Bayesian Inference and Life Testing Plan for the Weibull Distribution in Presence of Progressive Censoring

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This article deals with the Bayesian inference of unknown parameters of the progressively censored Weibull distribution. It is well known that for a Weibull distribution, while computing the Bayes estimates, the continuous conjugate joint prior distribution of the shape and scale parameters does not exist. In this article it is assumed that the shape parameter has a log-concave prior density function, and for the given shape parameter, the scale parameter has a conjugate prior distribution. As expected, when the shape parameter is unknown, the closed-form expressions of the Bayes estimators cannot be obtained. We use Lindley's approximation to compute the Bayes estimates and the Gibbs sampling procedure to calculate the credible intervals. For given priors, we also provide a methodology to compare two different censoring schemes and thus find the optimal Bayesian censoring scheme. Monte Carlo simulations are performed to observe the behavior of the proposed methods, and a data analysis is conducted for illustrative purposes.

KEY WORDS: Credible intervals; Fisher information matrix; Gibbs sampling; Log-concave density function; Markov chain Monte Carlo; Optimum censoring scheme; Prior distribution; Scale parameter; Shape parameter.

1. INTRODUCTION

The Weibull distribution is one of the most widely used distributions in reliability and survival analysis. A detailed discussion on it has been given by Johnson, Kotz, and Balakrishnan (1995, chap. 21). Because of its various shapes of the probability density function (pdf) and its convenient representation of the distribution/survival function, the Weibull distribution has been used very effectively for analyzing lifetime data, particularly when the data are censored, which is very common in most life testing experiments. Among the different censoring schemes, the progressive censoring scheme has received a considerable attention in the last few years, particularly in reliability analysis. It is a more general censoring mechanism than the traditional type II censoring. The recent review article by Balakrishnan (2007) provided details on progressive censoring scheme and on its different applications. Estimation and optimal progressive censoring plans for the Weibull distribution were discussed in detail by Balakrishnan and Aggarwala (2000), Viveros and Balakrishnan (1994), Balasooriya, Saw, and Gadag (2000) and Ng, Chan, and Balakrishnan (2004). Bayesian inference of the unknown parameters of a Weibull distribution in the presence of progressive censoring has not yet been studied. In this article, we explain that if we have a proper prior knowledge of the unknown parameters, then Bayesian inference has a clear advantage over the classical inferences. Moreover, we address the construction of the Bayesian optimal progressive censoring plan here when both the parameters are unknown.

It is assumed that the lifetimes of the units being tested, have a Weibull distribution function with the pdf

$$f(t; \alpha, \lambda) = \begin{cases} \alpha \lambda e^{-\lambda t^{\alpha}} t^{\alpha-1} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \quad (1)$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters. The Weibull distribution with the shape and scale parameters as α and λ is denoted by $WE(\alpha, \lambda)$. In this work, we first compute the Bayes estimates and construct the credible intervals of α and λ with respect to different priors, and compare these with the classical maximum likelihood estimators (MLEs) and with the confidence intervals based on the asymptotic distributions of the MLEs. Then, for given priors, we provide the methodology to compare two different sampling schemes and thus to compute the optimal progressive censoring plan for the Weibull distribution under the Bayesian setup. Recently, Zhang and Meeker (2005) described a Bayesian life testing plan for the Weibull distribution with known shape parameter and when the data are type II censored. The present article provides a more general life testing plan both in terms of censoring mechanism and model assumptions.

For the Bayesian inference and life testing plan, we need to assume some prior distribution(s) of the unknown parameter(s). If the shape parameter, α, is known, then the natural choice of the prior is the conjugate prior on the scale parameter, λ. Interestingly—although not surprisingly—in this case the explicit closed-form Bayes estimate of λ with respect to the squared error loss function can be obtained. Moreover, the corresponding credible interval also can be constructed explicitly.

It is well known that when α is unknown, the continuous conjugate priors do not exist (see, e.g., Kaminskiy and Krivtsov 2005). We use the same conjugate prior on λ even when α was unknown. No specific prior is assumed on α; it simply is assumed that the support of the prior is $(0, \infty)$ and that it has a

log-concave density function. Note that many common density functions are log-concave in nature; for example, the normal and lognormal distributions always have log-concave density functions. When the shape parameters are ≥ 1, then the gamma and Weibull distributions also have log-concave density functions. When α is unknown, as expected, the explicit expressions of the Bayes estimates cannot be obtained. We propose Lindley's approximation to construct the Bayes estimates and Markov chain Monte Carlo (MCMC) techniques to compute the credible intervals of the unknown parameters.

When the shape parameter is known in the type II censored data, Zhang and Meeker (2005) found an optimum design based on two criteria: a criterion based on large-sample approximate posterior precision factor and a criterion based on an exact relative posterior credibility interval length. Both criteria are based on the precision of estimating the pth quantile of this lifetime distribution. It is observed that the performances of both the criteria are very similar in nature. We propose two criteria that are modifications of the criterion based on large-sample approximate credible interval length. Our criteria also are based on the precision of estimating the pth quantile. Our first criterion depends on the pth quantile estimator; the second is based on the overall percentile estimator, similar to the idea proposed by Gupta and Kundu (2006). The second criterion does not depend on p, but it does depends on the sample size and the number of failures.

