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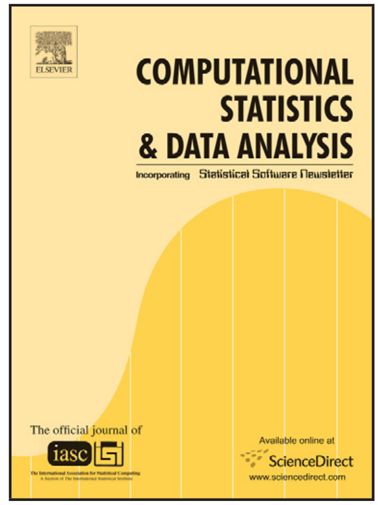

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Bayes estimators for reliability measures in geometric distribution model using masked system life test data

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Abstract

This paper presents Bayes estimators for the reliability measures of the individual components in a multi-component systems in the presence of masked system life test data. The life time distributions of the system components are assumed to be geometric with different parameters. Two sided Bayesian probability intervals of the parameters are also derived. Numerical simulation study is given in order to: (*i*) explain how one can apply the theoretical results obtained, (*ii*) study the influence of the sample size and masking level on the accuracy of point estimates.

Key Words: Bayes procedure, Geometric distribution, reliability measures, Masked data, competing risks.

1 Introduction

Continuous nonnegative life distributions such as exponential, Weibull, Pareto, gamma and lognormal are used to model the life time of devices in reliability, see for example [3] and [20] and the references therein.

Discrete distributions are extremely needed in reliability when the life time measurements are taken in discrete time. Discrete distributions provide better models when devices lives

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can be described by nonnegative integer valued random variables. For example, switches or circuit-breakers or those devises which operate in cycles or demands, for more details we refer for example to [14] and [15].

The geometric distribution is a common discrete distribution used to model the life time of a device in reliability. Lui [10] discussed both point and interval estimation of the reliability function for the geometric distribution. He derived the maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) for the reliability function of the geometric distribution. The hypothesis of testing of the equality of failure probability per time-unit among several comparison groups for geometric distribution are considered by Salvia [15] and Vit [21]. Casella and Berger [2] and Hoel [8] discussed some systematic on properties of the geometric distribution.

However estimating the reliability measures of the individual components in multi component system based on the system life test data is an interesting problem in reliability analysis. Such estimators can be extremely useful since they reflect the reliability measures of each component after assembly into a new configuration of a similar system. For the multi-component system, the life test data may contain the time to failure of the system along with information on the exact component causes the system failure. Under certain circumstances, the true component that may cause the system failure can not be identified. Instead, it may only be identified that the true component causes the system failure belongs to a subset of system components. In this cause, the cause of system failure is masked and then the system life test data collected is referred to be masked system life test data, see Miyakawa [12].

Recently, several papers studied the problem of estimating the parameters of the individual components in multi-component systems, using masked system life test data when the components lives are continuous distributed random variables. Under the assumption that the system's components have constant failure rates, Miyakawa [12] studied the problem of a 2-component series system. He derived closed form expressions for the maximum likelihood estimators for the parameters included based on masked data. Under the same assumptions, Usher and Hodgson [24] extended Miyakawa's results to a three-component series system. Guess, et al. [6] extended and clarified the derivation of the likelihood function under the assumption that masking is independent of the exact failure cause. Lin, et al. [9] derived the exact maximum likelihood estimates of the parameters included in the lives distribution of components with constant failure rates in a series system using masked system life test data. Sarhan [18] derived the maximum likelihood and Bayes estimates of the values of reliability of system's components in the case of n component series system under the same assumptions. Iterative maximum likelihood procedure is used by Usher [23] in the case of 2-component series system when the system's components life times have Weibull distributions. He illustrated the approach with a simple numerical example. Sarhan [17] derived the maximum likelihood estimators for the parameters included in the cases of 2-component and 3-component series systems under the assumption that the lives distributions of the components are Weibull. He derived closed-form expressions for maximum likelihood estimates in some particular cases, which generalize the results obtained by Usher and Hodgson [24]. Maximum likelihood and Bayes estimators of the reliability measures of the individual components in a series system are derived by Sarhan and El-Gohary [20] under the assumption that the lives of the system components have Pareto distributions. Sarhan [16] derived the maximum likelihood and Bayes estimators for the reliability of the system components in a series system when the lives of the components have linear failure rate distributions. Sarhan and El-Bassiouny [19] derived the maximum likelihood and Bayes estimators for the parameters included in a parallel system under the assumption that the life times of the system's components have complementary exponential distributions. Zhibin [25] studied the problem of estimating reliability of components in series and parallel system from masking system testing data. Some other related works have been done by Flehinger et al. [4], [5] and Zhibin [26, 27] and the references therein.

The main objective in this paper is to derive the Bayes estimators of the reliability measures (the failure rate, reliability function and the mean time to failure) of the individual components in a multi-component series system when the life time of each component has a geometric distribution, using masked system life test data. Since the problem of estimating the reliability measures when the system consists of more than two components is tedious, as we explained in the appendix, we will illustrate the problem on a series system consisting of two components.

The paper is organized as follows. Section 2 presents the model assumptions. The likelihood function of the available data is given in Section 2. The Bayes estimators of the reliability measures of the individual components are given in Section 3. Also the two-sided Bayesian probability intervals of the reliability measures of the individual components are derived in section 3. Section 4 gives numerical results and Section 5 concludes the paper. The proofs of the theorems which require lengthy derivations are given in the Appendix.

2 Model assumptions and likelihood function

In this section we present the main assumption on which the model is built. Also, we derive the likelihood function of the model presented based on the masked system life test data.

2.1 The model assumptions

The following assumptions are considered throughout this paper.

Assumptions 1.

1.1 The system is series with J, $J \geq 2$, independent components.

1.2 N identical systems are put on the life test. The test is terminated when n systems failed. That is the data are censored.

1.3 The random variables $X_{ij},\ j = 1, 2, \ldots, J;\ i = 1, 2, \ldots, N$ are independent with $X_{1j}, X_{2j}, \ldots, X_{Nj}$ being identical and having geometric distribution with parameter p_j.

