, therefore it needs to generate random deviate from the following probability density function

$$r_{AD}(x;\alpha) = \frac{t_{AD}(x;\alpha)}{\int_0^\infty t_{AD}(x;\alpha)dx} = \begin{cases} \frac{\alpha e x^{\alpha-1}}{e+\alpha} & \text{if} \quad 0 < x < 1, \\ \\ \frac{\alpha e}{e+\alpha}e^{-x} & \text{if} \quad \quad x > 1, \end{cases}$$

and using the standard acceptance-rejection method, the gamma random deviate can be easily obtained. Note that in this case the rejection proportion is $c_{AD} - 1$.

Best [3] modified Ahrens & Dieter's [1] method, by using the following majorization function;

$$t_B(x;\alpha) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{if} \quad 0 < x < d, \\ \\ \frac{1}{\Gamma(\alpha)}d^{\alpha-1}e^{-x} & \text{if} \quad \quad x > d. \end{cases} \tag{5}$$

Here d should be chosen in such a manner such that $c_B = \displaystyle\int_0^\infty t_B(x;\alpha)dx$ is minimum. It can be easily seen that in that case d must satisfy the following non-linear equation

$$d = e^{-d}(1-\alpha+d).$$

Best has suggested to use an approximation of d as $d = 0.07 + 0.75\sqrt{1-\alpha}$. Once d is known, the acceptance-rejection method can be easily used to generate gamma random deviate using the majorization function (5). It boils down to generate random deviate from the following probability density function

$$r_B(x;\alpha) = \frac{t_B(x;\alpha)}{\int_0^\infty t_B(x;\alpha)dx} = \begin{cases} \frac{\alpha x^{\alpha-1}}{bd^\alpha} & \text{if} \quad 0 < x < d, \\ \frac{\alpha}{bd} e^{-x} & \text{if} \quad\quad x > d, \end{cases}$$

here $b = 1 + \dfrac{e^{-d}\alpha}{d}$. It has been shown theoretically that Best's method has lower rejection proportion that Ahrens & Dieter's method.

3 Proposed Methodology

In this section, we provide the new gamma random number generator using the generalized exponential distribution. Observe that for all $x \geq 0$,

$$1 - e^{-x} \leq x. \tag{6}$$

Suppose $\beta = 1 - \alpha$, then for all $x \geq 0$, using (6), we have

$$\frac{x^{\alpha-1}e^{-x}}{(1-e^{-x})^{\alpha-1}} = \frac{e^{-x}(1-e^{-x})^\beta}{x^\beta} \leq e^{-x}. \tag{7}$$

Therefore,

$$\frac{\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}}{\frac{\alpha}{2}(1-e^{-x/2})^{\alpha-1}e^{-x/2}} = \frac{2e^{-x/2}x^{\alpha-1}}{\Gamma(\alpha+1)(1-e^{-x/2})^{\alpha-1}} \leq \frac{2^\alpha}{\Gamma(\alpha+1)}. \tag{8}$$

From (8), we obtain

$$f_{GA}(x;\alpha) \leq \frac{2^\alpha}{\Gamma(\alpha+1)} f_{GE}(x;\alpha,\frac{1}{2}). \tag{9}$$

Based on (9), we provide the following algorithm;

Algorithm 1:

1. Generate U from uniform $(0,1)$.

2. Compute $X = -2\ln(1 - U^{1/\alpha})$.

3. Generate V from uniform $(0,1)$ independent of U.

4. If $V \le \dfrac{X^{\alpha-1}e^{-X/2}}{2^{\alpha-1}(1-e^{-X/2})^{\alpha-1}}$ accept X, otherwise goto 1.

Although, (9) is true for all $x > 0$, it is observed that for $0 < x < 1$, the bound provided by (9) is quite sharp, but for $1 < x < \infty$, the bound is not very sharp. Therefore, we propose the following majorization function $t_1(x;\alpha)$ of $f_{GA}(x,\alpha)$, *i.e.*, for all $x > 0$, $f_{GA}(x,\alpha) \le t_1(x;\alpha)$, where

$$t_1(x;\alpha) = \begin{cases} \frac{2^\alpha}{\Gamma(\alpha+1)} f_{GE}(x;\alpha,\frac{1}{2}) & \text{if} \quad 0 < x < 1, \\ \\ \frac{1}{\Gamma(\alpha)} e^{-x} & \text{if} \quad x > 1. \end{cases} \tag{10}$$

Note that

$$\int_0^\infty t_1(x;\alpha)dx = \frac{1}{\Gamma(\alpha+1)}\left[2^\alpha\left(1-e^{-\frac{1}{2}}\right)^\alpha + \alpha e^{-1}\right] = c_1 \quad \text{(say)}.$$