The rest of the article is organized as follows. In Section 2 we provide the model assumptions, notations, and prior distributions. We cover Bayes estimates and construction of credible intervals using the MCMC techniques in Section 3. Numerical comparisons of the Bayes estimates and a data analysis are presented in Section 4. The methodology of constructing Bayesian life testing plan are presented in Section 5, and the article concludes in Section 6.

2. PROBLEM FORMULATION AND ASSUMPTIONS

In this section we first describe the prior information needed for the Bayesian analysis of the unknown parameter(s) and then briefly describe the progressive censoring scheme.

2.1 Prior Information

When the shape parameter is known, the scale parameter has a conjugate gamma prior. It is assumed that the prior distribution of λ is $Gamma(a, b)$, with pdf

$$\pi_1(\lambda|a, b) = \begin{cases} \dfrac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \leq 0, \end{cases} \quad (2)$$

with the hyperparameters $a > 0$, $b > 0$, and $\Gamma(a) = \int_0^\infty x^{a-1} \times e^{-x}\, dx$. In most practical applications with proper informative information on the Weibull scale parameter, the prior variance is usually finite.

Now we provide the prior information in the case when the shape parameter is unknown, which is more common in practice. It is known that in this case the Weibull distribution does not have a continuous conjugate joint prior distribution, although there exists a continuous-discrete joint prior distribution

(see Soland 1969). The continuous component of this distribution is related to the scale parameter, and the discrete one is related to the shape parameter. This method has been widely criticized because of its difficulty in application to real-life problems (see Kaminskiy and Krivtsov 2005), and thus is not addressed further.

Following the approach of Berger and Sun (1993) and Kundu and Gupta (2006), it is assumed that λ has the same prior distribution as defined in (2). No specific form of the prior $\pi_2(\alpha)$ on α is assumed here; however, it is assumed that the support of $\pi_2(\alpha)$ is $(0, \infty)$ and that it is independent of λ. Moreover, the pdf of $\pi_2(\alpha)$ is log-concave in nature. For computing the Bayes estimates or credible intervals, we need to assume specific form of $\pi_2(\alpha)$, which may depend on some hyperparameters. The choice of the hyperparameters has not been pursued here. We briefly discuss the progressive censoring scheme in the next section.

2.2 Progressive Censoring Scheme

Suppose that n identical units are put on a test, with the lifetimes of the n items are denoted by $T_1, \ldots, T_n$. It is assumed that the T_i's are independent and identically distributed random variables with pdf as in (1). The integer $m < n$ is prefixed, and $R_1, \ldots, R_m$ are m prefixed nonnegative integers satisfying $R_1 + \cdots + R_m + m = n$. At the time of the first failure, say t_1, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, t_2, R_2 of the remaining units are removed, and so on. Finally, at the time of the mth failure, the rest of the units, $R_m = n - R_1 - \cdots - R_{m-1} - m$, are removed. Therefore, we observe the data $\{(t_1, R_1), \ldots, (t_m, R_m)\}$ in a progressive censoring scheme. Although we have included $R_1, \ldots, R_m$ as part of the data, these are known in advance. Note that the usual type II censoring scheme is a special case of the progressive censoring scheme that can be obtained simply by taking $R_1 = \cdots = R_{m-1} = 0$.

3. BAYES ESTIMATES AND CREDIBLE INTERVALS

In this case, we provide the Bayes estimates of the unknown parameter(s) and the corresponding credible interval(s) for when the shape parameter is known and when the shape parameter is unknown, with respect to the prior distribution(s) described in Section 2. For computing the Bayes estimates, we assume mainly a squared error loss (SEL) function only; however, any other loss function can be easily incorporated.

3.1 Shape Parameter Known

The likelihood function of the observed sample $\{(t_1, R_1), \ldots, (t_m, R_m)\}$ is

$$l(data|\alpha, \lambda) \propto \alpha^m \lambda^m \prod_{i=1}^m t_i^{\alpha-1} e^{-\lambda \sum_{i=1}^m (R_i+1)t_i^\alpha}. \quad (3)$$

When α is known and λ has a prior pdf as given in (2), the posterior distribution of λ, given the *data* and the hyperparameters a and b, is $Gamma(a + m, b + \sum_{i=1}^m (R_i + 1)t_i^\alpha)$. Therefore, the Bayes estimate of λ with respect to the SEL is

$$\widehat{\lambda}_{Bayes} = \frac{a + m}{b + \sum_{i=1}^m (R_i + 1)t_i^\alpha}. \quad (4)$$

Because the posterior distribution of λ follows gamma, a credible interval of λ can be readily obtained. Moreover, if $a + m$ is a positive integer, then chi-squared table values also can be used for constructing a credible interval of λ. Next, we consider the more important case when the shape parameter is not known.