1.4 The observable quantities for the system i, which failed, on the test are: (i) the random variable T_i, represents the number of success trials of using system i to get its fist failure, and (ii) a set S_i of system's components that may cause the system i failed. But for the censored observation, we only observe X_i, $i = n+1, \cdots, N$. The data collected from this process are (X_1, S_1), $(X_2, S_2), \ldots$, (X_n, S_n), $(X_{n+1}, *), \cdots$, $(X_N, *)$. Here $(X, *)$ means the observation is censored.

1.5 Masking is s-independent of the true cause of system failure. That is, for all $\ell, j \in S_i$, $P(S_i = s_i | T_i = t_i, K_i = j) = P(S_i = s_i | T_i = t_i, K_i = \ell)$, where K_i denotes the index of the component causes the system i to fail.

1.6 The system may fail due to component 1 or component 2 or both components 1 and 2.

Based on the assumption (1.3), for $j = 1, 2, \ldots, J$, the random variables $X_{1j}, X_{2j}, \ldots, X_{Nj}$ can be written as a random variable X_j, $j = 1, 2, \ldots, J$, having geometric distribution with probability of success q_j. That is, $p_j = 1 - q_j$ denotes to the failure probability of component j 'per time-unit' or 'at each time', $0 < p_j < 1$, $j = 1, 2, \ldots, J$. That is, the probability mass function of X_j, $j = 1, 2, \ldots, J$, is given by

$$f_j(x) = p_j \, q_j^{x-1}\,, \; x = 1, 2, \ldots \tag{2.1}$$

The reliability function of X_j, $j = 1, 2, \ldots, J$, is

$$\bar{F}_j(x) = q_j^x\,, \; x = 1, 2, \ldots \tag{2.2}$$

The mathematical expectation of of X_j, say M_j, $j = 1, 2, \ldots, J$, is

$$M_j = E[X_j] = \frac{1}{p_j}\,. \tag{2.3}$$

There are two different definitions of the failure rate function of the discrete distributions. In that follows we present the two different forms of the failure rate functions denoted respectively by $\lambda_j(x)$ and $r_j(x)$, $j = 1, 2, \ldots, J$.

1. According to Barlow et al. [1], the hazard rate function of X_j, $j = 1, 2, \ldots, J$, is

$$\lambda_j(x) = P(X_j = x | X_j \geq x) = 1 - \frac{\bar{F}_j(x)}{\bar{F}_j(x-1)} = p_j\,. \tag{2.4}$$

2. According to Roy and Gupta [14] and Xie et al. [22], the hazard rate function of X_j, $j = 1, 2, \ldots, J$, is

$$r_j(x) = \ln \frac{\bar{F}_j(x-1)}{\bar{F}_j(x)} = -\ln(1 - p_j)\,. \tag{2.5}$$

There is a simple relation between $\lambda_j(x)$ and $r_j(x)$, Xie et al. [22],

$$\lambda_j(x) = 1 - e^{-r_j(x)}\,.$$

Our aim in this paper is to estimate the reliability measures of the individual components $\lambda_j(x) = p_j$, $r_j(x)$, $\bar{F}_j(x)$ and M_j based on masked system life test data.

2.2 The likelihood function

Based on the random sample (X_1, S_1), $(X_2, S_2), \ldots$, (X_n, S_n), $(X_{n+1}, *), \cdots$, $(X_N, *)$, the likelihood function is [6]

$$L(data; p_1, \ldots, p_m) = \prod_{i=1}^{n} \sum_{j \in S_i} f_j(x_i) \prod_{\ell=1, l \neq j}^{J} \bar{F}_\ell(x_i) \prod_{k=n+1}^{N} \bar{F}(x_k)\,. \tag{2.6}$$

where $\bar{F}(x) = \prod_{j=1}^{J} \bar{F}_k(x)$ is the survival function of the system. Then for the geometric distribution model, we have

$$L(data; p_1, \ldots, p_m) = \prod_{i=1}^{n} \sum_{j \in S_i} p_j (1-p_j)^{x_i - 1} \prod_{\ell=1, l \neq j}^{J} (1-p_\ell)^{x_i} \prod_{k=n+1}^{N} \prod_{j=1}^{J} (1-p_j)^{x_k}. \tag{2.7}$$

To show how the problem looks like and how it becomes more tedious when $J > 2$, we refer to the Appendix. From now and henceforth we assume that $J = 2$. That is the system consists of two components. In this case we need the following notations. Let n_1 be number observations when the component 1 causes the system failure. That is n_1 is the number of the observation when $S_i = \{1\}$. Let t_i be the observed value of X when $S_i = \{1\}$, $i = 1, 2, \ldots, n_1$. Let n_2 be number observations when the component 2 causes the system failure. That is n_2 is the number of the observation when $S_i = \{2\}$. Let y_i be the observed value of X when $S_i = \{2\}$, $i = 1, 2, \ldots, n_2$. Let n_0 be number observations when both components 1 and 2 cause the system failure. That is n_0 denotes the number of the observation when $S_i = \{0\}$. Here we mean by $S_0 = \{0\}$ that the cause of system i failure is due to both components 1 and 2. Let z_i be the observed value of X when $S_i = \{0\}$, $i = 1, 2, \ldots, n_0$. Also, let n_{12} be the number of observation when the cause of system failure is masked (either component 1 or component 2 or both components 1 and 2). That is n_{12} is the number of observation when $S_i = \{1, 2\}$. Let z_i be the observed value of X when $S_i = \{1, 2\}$, $i = 1, 2, \ldots, n_{12}$. Thus, the likelihood function (2.7), in this case, reduces to