We need to generate from the following density function;

$$r_1(x;\alpha) = \frac{1}{c_1}t_1(x;\alpha); \qquad x > 0,$$

which has the following distribution function;

$$R_1(x;\alpha) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \\ \frac{2^\alpha}{c_1\ \Gamma(\alpha+1)}\left(1-e^{-x/2}\right)^\alpha & \text{if} \quad 0 < x < 1, \\ \\ 1 - \frac{1}{c_1\ \Gamma(\alpha)}e^{-x} & \text{if} \quad x > 1. \end{cases} \tag{11}$$

Now we propose the following modified algorithm.

Algorithm 2:

Set $a = \dfrac{\left(1-e^{-1/2}\right)^\alpha}{(1-e^{-1/2})^\alpha + \frac{\alpha e^{-1}}{2^\alpha}}$ and $b = \left(1-e^{-1/2}\right)^\alpha + \dfrac{\alpha e^{-1}}{2^\alpha}$.

1. Generate U from uniform $(0,1)$.

2. If $U \le a$, then $X = -2\ln\left[1-(Ub)^{\frac{1}{\alpha}}\right]$, otherwise $X = -\ln\left[\frac{2^\alpha}{\alpha}b(1-U)\right]$.

3. Generate V from uniform $(0,1)$. If $X \le 1$, check whether $V \le \frac{X^{\alpha-1}e^{-X/2}}{2^{\alpha-1}(1-e^{-X/2})^{\alpha-1}}$. If true return X, otherwise go back to 1. If $X > 1$, check whether $V \le X^{\alpha-1}$. If true return X, otherwise go back to 1.

Note that the majorization function $t(x;\alpha)$ has two part envelop with the change point at 1, similar to the Ahrens and Dieter's [1] algorithm. But it was suggested by Atkinson and Pearce [2] and later implemented by Best [3] that the change point might depend on the shape parameter α and it enhanced the performance. Using that idea, we propose the following majorization function for $f_{GA}(x;\alpha) \le t_2(x;\alpha)$, where

$$t_2(x;\alpha) = \begin{cases} \frac{2^\alpha}{\Gamma(\alpha+1)} f_{GE}(x;\alpha,\frac{1}{2}) & \text{if} \quad 0 < x < d_\alpha, \\ \\ \frac{1}{\Gamma(\alpha)} d_\alpha^{\alpha-1} e^{-x} & \text{if} \quad x > d_\alpha, \end{cases} \tag{12}$$

where d_α is chosen before. How to choose the optimum d_α will be discussed shortly. Note that for $d_\alpha = 1$, $t_1(x;\alpha) = t_2(x;\alpha)$. For $0 < \alpha < 1$, the majorization is obvious. Now

$$\int_0^\infty t_2(x;\alpha)dx = \frac{1}{\Gamma(\alpha+1)}\left[2^\alpha\left(1-e^{-\frac{d_\alpha}{2}}\right)^\alpha + \alpha d_\alpha^{\alpha-1}e^{-d_\alpha}\right] = c_2 \quad \text{(say)}.$$

Therefore, we need to generate from the following distribution function

$$R_2(x;\alpha) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \\ \frac{2^\alpha}{c_2\Gamma(\alpha+1)}\left(1-e^{-x/2}\right)^\alpha & \text{if} \quad 0 < x < d_\alpha, \\ \\ 1 - \frac{1}{c_2\Gamma(\alpha)} d_\alpha^{\alpha-1}e^{-x} & \text{if} \quad x > d_\alpha. \end{cases} \tag{13}$$

Now we discuss the optimum choice of d_α. Note that $c_2 \ge 1$ is the normalizing constant and $c_2 - 1$ denotes the rejection proportion. Therefore, we should choose d_α, so that c_2 is minimum for a given α. It has to be obtained iteratively by solving a non-linear equation. We denote the optimum d_α as d_α^o. We have performed a non-linear regression between α and

d_α^o and based on that we are suggesting the following approximation of the optimum d_α^o as d_α^*, where

$$d_\alpha^* = 1.0334 - 0.0766 e^{2.2942\alpha}. \tag{14}$$

Therefore, now we have the final algorithm;

ALGORITHM 3:

Set $d = 1.0334 - 0.0766e^{2.2942\alpha}$, $a = 2^\alpha \left(1 - e^{-\frac{d}{2}}\right)^\alpha$, $b = \alpha d^{\alpha-1} e^{-d}$ and $c = a + b$.