3.2 Shape Parameter Unknown

In this section we describe how to obtain the Bayes estimates and the corresponding credible intervals of α and λ when both are unknown. We assume that α and λ have the joint prior distribution as described in the previous section. The likelihood function of the observed sample is same as (3). Using the joint prior distribution of α and λ, we obtain the joint distribution of the *data*, α, and λ as

$$l(data|\alpha, \lambda) \times \pi_1(\lambda|a, b) \times \pi_2(\alpha). \tag{5}$$

Based on (5), the joint posterior density of α and λ given the data is

$$l(\alpha, \lambda|data)$$
$$= \frac{l(data|\alpha, \lambda) \times \pi_1(\lambda|a, b) \times \pi_2(\alpha)}{\int_0^\infty \int_0^\infty l(data|\alpha, \lambda) \times \pi_1(\lambda|a, b) \times \pi_2(\alpha) \, d\alpha \, d\lambda}; \tag{6}$$

therefore, the Bayes estimate of any function of α and λ say $g(\alpha, \lambda)$, under the squared error loss function is

$$\hat{g}_B(\alpha, \lambda)$$
$$= E_{\alpha, \lambda|data}(g(\alpha, \lambda))$$
$$= \frac{\int_0^\infty \int_0^\infty g(\alpha, \lambda)l(data|\alpha, \lambda)\pi_1(\lambda|a, b)\pi_2(\alpha) \, d\alpha \, d\lambda}{\int_0^\infty \int_0^\infty l(data|\alpha, \lambda)\pi_1(\lambda|a, b)\pi_2(\alpha) \, d\alpha \, d\lambda}. \tag{7}$$

It is not possible to compute (7) analytically even when $\pi_2(\alpha)$ is known explicitly. Therefore, we propose two approaches to approximate (7): Lindley's approximation and the MCMC technique.

3.2.1 Lindley's Approximation. To calculate the Bayes estimates, we need to specify the prior on α. We assume that λ has a *Gamma(a, b)* prior, and that α has a *Gamma(c, d)* prior, and that α and λ are independently distributed. Now, based on the foregoing priors, we compute the approximate Bayes estimates of α and λ using Lindley's approximation technique.

Lindley (1980) proposed his procedure to approximate the ratio of two integrals such as in (7). This approach has been used by several authors to obtain the approximate Bayes estimates (for details, see Lindley 1980 or Press 2001). Based on Lindley's approximation, the approximate Bayes estimates of α and λ under the squared error loss function are

$$\hat{\alpha}_B = \hat{\alpha} + \frac{1}{2}\left[\left(\frac{2m}{\hat{\alpha}^3} - \hat{\lambda}\sum_{i=1}^m (R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^3\right)\tau_{11}^2\right.$$
$$\left. + \frac{2m}{\hat{\lambda}^3}\tau_{21}\tau_{22} - 3\tau_{11}\tau_{12}\sum_{i=1}^m (R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^2\right]$$
$$+ \tau_{11}\left(\frac{c-1}{\hat{\alpha}} - d\right) + \tau_{12}\left(\frac{a-1}{\hat{\lambda}} - b\right) \tag{8}$$

and

$$\hat{\lambda}_B = \hat{\lambda} + \frac{1}{2}\left[\left(\frac{2m}{\hat{\alpha}^3} - \hat{\lambda}\sum_{i=1}^m (R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^3\right)\tau_{11}\tau_{12}\right.$$
$$\left. + \frac{2m}{\hat{\lambda}^3}\tau_{22}^2 - (\tau_{11}\tau_{22} + 2\tau_{12}^2)\left(\sum_{i=1}^n (R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^2\right)\right]$$
$$+ \tau_{21}\left(\frac{c-1}{\hat{\alpha}} - d\right) + \tau_{22}\left(\frac{a-1}{\hat{\lambda}} - b\right). \tag{9}$$

Here

$$\tau_{11} = \frac{W}{UW - V^2}, \qquad \tau_{22} = \frac{U}{UW - V^2},$$
$$\tau_{12} = \tau_{21} = -\frac{V}{UW - V^2},$$
$$U = \frac{m}{\hat{\alpha}^2} + \hat{\lambda}\sum_{i=1}^m (R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^2, \tag{10}$$
$$V = \sum_{i=1}^m (R_i + 1)t_i^{\hat{\alpha}}\ln t_i, \quad \text{and} \quad W = \frac{m}{\hat{\lambda}^2}.$$

Moreover, $\hat{\alpha}$ and $\hat{\lambda}$ denote the MLEs of α and λ. Lindley's approximations of the expressions (8) and (9) are summarized in Appendix A. Note that although we can compute the approximate Bayes estimates of α and λ using Lindley's approximation, we cannot compute the credible intervals though this approach. We propose using the MCMC technique to compute Bayes estimates of the unknown parameters and to construct the corresponding credible intervals.

3.2.2 Gibbs Sampling. We propose using the Gibbs sampling procedure to generate a sample from the posterior density function $l(\alpha, \lambda|data)$ and in turn compute the Bayes estimates and also construct the corresponding credible intervals based on the generated posterior sample. For generating a sample from the posterior distribution, we have assumed that the pdf of $\pi_1(\lambda|a, b)$ has the form of (2), that the pdf of $\pi_2(\alpha)$ is log-concave, and that the two pdf's are independent. We need the following results for further development.

Theorem 1. The conditional density of λ, given α and the *data*, is *Gamma($a + m, b + \sum_{i=1}^m (R_i + 1)t_i^\alpha$)*.

Proof. It is trivial and thus is not provided.

Theorem 2. The conditional density of α, given the *data*, is log-concave.

Proof. See Appendix B.

Now, using Theorems 1 and 2 and following the idea of Geman and Geman (1984), we propose the following scheme to generate (α, λ) from the posterior density function and in turn obtain the Bayes estimates and the corresponding credible intervals.