$$\begin{aligned} L(data; p_1, p_2) \;=\; & \prod_{i=1}^{n_1} p_1 q_1^{t_i - 1} q_2^{t_i} \prod_{j=1}^{n_2} p_2 q_2^{y_j - 1} q_1^{y_j} \prod_{j=1}^{n_0} p_1 p_2 (q_1 q_2)^{z_j - 1} \prod_{k=n+1}^{N} (q_1 q_2)^{x_k} \\ & \prod_{\ell=1}^{n_{12}} \left(p_1 q_1^{z_\ell - 1} q_2^{z_\ell} + p_2 q_2^{z_\ell - 1} q_1^{z_\ell} + p_1 p_2 (q_1 q_2)^{z_\ell - 1} \right), \end{aligned}$$

where $q_j = 1 - p_j$, $j = 1, 2$.
After making some algebraic simplifications, one get

$$L(data; p_1, p_2) = p_1^{n_1 + n_0} p_2^{n_2 + n_0} (1-p_1)^{T - n_1 - n_{12}} (1-p_2)^{T - n_2 - n_{12}} \Big(p_1 + p_2 - p_1 p_2 \Big)^{n_{12}}, \tag{2.8}$$

where $T = \sum_{i=1}^{N} x_i$.

3 Bayes analysis

In this section we present and illustrate the methodology for obtaining the Bayes estimators for the reliability measures of the individual components in the system previously described. To do that, the following additional assumptions are needed:

Assumptions 2:

2.1 The parameters p_1 and p_2 behave as independent random variables.

2.2 The random variable p_j has Beta prior distribution with known shape and scale parameters α_j and β_j, $j = 1, 2$. That is, the prior probability density function (pdf) of p_j, $j = 1, 2$, takes the following form

$$g_j(p_j) = \frac{1}{B(\alpha_j,\, \beta_j)}\, p_j^{\alpha_j - 1}(1 - p_j)^{\beta_j - 1}\,,\ 0 < p_j < 1. \tag{3.1}$$

2.3 The loss incurred when p_1 and p_2 are estimated, respectively, by $\hat{p}_1$ and $\hat{p}_2$ is a quadratic. Namely,

$$l((p_1, p_2), (\hat{p}_1, \hat{p}_2) = k_1(\hat{p}_1 - p_1)^2 + k_2(\hat{p}_2 - p_2)^2,\ k_1, k_2 > 0. \tag{3.2}$$

The Beta prior distribution is assumed not only to give nicely results but also permits closed forms of the required estimators in terms of Beta functions.

The following corollary can easily be proved by using binomial expansion of $\Big(p_1 + p_2 - p_1 p_2\Big)^{n_{12}}$.

Corollary 3.1 *The likelihood function (2.8) can be written as*

$$L(data; p_1, p_2) = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\, p_1^{n_{12}+n_1+n_0+j-i}\, (1 - p_1)^{T-n_1-n_{12}} \times p_2^{n_{12}+n_2+n_0-j}\, (1 - p_2)^{T-n_2-n_{12}}. \tag{3.3}$$

The following Theorem gives the joint posterior pdf of p_1, p_2.

Theorem 3.1 *Based on the assumptions 2.1 to 2.3, the joint posterior pdf of p_1, p_2 is*

$$g(p_1, p_2) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{\ell=1}^{2} p_\ell^{a_\ell - 1}\, (1 - p_\ell)^{b_\ell - 1},\ 0 < p_1, p_2 < 1\,, \tag{3.4}$$

where $a_\ell = n_{12} + n_0 + n_\ell + \alpha_\ell - (-1)^\ell j - (2 - \ell)i$, $\quad b_\ell = T + \beta_\ell - n_\ell - n_{12},\ \ell = 1, 2$ *and*

$$I_0 = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{\ell=1}^{2} B(a_\ell, b_\ell)\,. \tag{3.5}$$

Proof. *See the Appendix.*

The following Corollary gives the marginal posterior pdf of p_ℓ, $\ell = 1, 2$.

Corollary 3.2 *The marginal posterior pdf's of p_1 and p_2 are give, respectively, by*

$$g_1(p_1|data) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B(a_2, b_2)\, p_1^{a_1-1} (1-p_1)^{b_1-1},\ 0 < p_1 < 1, \tag{3.6}$$

and

$$g_2(p_2|data) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B(a_1, b_1)\, p_2^{a_2-1} (1-p_2)^{b_2-1},\ 0 < p_2 < 1. \tag{3.7}$$

Proof. *According to the very well known relation between the joint and marginal pdfs given by*

$$g_\ell(p_\ell|data) = \int_0^1 g(p_1, p_2|data) \Big[\delta_{\ell 1}\, dp_2 + \delta_{\ell 2}\, dp_1\Big]$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, one can deduce $g_\ell(p_\ell|data)$ as given in the above relations after making very simple calculations, which completes the proof. □

The following Corollary presents the marginal posterior moments of p_ℓ, say $\mu_\ell^{(m)}$, $\ell = 1, 2$ and $m = 1, 2, \ldots$.

Corollary 3.3 *The following statements are fulfilled for all $m = 1, 2, \ldots$:*

$$\mu_\ell^{(m)} = \frac{J_\ell^{(m)}}{I_0}, \tag{3.8}$$

where

$$J_\ell^{(m)} = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k + m\delta_{k\ell}, b_k)\,. \tag{3.9}$$

Proof. *See the Appendix.*

Now we are ready to introduce a Theorem which gives the Bayes estimators for $p_\ell = \lambda_\ell(x)$, $\ell = 1, 2$.