1. Generate U from uniform $(0, 1)$.
2. If $U \leq \frac{a}{a+b}$, then $X = -2 \ln \left[1 - \frac{(cU)^{1/\alpha}}{2}\right]$, otherwise $X = -\ln \left[\frac{c(1-U)}{\alpha d^{\alpha-1}}\right]$.
3. Generate V from uniform $(0, 1)$. If $X \leq d$, check whether $V \leq \frac{X^{\alpha-1} e^{-X/2}}{2^{\alpha-1}(1 - e^{-X/2})^{\alpha-1}}$. If true return X, otherwise go back to 1. If $X > d$, check whether $V \leq \left(\frac{d}{X}\right)^{1-\alpha}$. If true return X, otherwise go back to 1.

4 Numerical Comparison

In this section we compare different methods numerically. Note that all the methods discussed here, are based on the acceptance-rejection principle. We compare different methods in terms of their rejection proportion or equivalently the expected number of the uniform random deviates to be generated to produce one gamma variate, see Law and Kelton [12]. We report the different expected numbers below:

1. Ahrens and Dieter Method:
$$c_{AD} = \frac{1 + e^{-1}\alpha}{\Gamma(\alpha + 1)}.$$
2. Best Method:
$$c_B = \frac{(d + e^{-d}\alpha) d^{\alpha-1}}{\Gamma(\alpha + 1)}.$$

3. Algorithm 1:

$$c_1 = \frac{2^\alpha}{\Gamma(\alpha+1)}.$$

4. Algorithm 2:

$$c_2 = \frac{1}{\Gamma(\alpha+1)} \left[2^\alpha \left(1 - e^{-\frac{1}{2}}\right)^\alpha + \alpha e^{-1} \right].$$

5. Algorithm 3:

$$c_3 = \frac{1}{\Gamma(\alpha+1)} \left[2^\alpha \left(1 - e^{-\frac{d_\alpha^*}{2}}\right)^\alpha + \alpha d_\alpha^{*(\alpha-1)} e^{-d_\alpha^*} \right].$$

We report $\Gamma(\alpha+1)c$ for different values of α in Table 1. We also report $\Gamma(\alpha+1)c_4$, where c_4 is obtained from c_3, by replacing d_α^* with d_α^o. We further provide the graphs of the performances of the different methods in Figure 1 for a much finer range of α in $(0, 1)$ (here α varies between 0.01 to 0.99 with an increment 0.01). From the table and also from the graphs it is clear that our proposed final algorithm Algorithm 3 has lower rejection proportion than Best's method.

Finally we just want to see how our proposed approximation, d_α^*, of d_α^o works. For that purpose we have plotted in Figure 2 the performance of the Algorithm 3 and also the performance of the method obtained by replacing d_α^* with d_α^o. We obtained d_α^o numerically and name it as 'Optimum Algorithm'. From Figure 2 it is clear that the proposed approximation works very well for $0 < \alpha < 0.9$. When α is very close to 1, the approximation is not that good.

5 Conclusions

In this paper we have proposed a new algorithm of generating gamma variates using generalized exponential distribution for the shape parameter less than 1. It is observed that our algorithm has lower rejection proportion than the popular Ahrens-Dieter or Best method. Our method is not very difficult to implement also. Moreover, like any other gamma gener-

Table 1: The expected numbers multiplied by $\Gamma(\alpha+1)$ for different methods and for different values of α are reported

α	$\Gamma(\alpha+1)c_{AD}$	$\Gamma(\alpha+1)c_B$	$\Gamma(\alpha+1)c_1$	$\Gamma(\alpha+1)c_2$	$\Gamma(\alpha+1)c_3$	$\Gamma(\alpha+1)c_4$
.1000	1.0368	1.0328	1.0718	1.0131	1.0129	1.0129
.2000	1.0736	1.0630	1.1487	1.0268	1.0261	1.0261
.3000	1.1104	1.0897	1.2311	1.0410	1.0392	1.0392
.4000	1.1472	1.1121	1.3195	1.0558	1.0517	1.0517
.5000	1.1839	1.1289	1.4142	1.0710	1.0632	1.0632
.6000	1.2207	1.1383	1.5157	1.0868	1.0725	1.0725
.7000	1.2575	1.1381	1.6245	1.1031	1.0780	1.0780
.8000	1.2943	1.1246	1.7411	1.1199	1.0769	1.0768
.9000	1.3311	1.0906	1.8661	1.1371	1.0625	1.0624

ator for shape parameter less than one, our method also can be used for generating normal random numbers through chi-square generator.