Algorithm 1.
Step 1: Generate α_1 from the log-concave density $l(\cdot|data)$ as given in (B.2) in Appendix B using the method proposed by Devroye (1984).
Step 2: Generate λ_1 from $\pi_1(\cdot|\alpha, data)$ as given in Theorem 1.

Step 3: Repeat Steps 1 and 2 M times and obtain $(\alpha_1, \lambda_1), \ldots,$ (α_M, λ_M).

Step 4: Obtain the Bayes estimates of α and λ with respect to the SEL function as

$$\widehat{E}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}\alpha_k \qquad \text{and}$$

$$\widehat{E}(\lambda|data) = \frac{1}{M}\sum_{k=1}^{M}\lambda_k.$$

Step 5: Obtain the posterior variances of α and λ as

$$\widehat{V}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}(\alpha_k - \widehat{E}(\alpha|data))^2 \qquad \text{and}$$

$$\widehat{V}(\lambda|data) = \frac{1}{M}\sum_{k=1}^{M}(\lambda_k - \widehat{E}(\lambda|data))^2.$$

Step 6: To compute the credible intervals of α and λ, order $\alpha_1, \ldots, \alpha_M$ and $\lambda_1, \ldots, \lambda_M$ as $\alpha_{(1)} < \cdots < \alpha_{(M)}$ and $\lambda_{(1)} < \cdots < \lambda_{(M)}$. Then the $100(1-2\beta)\%$ symmetric credible intervals of α and λ become

$$\left(\alpha_{(M\beta)}, \alpha_{(M(1-\beta))}\right) \qquad \text{and} \qquad \left(\lambda_{(M\beta)}, \lambda_{(M(1-\beta))}\right).$$

4. SIMULATIONS AND DATA ANALYSIS

4.1 Simulations

In this section we report some numerical experiments performed to evaluate the behavior of the proposed methods for different sample sizes, different sampling schemes, different parameter values, and different priors. All of the computations were performed at IIT Kanpur using a Pentium IV processor. For random number generation, we used the RAN2 program of Press, Flannery, Teukolsky, and Vetterling (1991), with all codes written in FORTRAN-77.

We used different sample sizes (n), different effective sample sizes (m), different hyperparameters (a, b, c, d), and different sampling schemes (i.e., different R_i values). We used two sets of parameter values—$\alpha = 1, \lambda = 1$ and $\alpha = 1.5, \lambda = 1.0$—mainly to compare the MLEs and different Bayes estimators and also to explore their effects on different parameter values.

To generate progressively censored Weibull samples, we used the algorithm proposed by Balakrishnan and Aggarwala (2000). First, we used the noninformative gamma priors for both the shape and scale parameters, that is, when the hyperparameters are 0. We call it prior 0: $a = b = c = d = 0$. Note that as the hyperparameters go to 0, the prior density becomes inversely proportional to its argument and also becomes improper. This density is commonly used as an improper prior for parameters in the range of 0 to infinity, and this prior is not specifically related to the gamma density. Moreover, it should be mentioned that even if we take noninformative gamma priors, that are not log-concave, the posterior density of α is still log-concave.

For computing Bayes estimators, other than prior 0, we also used different informative priors, including prior 1, $a = b = c = d = 5$; prior 2, $a = b = c = d = 1$, when $\alpha = 1$ and $\lambda = 1$;

prior 3, $a = b = 5$, $c = 2.25$, and $d = 1.5$, when $\alpha = 1.5$ and $\lambda = 1$. Purposefully, we kept the prior means the same as the original means, although we have seen (not reported here) that the overall behavior remains the same. Note that when $\alpha = 1$ and $\lambda = 1$, prior 1 is more informative than prior 2, because the variance of prior 1 is smaller than that of prior 2, and both are more informative than the noninformative priors. In all cases, we used the squared error loss function to compute the Bayes estimates. We computed the Bayes estimates using Lindley's approximations. We also computed the Bayes estimates and 95% credible intervals based on 10,000 MCMC samples.

We report the average Bayes estimates, mean squared errors (MSEs), coverage percentages, and average confidence interval lengths based on 1,000 replications. For comparison purposes, we also compute the MLEs and the 95% confidence intervals based on the observed Fisher information matrix. We consider the following sampling schemes:

Scheme 1: $n = 20$, $m = 15$, $R_1 = \cdots = R_{14} = 0$, $R_{15} = 5$; (20, 15, 14*0, 5)

Scheme 2: $n = 20$, $m = 15$, $R_1 = 5$, $R_2 = \cdots = R_{15} = 0$; (20, 15, 5, 14*0)

Scheme 3: $n = 30$, $m = 15$, $R_1 = \cdots = R_{14} = 0$, $R_{15} = 15$; (30, 15, 14*0, 15)

Scheme 4: $n = 30$, $m = 15$, $R_1 = 15$, $R_2 = \cdots = R_{15} = 0$; (30, 15, 15, 14*0)

Scheme 5: $n = 30$, $m = 15$, $R_1 = \cdots = R_{15} = 1$; (30, 15, 1*15)

Scheme 6: $n = 30$, $m = 20$, $R_1 = \cdots = R_{19} = 0$, $R_{20} = 10$; (30, 20, 19*0, 10)

Scheme 7: $n = 30$, $m = 20$, $R_1 = 10$, $R_2 = \cdots = R_{20} = 0$; (30, 20, 10, 19*0).