Theorem 3.2 *Under the group of assumptions 1 and 2:*

1. *The Bayes estimator for p_ℓ, $\ell = 1, 2$, is*

$$\hat{p}_\ell = \frac{J_\ell^{(1)}}{I_0}\,. \tag{3.10}$$

2. *The minimum posterior risk associated with $\hat{p}_\ell$, say $R_{\hat{p}_\ell}$, is*

$$R_{\hat{p}_\ell} = \frac{J_\ell^{(2)}}{I_0} - \left(\frac{J_\ell^{(1)}}{I_0}\right)^2. \tag{3.11}$$

Proof. *See the Appendix.*

Theorem 3.3 *Under the group of assumptions 1 and 2:*

1. *The Bayes estimator for the reliability function $\bar{F}_\ell(x_0)$, $\ell = 1, 2$ is*

$$\widehat{\bar{F}}_\ell(x_0) = \frac{K_\ell^{(1)}}{I_0}. \tag{3.12}$$

2. *The minimum posterior risk associated with $\widehat{\bar{F}}_\ell(x_0)$, say $R_{\widehat{\bar{F}}_\ell}$, is*

$$R_{\widehat{\bar{F}}_\ell} = \frac{K_\ell^{(2)}}{I_0} - \left(\frac{K_\ell^{(1)}}{I_0}\right)^2. \tag{3.13}$$

where for $\ell = 1, 2$, and $m = 1, 2, \ldots$

$$K_\ell^{(m)} = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k, b_k + x_0\delta_{k\ell}). \tag{3.14}$$

Proof. *See the Appendix.*

The following Theorem presents the Bayes estimators of the failure rates of the system components, $r_\ell(x_0)$, $\ell = 1, 2$.

Theorem 3.4 *Under the previous assumptions:*

1. *The Bayes estimator for $r_\ell(x_0)$, $\ell = 1, 2$, is*

$$\hat{r}_\ell(x_0) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \Big[\psi(a_\ell + b_\ell) - \psi(b_\ell)\Big] \prod_{k=1}^{2} B(a_k, b_k), \tag{3.15}$$

2. *The minimum posterior risk associated with $\hat{r}_\ell(x_0)$ is*

$$\begin{aligned} R_{\hat{r}_\ell} &= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k, b_k) \\ &\quad \times \left\{ \Big[\psi(b_\ell) - \psi(a_\ell + b_\ell)\Big]^2 + \psi'(b_\ell) - \psi'(a_\ell + b_\ell) \right\} \\ &\quad - \left\{ \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \Big[\psi(b_\ell) - \psi(a_\ell + b_\ell)\Big] \prod_{k=1}^{2} B(a_k, b_k) \right\}^2, \end{aligned} \tag{3.16}$$

where $\psi(z)$ and $\psi'(z)$ are diagamma and polygamma functions defined, respectively, as $\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$ and $\psi'(z) = \frac{d\psi(z)}{dz}$.

Proof. *See the Appendix.*

Theorem 3.5 *Under the groups of assumptions 1, 2:*

1. *The Bayes estimator for M_ℓ, $\ell = 1, 2$, is*

$$\hat{M}_\ell = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k - \delta_{\ell k}, b_k)\,, \tag{3.17}$$

2. *The minimum posterior risk associated with $\hat{M}_\ell$, $\ell = 1, 2$, is*

$$\begin{aligned} R_{\hat{M}_\ell} &= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k - 2\delta_{\ell k}, b_k) \\ &\quad - \left\{ \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k - \delta_{\ell k}, b_k) \right\}^2 . \end{aligned} \tag{3.18}$$

Proof. *The proof of this theorem is similar to the proof of Theorem 3.2.* □

In what follows we derive the two sided Bayesian probability intervals for the unknown parameters p_1 and p_2

Two sided Bayesian probability intervals: Once the posterior probability density function $g(\theta|data)$ of the unknown parameter θ is derived, the $100(1-\alpha)\%$ two-sided Bayesian probability interval, shortly denoted by $100(1-\alpha)\%$TBPI, $(u,\, v)$ can be derived by solving the following two equations, with respect to u and v [see Martz and Waller [11]]:

$$\frac{\alpha}{2} = \int_0^u g(\theta|data)\, d\theta\,, \tag{3.19}$$

$$\frac{\alpha}{2} = \int_v^\infty g(\theta|data)\, d\theta\,, \tag{3.20}$$

Substituting from (3.6) into (3.19) and (3.20), the $100(1-\alpha)\%$TBPI of p_1, say $(u_1,\, v_1)$ can be obtained by solving the following equations with respect to u_1 and v_1:

$$\frac{\alpha}{2} = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B_{u_1}(a_1, b_1) B(a_2, b_2)\,, \tag{3.21}$$

$$1 - \frac{\alpha}{2} = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B_{v_1}(a_1, b_1) B(a_2, b_2)\,, \tag{3.22}$$

Here $B_x(a, b)$ is the incomplete beta function defined by $B_x(a, b) = \int_0^x w^{a-1}(1-w)^{b-1} dw$.

Equations (3.21) and (3.22) do not yield explicit solutions for u_1 and v_1 and have to be solved numerically to obtain (u_1, v_1).

Similarly, from (3.7) and (3.19) and (3.20), one can get the $100(1-\alpha)\%$TBPI of p_2, say $(u_2,\, v_2)$ by solving the following equations with respect to u_2 and v_2:

$$\frac{\alpha}{2} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}B(a_1,b_1)B_{u_2}(a_2,b_2)\,, \qquad (3.23)$$

$$1-\frac{\alpha}{2} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}B(a_1,b_1)B_{v_2}(a_2,b_2)\,, \qquad (3.24)$$

Also we can derive the $100(1-\alpha)\%$TBPI of the rest of the reliability measures of the individual components by firstly deriving the posterior probability density function of the reliability measure, using transformation method and the probability density function of p_1 or p_2, then solving the equations derived by gathering the probability density function obtained with the relations (3.19) and (3.20).

4 Simulation study

In this section we present two numerical examples to illustrate the theoretical results obtained in the previous sections. It is assumed in these examples that there exists a series system with two independent components, where $f_j(x) = p_j(1-p_j)^{x-1}$, $j = 1, 2$ and $p_1 = 0.1$ and $p_2 = 0.12$. For each system we generate a pair of random variables X_1 and X_2 from f_1 and f_2, respectively. Then we calculate $X = \min(X_1, X_2)$ and record the index of the minimum if it is available. This step is repeated n times to simulate a random sample (X_1, S_1), $(X_2, S_2), \cdots, (X_n, S_n)$ with size n from the underlying system. Note that X_i and S_i represent random successive trails of system i before the first failure and the set contains the index of component causes its failure, respectively. In the case of masking data we randomly masking about 50% of the observations. The parameters p_1 and p_2 behave as random variables with beta prior distributions with parameters $(7.22, 48.33)$ and $(2.85, 25.89)$, respectively. The values of the parameters of the prior distributions are determined by following the technique given in Martz and Waller [11].