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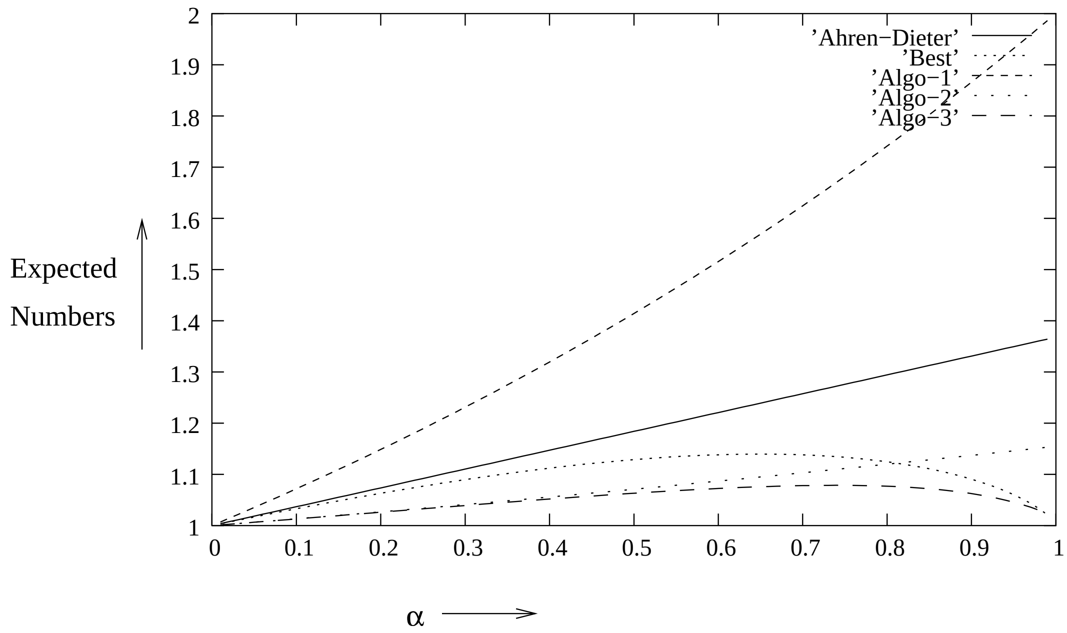

Figure 1: Expected numbers multiplied by $\Gamma(\alpha+1)$ are reported for different methods

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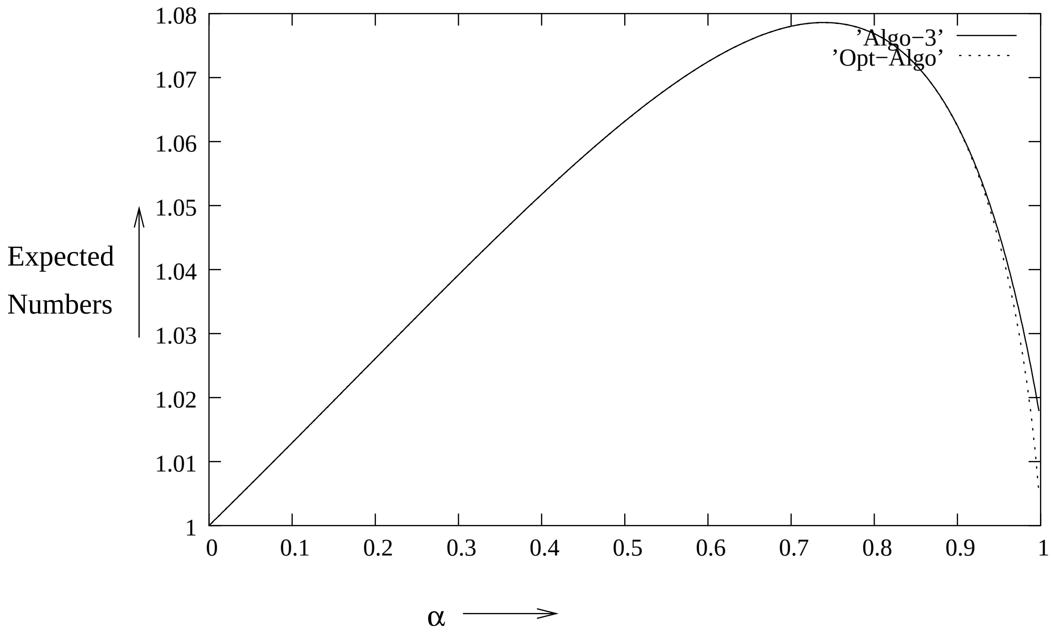

Figure 2: Expected numbers multiplied by $\Gamma(\alpha + 1)$ are reported for Algorithm 3 and for Optimum Algorithm methods

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