Note that Schemes 1, 3, and 6 are the usual type II censoring schemes for fixed n and m; that is, $n - m$ items are removed at the time of the mth failure. Schemes 2, 4, and 7 are just the opposite of the type II censoring schemes in the sense for fixed n and m, $n - m$ items are removed at the time of the first failure. It is well known that for fixed n and m, the expected experimental time of the type II censoring scheme (i.e., for Schemes 1, 3, and 6) are less than that for Schemes 2, 4, and 7. In fact, the expected time of any other censoring scheme (for fixed m and n) is always between these two extremes; for example, the expected experimental time of Scheme 5 lies between Schemes 3 and 4. Finally, we mention that for $\alpha = 1$ and $\lambda = 1$, we used the same 1,000 replicates to compute different estimates for each scheme and similarly for $\alpha = 1.5$ and $\lambda = 1$, to avoid sampling bias.

The results for $\alpha = 1$ and $\lambda = 1$ are reported in Tables 1–4, and the results for $\alpha = 1.5$ and $\lambda = 1$ are reported in Tables 5 and 6. Tables 1 and 3 report the results based on MLEs and the Bayes estimators (using both the Gibbs sampling procedure and Lindley's approximations) using noninformative priors on both α and λ. Tables 2 and 4 report the results based on two different informative priors, prior 1 and prior 2. Because also in these cases the results based on Lindley's approximations and using the Gibbs sampling procedure are quite similar in nature, we report only the results based on the Gibbs sampling procedure. For $\alpha = 1.5$ and $\lambda = 1$, the average values of the relative Bayes estimates ($\frac{\hat{\alpha}}{\alpha}$) and corresponding MSEs, based on prior 0 and prior 3 are reported in Table 5. The average relative credible intervals and corresponding coverage percentages are reported in

Table 1. Average values of the different estimators and the corresponding MSEs when $\alpha = 1$ and $\lambda = 1$

Sampling scheme	MLE		Bayes (MCMC)		Bayes (Lindley)	
	α	λ	α	λ	α	λ
1	1.1071	1.0894	1.0966	1.0760	1.1009	1.0838
(20, 15, 14*0, 5)	(.0913)	(.1386)	(.0866)	(.1371)	(.0895)	(.1411)
2	1.0886	1.0396	1.0784	1.0246	1.0825	1.0300
(20, 15, 5, 14*0)	(.0613)	(.1063)	(.0586)	(.0999)	(.0602)	(.1008)
3	1.1170	1.1971	1.1081	1.2065	1.1131	1.2148
(30, 15, 14*0, 15)	(.0991)	(.3473)	(.0951)	(.4207)	(.0974)	(.3976)
4	1.0861	1.0610	1.0817	1.0405	1.0847	1.0464
(30, 15, 15, 14*0)	(.0530)	(.0951)	(.0520)	(.0889)	(.0532)	(.0900)
5	1.1080	1.1541	1.0938	1.1363	1.0976	1.1430
(30, 15, 1*15)	(.0786)	(.1936)	(.0719)	(.2011)	(.0755)	(.1945)
6	1.0924	1.0942	1.0847	1.0859	1.0883	1.0912
(30, 20, 19*0, 10)	(.0648)	(.0983)	(.0626)	(.0973)	(.0640)	(.0997)
7	1.0592	1.0591	1.0529	1.0459	1.0553	1.0506
(30, 20, 10, 19*0)	(.0426)	(.0785)	(.0408)	(.0742)	(.0424)	(.0750)

NOTE: Corresponding to each scheme, the first figure represents the average estimates, with the corresponding MSEs reported below it in parentheses.

Table 6. In this case the results based on the MLEs and Lindley's approximations are not reported. The performance of the MLEs is very similar to that of the Bayes estimators based on prior 0. Similarly, the performance of the Lindley estimators is very similar to that of the corresponding Bayes estimators using the Gibbs sampling procedure.

Some of the foregoing points are made very clear in Tables 1 and 3. It can be seen that the performance of the MLEs is quite close to that of the Bayes estimators with respect to the non-informative priors, as expected. Thus, if we have no prior information on the unknown parameters, then it is always better to use the MLEs rather than the Bayes estimators, because the Bayes estimators are computationally more expensive. Comparing the two Bayes estimators based on two informative priors clearly shows that the Bayes estimators based on prior 1 perform better than the Bayes estimators based on prior 2, in terms of both MSEs and lengths of the credible intervals. But both of these perform better than the Bayes estimators obtained using noninformative priors. From Tables 5 and 6, it is clear that the Bayes estimators based on informative priors perform much better than the Bayes estimators based on noninformative priors or MLEs in terms of biases, MSEs, and lengths of credible intervals.