Example 4.1 In this example, we generate a random sample with size $N = n = 15$ from the underlying model. Table 1 shows the data generated. Then this data is used to calculate: (i) the point estimates of p_1 and p_2, (ii) the percentage errors associated with the point estimates obtained, (iii) 95% TBPI for each parameter, when the parameters have beta and noninformative prior distributions. Also, the prior and posterior probability density functions of the parameters p_1 and p_2 are plotted when the prior density functions are beta. Figures 1 and 2 show these functions for p_1 and p_2, respectively.

Note that the percentage error associated with the point estimate of w, say $PE_{\hat{w}}$, is given by the following formula:

$$PE_{\hat{w}} = \frac{|\text{exact value of } w - \text{estimated value of } w|}{\text{exact value of } w} \times 100\% \qquad (4.1)$$

Table 2 gives the the point estimates of p_1, p_2, $\mathrm{PE}_{\hat{p}_1}$, $\mathrm{PE}_{\hat{p}_2}$, and 95% TBPI for p_1 and p_2.

Table 1. The simulated data for example 1.

i	t_i	No masking S_i	General masking S_i	i	t_i	No masking S_i	General masking S_i
1	5	$\{1\}$	$\{1,2\}$	9	1	$\{1\}$	$\{1,2\}$
2	7	$\{1\}$	$\{1\}$	10	2	$\{1\}$	$\{1,2\}$
3	2	$\{2\}$	$\{2\}$	11	1	$\{1\}$	$\{1,2\}$
4	7	$\{2\}$	$\{2\}$	12	1	$\{0\}$	$\{1,2\}$
5	1	$\{2\}$	$\{2\}$	13	8	$\{2\}$	$\{2\}$
6	2	$\{1\}$	$\{1\}$	14	12	$\{2\}$	$\{2\}$
7	4	$\{2\}$	$\{2\}$	15	12	$\{1\}$	$\{1,2\}$
8	2	$\{2\}$	$\{2\}$				

Note in table 1 that i denotes to the system number, t_i denotes to the number of successive trails of system i before its first failure, and S_i denotes the set of components that may cause the system i to fail. Further, $S_i = \{0\}$ means that the system i fails due to both components 1 and 2.

Table 2. Point estimates, PE and 95% TBPI for p_1 and p_2.

Parameter	Beta			noninformative		
	Estimate	PE	95% TBPI	Estimate	PE	95% TBPI
	No masking					
p_1	0.112	12.16	(0.101,0.221)	0.118	17.65	(0.101,0.221)
p_2	0.123	2.66	(0.101,0.221)	0.118	1.96	(0.091,0.211)
	General masking					
p_1	0.074	26.27	(0.07,0.231)	0.053	47.52	(0.06,0.271)
p_2	0.147	22.48	(0.131,0.291)	0.171	42.83	(0.131,0.361)

Based on the results shown in table 2, one can conclude that follows, for the current example:

i. The percentage error when the there is no masking in the observations is smaller than its value when there is no masking.

ii. The percentage error associated with the point estimate when the prior distribution is beta is smaller that one associated with the point estimate when the prior distribution is noninformative.

Therefore, one may say that: (*i*) beta prior distributions peforms better than the noninformative ones in the sense of giving point estimates with smaller percentage errors, (*ii*) a sample with known cause of system failure gives better estimates, in the same sense, than that obtained when the masking takes place.

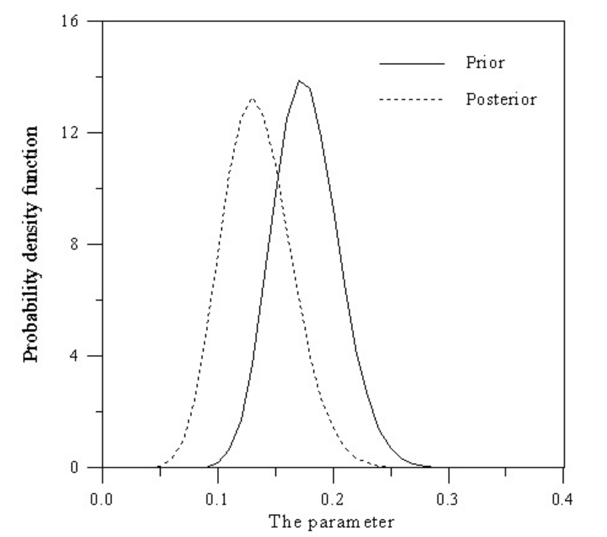

Fig. 1. The prior and posterior pdf's of p_1.

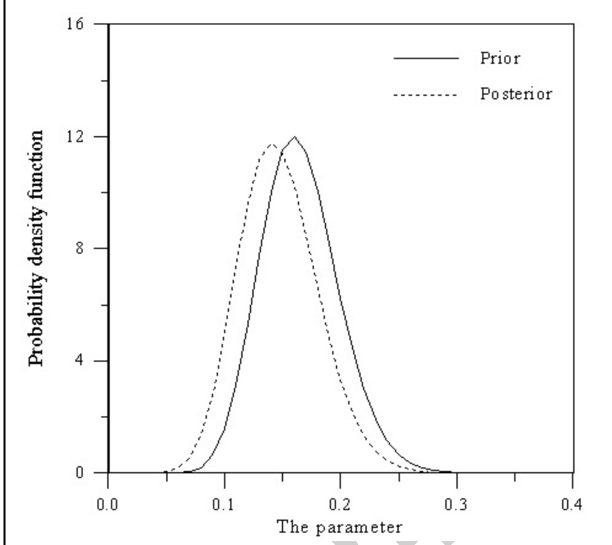

Fig. 2. The prior and posterior pdf's of p_2.