Table 2. Average values of the Bayes estimators with respect to different priors, and their corresponding MSEs when $\alpha = 1$ and $\lambda = 1$

Sampling scheme	Bayes (MCMC) Prior 1		Bayes (MCMC) Prior 2	
	α	λ	α	λ
1	1.0504	1.0604	1.0825	1.0700
(20, 15, 14*0, 5)	(.0411)	(.0910)	(.0615)	(.1142)
2	1.0471	1.0338	1.0698	1.0273
(20, 15, 5, 14*0)	(.0334)	(.0922)	(.0512)	(.0976)
3	1.0408	1.0959	1.0807	1.1496
(30, 15, 14*0, 15)	(.0445)	(.1048)	(.0671)	(.1908)
4	1.0541	1.0464	1.0744	1.0421
(30, 15, 15, 14*0)	(.0315)	(.0822)	(.0463)	(.0870)
5	1.0562	1.0968	1.0804	1.1202
(30, 15, 1*15)	(.0333)	(.1018)	(.0592)	(.1508)
6	1.0509	1.0690	1.0752	1.0805
(30, 20, 19*0, 10)	(.0351)	(.0736)	(.0544)	(.0882)
7	1.0362	1.0507	1.0486	1.0472
(30, 20, 10, 19*0)	(.0271)	(.0704)	(.0372)	(.0731)

NOTE: Corresponding to each scheme, the first figure represents the average estimates, with the corresponding MSEs reported below it in parentheses.

Table 3. Average confidence interval/credible interval lengths and the coverage percentages when $\alpha = 1$ and $\lambda = 1$

Sampling scheme	MLE		Bayes (MCMC)	
	α	λ	α	λ
1	.9811	1.1525	.9763	1.1461
(20, 15, 14*0, 5)	(.96)	(.95)	(.94)	(.94)
2	.8393	1.1025	.8350	1.0876
(20, 15, 5, 14*0)	(.95)	(.93)	(.94)	(.94)
3	1.0446	1.5474	1.0413	1.6508
(30, 15, 14*0, 15)	(.96)	(.97)	(.95)	(.94)
4	.7939	1.1074	.7920	1.0891
(30, 15, 15, 14*0)	(.95)	(.95)	(.94)	(.94)
5	.8863	1.2783	.8813	1.2731
(30, 15, 1*15)	(.95)	(.97)	(.94)	(.95)
6	.8559	1.0188	.8521	1.0163
(30, 20, 19*0, 10)	(.95)	(.96)	(.93)	(.94)
7	.6992	.9639	.6962	.9534
(30, 20, 10, 19*0)	(.95)	(.94)	(.94)	(.94)

NOTE: Corresponding to each scheme, the first figure represents the average confidence interval/credible interval length, with the corresponding coverage percentage reported below it in parentheses.

Table 4. Average credible interval lengths and the coverage percentages of the two Bayes estimators when $\alpha = 1$ and $\lambda = 1$

Sampling scheme	Prior 1 (MCMC)		Prior 2 (MCMC)	
	α	λ	α	λ
1	.8308	1.1068	.9382	1.1298
(20, 15, 14*0, 5)	(.98)	(.96)	(.95)	(.95)
2	.7345	1.0842	.8100	1.0873
(20, 15, 5, 14*0)	(.97)	(.95)	(.95)	(.94)
3	.8295	1.2919	.9686	1.4516
(30, 15, 14*0, 15)	(.98)	(.97)	(.96)	(.95)
4	.7069	1.0862	.7714	1.0885
(30, 15, 15, 14*0)	(.97)	(.95)	(.95)	(.95)
5	.7627	1.1899	.8499	1.2381
(30, 15, 1*15)	(.97)	(.97)	(.95)	(.96)
6	.7482	.9831	.8259	1.0057
(30, 20, 19*0, 10)	(.96)	(.96)	(.95)	(.95)
7	.6374	.9510	.6823	.9529
(30, 20, 10, 19*0)	(.97)	(.95)	(.95)	(.95)

NOTE: Corresponding to each scheme, the first figure represents the average credible length, with the corresponding coverage percentage reported below it in parentheses.

From Tables 1–4, comparing the schemes 1 and 2, 3 and 4, and 6 and 7, it is clear that the biases, MSEs, and average confidence interval lengths/credible interval lengths of the MLEs and Bayes estimators for both parameters are greater for the censoring schemes 1, 3, and 6 than the censoring schemes 2, 4, and 7. This may not be very surprising, because the expected duration of the experiments is greater for censoring schemes 2, 4, and 7 than for the censoring schemes 1, 3, and 6. Thus the data obtained by the censoring schemes 2, 4, and 7 would be expected to provide more information about the unknown parameters than the data obtained by censoring schemes 1, 3, and 6. Com-

Table 5. Average values of the relative Bayes estimators with respect to different priors and their MSEs when $\alpha = 1.5$ and $\beta = 1$

Sampling scheme	Bayes (MCMC) Prior 0		Bayes (MCMC) Prior 3	
	α	λ	α	λ
1	1.1071	1.0827	1.0695	1.0598
(20, 15, 14*0, 5)	(.0898)	(.2155)	(.0495)	(.1032)
2	1.0849	1.0581	1.0551	1.0425
(20, 15, 5, 14*0)	(.0619)	(.1106)	(.0390)	(.0973)
3	1.1175	1.2137	1.0616	1.0891
(30, 15, 14*0, 15)	(.1039)	(.5449)	(.0510)	(.1049)
4	1.0744	1.0502	1.0546	1.0530
(30, 15, 15, 14*0)	(.0520)	(.1085)	(.0356)	(.0950)
5	1.0924	1.1127	1.0615	1.0823
(30, 15, 1*15)	(.0713)	(.1658)	(.0423)	(.0903)
6	1.0872	1.0888	1.0623	1.0700
(30, 20, 19*0, 10)	(.0620)	(.1023)	(.0400)	(.0700)
7	1.0566	1.0302	1.0439	1.0343
(30, 20, 10, 19*0)	(.0377)	(.0708)	(.0282)	(.0664)

NOTE: Corresponding to each scheme, the first figure represents the average value of the relative estimate, with the corresponding MSE reported below it in parentheses.