Example 4.2 In order study the influence of the sample size and the masking level on the accuracy of the point estimates, a large simulation study is carried out according to the following scheme

1. Specify the value of sample size $N = n$, no censoring.
2. Generate a random sample with size n of the system life time, $(t_1, s_1), \ldots, (t_n, s_n)$. Then the values of n_0, n_1, n_2 and n_{12} are determined.
3. Calculate $\hat{p}_1$, $\hat{p}_2$.
4. Calculate the squared error for each estimate, $(p_\ell - \hat{p}_\ell)^2$, $\ell = 1, 2$.
5. Repeat steps (2-4) 3000 times.
6. Calculate the mean squared error, MSE, associated with each estimate of p_ℓ, $\ell = 1, 2$, according to the following relation

$$\text{MSE}_{\hat{p}_\ell} = \frac{\sum_{i=1}^{3000}(p_\ell - \hat{p}_\ell^{(i)})^2}{3000},$$

   where $\hat{p}_\ell^{(i)}$ is the Bayes estimate of p_ℓ using sample number i.
7. Steps 1-6 are done for $n = 5, 10, 15, \ldots, 50$.
8. Steps 1-7 are repeated for the following three cases:
   (a) No masking case, it is assumed in this case that the cause of failure for each system failure is known.

(b) General masking-1 case, in this cases it is assumed that 20% of the observations are masked.

(c) General masking-2 case, in this cases it is assumed that 50% of the observations are masked.

Table 3 shows the values of the MSEs obtained from all cases proposed.

Table 3. The MSE$\times 10^4$ associated with estimates of p_1 and p_2.

	No masking		General masking-1		General masking-2	
n	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_1$	$\hat{p}_2$
5	7.528	9.128	8.437	9.969	8.440	10.01
10	6.026	8.472	6.787	9.792	7.307	9.904
15	4.779	7.749	5.616	8.021	5.776	9.267
20	4.282	6.265	5.467	7.051	5.496	8.267
25	3.620	5.845	4.263	6.364	4.805	7.468
30	3.248	5.058	3.953	5.940	4.314	6.986
35	3.118	4.641	3.539	4.978	4.339	6.569
40	2.745	4.393	3.178	4.732	3.801	6.499
45	2.693	3.843	3.163	4.499	3.564	5.646
50	2.476	3.696	3.162	4.498	3.666	5.635

According to the results shown in Table 3, one can conclude that:

(i) Each of MSE decreases with the increasing the sample size.

(ii) MSE takes its minimum when for the no masking case at all possible sample sizes.

(iii) MSE increases with increasing the masking level, for all sample sizes.

5 Conclusion

This paper presented the Bayes estimators for the reliability measures of the individual components in a two-component series system in the presence of masked system life test data. We studied the problem when the life times of the system components follow geometric distribution with different parameters. We explained how the case when the system consists of more than two components works. We assumed the non-informative and beta prior distributions families of the prior distributions of the unknown parameters. We derived the two sided Bayesian probability intervals of the unknown parameters. We presented numerical simulation studies in order to: (i) explain how the theoretical results obtained can be applied, (ii) how the masking level and the sample size affect the accuracy of point estimates. We observed that the point estimates become more accurate when the making level is small and the sample size is large. For further research, one can try to use the maximum likelihood and Bayesian approaches, to estimate the unknown parameters and the reliability measures for the systems which consist of more than two components. Also, one can try to solve the same problem when the data are progressively censored.

Appendix

In this appendix we discuss the extension of the problem from the case when $J = 2$ to the case when $J > 2$. Let us start with the case when $J = 3$ (the system consists of three components). In this case, we need the following notations. Let n_j be number observations when the component j, $j = 1, 2, 3$, causes the system failure. Let $t_{j,i}$ be the observed value of X when $S_i = \{j\}$, $i = 1, 2, \ldots, n_j$. Let $n_{\ell k}$ denote the number observations when either component ℓ or component k causes the system failure (partial masking). That is, $n_{\ell k}$ denotes the number of the observation when $S_i = \{\ell, k\}$, $\ell < k \in \{1, 2, 3\}$ and let $t_{\ell k,i}$ be the observed value of X in this case, $i = 1, 2, \cdots, n_{\ell k}$. Let n_{123} be the number observations when the cause of system failure is completely masked and $t_{123,i}$ be the observed value of X in this case, $i = 1, 2, \cdots, n_{123}$. That is, n_{123} denotes the number of the observation when $S_i = \{1, 2, 3\}$. Let $n_{0\ell k}$ be number observations when both components ℓ and k cause the system failure and let $t_{0\ell k,i}$ be the observed value of X in this case, $i = 1, 2, \cdots, n_{0\ell k}$. That is, $n_{0\ell k}$ denotes the number of the observation when $S_i = \{\ell, k\}_0$, $\ell < k \in \{1, 2, 3\}$. Finally, let n_{0123} denote the number observations when the components 1, 2 and 3 cause the system failure and let $t_{0123,i}$ be the observed value of X in this case, $i = 1, 2, \cdots, n_{0123}$. That is, n_{0123} denotes the number of the observation when $S_i = \{1, 2, 3\}_0$. It is so important to note that there is a difference between $\{\ell, k\}$ and $\{\ell, k\}_0$ and between $\{1, 2, 3\}$ and $\{1, 2, 3\}_0$.
Based on the above notations, the likelihood function (2.7) takes the following form

$$\begin{aligned}
L(data; \mathbf{p}) \;=\; & \prod_{i=1}^{n_j} \prod_{j=1}^{3} p_j q_j^{t_{j,i}-1} \prod_{\ell=1,\ell\neq j}^{3} q_\ell^{t_{j,i}} \\
& \prod_{i=1}^{n_{\ell k}} \prod_{1\leq \ell<k\leq 3} \left[p_\ell q_\ell^{t_{\ell k,i}-1} \prod_{j=1,j\neq \ell}^{3} q_j^{t_{\ell k,i}} + p_k q_k^{t_{\ell k,i}-1} \prod_{j=1,j\neq k}^{3} q_j^{t_{\ell k,i}} + p_\ell p_k (q_\ell q_k)^{t_{\ell k,i}-1} \right] \\
& \prod_{i=1}^{n_{123}} \sum_{j=1}^{3} \left[p_j q_j^{t_{123,i}-1} \prod_{\ell=1,\ell\neq j}^{3} q_\ell^{t_{123,i}} + \sum_{1\leq \ell<k\leq 3} p_\ell p_k (q_\ell q_k)^{t_{123,i}-1} \right] \\
& (p_1 p_2 p_3)^{n_{0123}} \prod_{i=1}^{n_{0\ell k}} \prod_{1\leq \ell<k\leq 3} \left[p_\ell p_k (q_\ell q_k)^{t_{0\ell k,i}-1} \right] \\
& (q_1 q_2 q_3)^{\sum_{i=1}^{n_{0123}} t_{0123,i} + \sum_{i=n+1}^{N} x_i - n_{0123}} q_1^{\sum_{i=1}^{n_{023}} t_{023,i}} q_2^{\sum_{i=1}^{n_{013}} t_{013,i}} q_3^{\sum_{i=1}^{n_{012}} t_{012,i}},
\end{aligned}$$

where $\mathbf{p} = (p_1, p_2, p_3)$.
To get the Bayes estimates of the parameters p_1, p_2 and p_2, we have to use the binomial expansion to expand the following expressions

$$\prod_{i=1}^{n_{\ell k}} \prod_{1\leq \ell<k\leq 3} \left[p_\ell q_\ell^{t_{\ell k,i}-1} \prod_{j=1,j\neq \ell}^{3} q_j^{t_{\ell k,i}} + p_k q_k^{t_{\ell k,i}-1} \prod_{j=1,j\neq k}^{3} q_j^{t_{\ell k,i}} + p_\ell p_k (q_\ell q_k)^{t_{\ell k,i}-1} \right]$$

and

$$\prod_{i=1}^{n_{123}} \sum_{j=1}^{3} \left[p_j q_j^{t_{123,i}-1} \prod_{\ell=1,\ell\neq j}^{3} q_\ell^{t_{123,i}} + \sum_{1\leq \ell<k\leq 3} p_\ell p_k (q_\ell q_k)^{t_{123,i}-1} \right]$$

in terms of p_j, $j = 1, 2, 3$. Then, we can continue as we did for the case $J = 2$. In a similar way we can formulate the problem when $J > 3$. Unfortunately, as it seems the problem becomes more complicated for a large J, for this reason we illustrated it when $J = 2$.

Proof of Theorem 3.1:
Based on the assumptions 2.1 and 2.2, the joint prior pdf of p_1 and p_2 becomes

$$g(p_1, p_2) = \frac{p_1^{\alpha_1 - 1}(1 - p_1)^{\beta_1 - 1}\, p_2^{\alpha_2 - 1}(1 - p_2)^{\beta_2 - 1}}{B(\alpha_1, \beta_1)\, B(\alpha_2, \beta_2)}\,, 0 < p_1, p_2 < 1 \tag{5.2}$$

But the joint posterior pdf of p_1, p_2 is related with their joint prior pdf and the likelihood function according to the following relation, see Martz and Waller [11],

$$g(p_1, p_2|data) = \frac{g(p_1, p_2)L(data; p_1, p_2)}{\int_0^1 \int_0^1 g(p_1, p_2)L(data; p_1, p_2)\, dp_1\, dp_2} \tag{5.3}$$

Substituting from equations (3.3) and (5.2) into (5.3) we get

$$g(p_1, p_2|data) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12} - i} \prod_{\ell=1}^{2} p_\ell^{a_\ell - 1}\, (1 - p_\ell)^{b_\ell - 1},\ 0 < p_1, p_2 < 1$$

where

$$I_0 = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12} - i} \prod_{\ell=1}^{2} \int_0^1 p_\ell^{a_\ell - 1}\, (1 - p_\ell)^{b_\ell - 1}\, dp_\ell \tag{5.4}$$

But

$$\int_0^1 p_\ell^{a_\ell - 1}\, (1 - p_\ell)^{b_\ell - 1}\, dp_\ell = B(a_\ell, b_\ell) \tag{5.5}$$

Using (5.4) and (5.5), one get I_0 as given by (3.5) which completes the proof. □

Proof of Theorem 3.3:
The marginal posterior mth moment of p_ℓ is related with the marginal posterior pdf of p_ℓ, $\ell = 1, 2$, according to the following relation

$$\mu_\ell^{(m)} = \int_0^1 p_\ell^m\, g_\ell(p_\ell|data)\, dp_\ell\,. \tag{5.6}$$

For $\ell = 1$, substituting from (3.6) into (5.6) we have

$$\begin{aligned}
\mu_1^{(m)} &= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12} - i}\, B(a_2, b_2) \int_0^1 p^{m + a_1 - 1}(1 - p_1)^{b_1 - 1}\, dp_1 \\
&= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12} - i}\, B(a_2, b_2) B(a_1 + m, b_1) \\
&= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12} - i} \prod_{k=1}^{2} B(a_k + m\delta_{k1}, b_k)
\end{aligned}$$

Similarly for $\ell = 2$, we can derive

$$\mu_2^{(m)} = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \begin{pmatrix} n_{12} \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k + m\delta_{k2}, b_k)$$

which completes the proof. □

The proof of theorem 3.2:
Under the squared error loss, the Bayes estimator for an unknown parameter is defined as its posterior posterior expectation and the associated minimum posterior risk is the posterior variance, see Martz an Waller [11]). That is, the Bayes estimator for p_ℓ is

$$\hat{p}_\ell = E\left[p_\ell|data\right] = \mu_\ell^{(1)}\,,\ \ell = 1, 2\,. \tag{5.7}$$

and the minimum posterior risk associated with $\hat{p}_\ell$ is

$$R_{\hat{p}_\ell} = \mathrm{Var}\left[p_\ell|data\right] = \mu_\ell^{(2)} - \left[\mu_\ell^{(1)}\right]^2\,,\ \ell = 1, 2 \tag{5.8}$$