Table 6. Average relative credible interval lengths and the coverage percentages of the Bayes estimators with respect to different priors when $\alpha = 1.5$ and $\beta = 1$

Sampling scheme	Prior 0 (MCMC)		Prior 3 (MCMC)	
	α	λ	α	λ
1	.9850	1.1696	.8959	1.1144
(20, 15, 14*0, 5)	(.94)	(.94)	(.96)	(.95)
2	.8384	1.1192	.7753	1.0972
(20, 15, 5, 14*0)	(.95)	(.95)	(.96)	(.96)
3	1.0493	1.6924	.8990	1.3035
(30, 15, 14*0, 15)	(.93)	(.93)	(.97)	(.96)
4	.7869	1.1015	.7374	1.0988
(30, 15, 15, 14*0)	(.94)	(.93)	(.96)	(.94)
5	.8813	1.2365	.8099	1.1763
(30, 15, 1*15)	(.94)	(.95)	(.96)	(.96)
6	.8540	1.0204	.7929	.9911
(30, 20, 19*0, 10)	(.94)	(.95)	(.96)	(.96)
7	.6969	.9412	.6624	.9392
(30, 20, 10, 19*0)	(.95)	(.93)	(.96)	(.94)

NOTE: Corresponding to each scheme, the first figure represents the average credible length, with the corresponding coverage percentage reported below it in parentheses.

paring schemes 1 and 3 and 2 and 4 shows that for fixed R_i's (within specified limits) and m, as the sample size n increases, the biases or MSEs may not decrease. Interestingly, comparing schemes 3 and 6 and 4 and 7, shows that as the effective sample size m increases, the biases, MSEs, and average confidence interval lengths/credible interval lengths decrease for both estimators in all cases. This indicates that the effective sample size plays a more important role than the actual sample size in determining the efficiency of the parameters. Comparing schemes 3, 4, and 5 shows that the performance of scheme 4 is best followed by scheme 5 and then scheme 3, in terms of minimum biases, MSEs, and confidence interval lengths/credible interval lengths for both the parameters, as expected. From Tables 5 and 6, it is clear that the general behavior does not change even if the true parameter values change.

4.2 Data Analysis

For illustrative purposes, we performed a real data analysis. The original data, from Lawless (1982), represent the failure or censoring times for 36 appliances subjected to an automatic life test. A progressively censored sample of the scheme $n = 36$, $m = 10$, $R_1 = \cdots = R_9 = 2$, and $R_{10} = 8$ was generated and analyzed by Kundu and Joarder (2006) using an exponential model. The following progressively censored data were observed: 11, 35, 49, 170, 329, 958, 1,925, 2,223, 2,400, and 2,568. Here we reanalyze the data using the Weibull failure model. For computational ease, we divide all of the values by 100.

Figure 1 plots the profile log-likelihood function of α. It is a unimodal function. We use the fixed-point type algorithm to
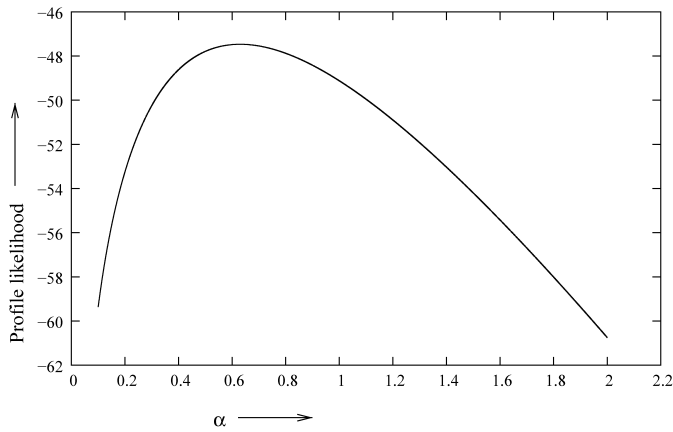
Figure 1. Profile log-likelihood function of α.

compute the MLE as suggested by Kundu and Gupta (2006), with the initial guess of α as 1. The algorithm stops after 18 iterations and the maximum likelihood estimates of α and λ are .6298 and .0627. The 95% confidence intervals of α and λ based on the empirical Fisher information matrix are (.2893, .9703) and (−.0093, .1348).