Substituting from (3.8) into (5.7) and (5.8) one can reach the proof of the Theorem. □

Proof of Theorem 3.3:
The Bayes estimator for a function of p_ℓ and the associated minimum posterior risk are defined respectively as the posterior expectation and variance of that function, see Martz and Waller [11]. That is, the Bayes estimator for $\bar{F}_\ell(x_0)$, $\ell = 1, 2$, is

$$\widehat{\bar{F}}_\ell(x_0) = E\left[\bar{F}_\ell(x_0)|data\right] = \int_0^1 (1-p_\ell)^{x_0}\, g_\ell(p_\ell|data)\, dp_\ell\,. \tag{5.9}$$

and the associated minimum posterior risk is

$$\begin{aligned} R_{\widehat{\bar{F}}_\ell} &= \mathrm{Var}\left[\bar{F}_\ell(x_0)|data\right] \\ &= E\left[\left\{\bar{F}_\ell(x_0)\right\}^2|data\right] - \left\{E\left[\bar{F}_\ell(x_0)|data\right]\right\}^2 \\ &= \int_0^1 (1-p_\ell)^{2\,x_0}\, g_\ell(p_\ell|data)\, dp_\ell - \left\{\int_0^1 (1-p_\ell)^{x_0}\, g_\ell(p_\ell|data)\, dp_\ell\right\}^2\,. \end{aligned} \tag{5.10}$$

Substituting from (3.6) and (3.7) into (5.9) and (5.10) and making some simple integrations, one can complete the proof. □

Proof of Theorem 3.4:
As before, the Bayes estimator for $r_\ell(x_0)$ is given by

$$\hat{r}_\ell(x_0) = \int_0^1 r_\ell(x_0) g_\ell(p_\ell|data)\, dp_\ell$$

Substituting from (3.6) into the above relation, we get

$$\begin{aligned}\hat{r}_\ell(x_0) &= \int_0^1 \Big[-\ln(1-p_\ell)\Big]\, g_\ell(p_\ell|data)\, dp_\ell \\ &= -E\Big[\ln(1-p_\ell)|data\Big] \qquad (5.11)\end{aligned}$$

where

$$E\Big[\ln(1-p_\ell)|data\Big] = \int_0^1 \ln(1-p_\ell)\, g_\ell(p_\ell|data)\, dp_\ell \qquad (5.12)$$

For $\ell = 1$, using the form of the function $g_1(p_1|data)$ together with (5.12), one can derive that

$$E\Big[\ln(1-p_1)|data\Big] = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)\,\kappa(a_1,b_1) \qquad (5.13)$$

where

$$\kappa(a_1,b_1) = \int_0^1 p_1^{a_1-1}(1-p_1)^{b_1-1}\ln(1-p_1)\, dp_1$$

Let $x = 1-p_1$, then

$$\kappa(a_1,b_1) = \int_0^1 x^{b_1-1}(1-x)^{a_1-1}\ln x\, dx$$

According to formula 4.253 (Gradshteyn and Ryzhik [7], p. 538), we have

$$\kappa(a_1,b_1) = B(a_1,b_1)\Big[\psi(b_1)-\psi(a_1+b_1)\Big] \qquad (5.14)$$

Substituting from (5.14) into (5.13), one can get

$$E\Big[\ln(1-p_1)|data\Big] = \frac{-1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\Big[\psi(a_1+b_1)-\psi(b_1)\Big]\prod_{k=1}^{2}B(a_k,b_k)\,, \qquad (5.15)$$

Substituting from (5.15) into (5.11) we get $\hat{r}_1(x_0)$ as given by (3.15) when $\ell = 1$. Similarly we can prove that the relation (3.15) is correct for $\ell = 2$. Let us now prove that the relation (3.16) is fulfilled. As it was stated before, the minimum posterior risk associated with $\hat{r}_\ell(x_0)$ is the posterior variance of $r_\ell(x_0)$. That is,

$$\begin{aligned}R_{\hat{r}_\ell} &= E\Big[r_\ell^2(x_0)|data\Big] - \Big\{E\Big[r_\ell(x_0)|data\Big]\Big\}^2 \\ &= E\Big[(-\ln(1-p_\ell))^2|data\Big] - \Big\{E\Big[-\ln(1-p_\ell)|data\Big]\Big\}^2 \\ &= E\Big[\{\ln(1-p_\ell)\}^2|data\Big] - \Big\{E\Big[\ln(1-p_\ell)|data\Big]\Big\}^2 \qquad (5.16)\end{aligned}$$

For $\ell = 1$, $E\Big[\ln(1-p_1)|data\Big]$ is given by (5.15) and $E\Big[\{\ln(1-p_1)\}^2|data\Big]$ can be derived as follows

$$E\Big[\{\ln(1-p_1)\}^2|data\Big] = \int_0^1 \{\ln(1-p_1)\}^2 g_1(p_1|data)\,dp_1 \tag{5.17}$$

Substituting from (3.6) into (5.16), one can get

$$E\Big[\{\ln(1-p_1)\}^2|data\Big] = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}B(a_2,b_2)\,\mu(a_1,b_1), \tag{5.18}$$

where

$$\mu(a_1,b_1) = \int_0^1 p_1^{a_1-1}(1-p_1)^{b_1-1}\ln^2(1-p_1)\,dp_1$$

Let $x = 1-p_1$, then

$$\mu(a_1,b_1) = \int_0^1 x^{b_1-1}(1-x)^{a_1-1}\ln^2 x\,dx$$

According to formula 4.261.17 (Gradshteyn and Ryzhik, [7], p. 541), we have

$$\mu(a_1,b_1) = B(a_1,b_1)\Big[\{\psi(b_1)-\psi(a_1+b_1)\}^2 + \psi^{'}(b_1) - \psi^{'}(a_1+b_1)\Big] \tag{5.19}$$

Substituting from (5.19) into (5.18), then using the result obtained together with relations (5.15) and (5.16) one can get R_{r_1} as given by (3.16) when $\ell = 1$. Similarly one prove the case when $\ell = 2$ which completes the proof. □

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