Now we would like to compute the Bayes estimates of α and λ. Because we have no prior information about the unknown parameters, we assume the noninformative gamma priors of both the unknown parameters. Note that both the noninformative priors also are improper also, but the corresponding posterior density functions of α and λ are proper. Based on the data and the prior distributions, the posterior density function of α is provided in (B.2). Figure 2 plots the posterior density function of α. The shape of the posterior density function closely resembles that of the gamma density function, so we approximate the posterior density function of α with the gamma density function by equating the first two moments. The approximate gamma density function has shape and scale parameters 12.9378 and 20.5994. Figure 2 also plots the approximate posterior density function also; clearly, the two cannot be distinguished. Therefore, the approximation clearly works very well in this case. Finally, Figure 2 plots a histogram based on 10,000 MCMC samples. Based on the MCMC samples, the Bayes estimates of α and λ become .6223 and .0679. The 95% credible intervals for α and λ are (.3344, 1.0043) and (.0165, .1642). Based on Lindley's approximation, the Bayes estimates of α and λ are

Figure 2. Posterior density function, approximate posterior density function, and the generated MCMC samples of α.

.6283 and .0699. Therefore, the MLEs and the Bayes estimates are quite close with respect to the noninformative priors.

5. OPTIMAL PROGRESSIVE CENSORING SCHEME

In practice, it is very important to choose the "optimum" censoring scheme in a class of possible schemes. Here possible schemes mean, for fixed sample size n and fixed effective sample size m, the different choices of $\{R_1, \ldots, R_m\}$ such that $\sum_{i=1}^{m} R_i + m = n$. In this section we assume that m and n are fixed. Now, when comparing two different progressive censoring schemes, say $P_1 = \{R_1^1, \ldots, R_m^1\}$ and $P_2 = \{R_1^2, \ldots, R_m^2\}$, where $\sum_{i=1}^{m} R_i^1 + m = \sum_{i=1}^{m} R_i^2 + m = n$, P_1 is said to better than P_2 if P_1 provides more information about the unknown parameters than P_2. In the next section we propose different criteria to compare two different censoring schemes based on their information contents.

5.1 Precision Criteria

In this section we propose two criteria used to compare two censoring schemes to determine the optimum one. Both of these criteria are based on the estimation precision of the pth $(0 < p < 1)$ quantile. Note that the pth quantile of the Weibull distribution with pdf (1) is $T_p = [-\frac{1}{\lambda} \ln(1-p)]^{1/\alpha}$. Our criteria are based on the precision of estimating $\ln T_p$, as was used by Zhang and Meeker (2005) as well. When the shape parameter is known, it is natural to estimate the posterior variance of $\ln T_p$. In fact, it can be easily shown that for known shape parameter, the posterior variance of $\ln T_p$ is independent of the sample, and when the shape parameter is unknown, the posterior variance of $\ln T_p$ depends on the sample observed. Therefore, the posterior variance of $\ln T_p$ cannot be used as a criterion for finding the optimum life testing plan. Our criterion 1 is

$$C_1(P) = \frac{E_{data}\{V_{posterior(P)}(\ln T_p)\}}{E_{data}\{V_{posterior(C)}(\ln T_p)\}}, \qquad (11)$$

where $P = \{R_1, \ldots, R_m\}$ denotes the reliability plan and $V_{posterior(P)}(\ln T_p)$ and $V_{posterior(P)}(\ln T_p)$ denote the posterior variance of $\ln T_p$ for the given reliability plan and for complete sample. Clearly, $C_1(P)$ depends on the quantile p but is independent of the sample; therefore, according to criterion 1, censoring scheme P_1 is better than P_2 if $C_1(P_1) < C_1(P_2)$.

One drawback of the criterion 1 is that it is a function of the quantile point p. We propose the following criterion using the idea of Gupta and Kundu (2006), which is independent of p. Thus we have criterion 2,

$$C_2(P) = \frac{E_{data} \int_0^1 V_{posterior(P)}(\ln T_p) \, dW(p)}{E_{data} \int_0^1 V_{posterior(P)}(\ln T_p) \, dW(p)}, \qquad (12)$$

where the reliability sampling plan P, $V_{posterior(P)}(\ln T_p)$, and $V_{posterior(P)}(\ln T_p)$ are same as before. In this case $0 \leq W(p) \leq 1$ is a nonnegative weight function defined on [0, 1] and it has to be decided before hand depending on the problem. For example, if someone is interested to give more stress at the center then more weight should be given at $p = .5$, on the other hand if tail probabilities are more important then proper weight can be given for large p. In this case also similarly as before, P_1 is better than P_2, if $C_2(P_1) < C_2(P_2)$.
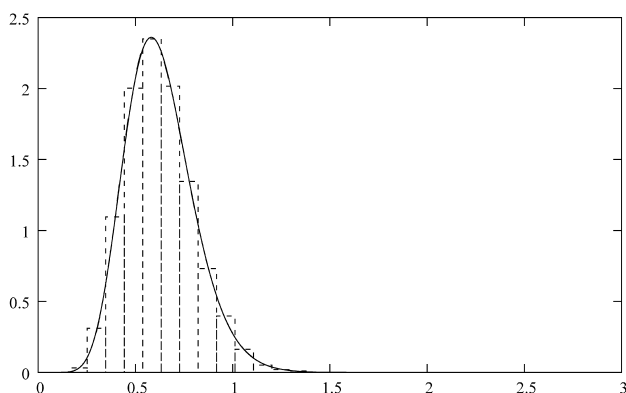
It is clear from the expressions of (11) and (12) that they are not very easy to compute. We use Lindley's approximations and simulation techniques to approximate (11) and (12). The details will be explained later.

5.2 Optimum Scheme: Shape Parameter Known

In this section we provide the optimum censoring scheme when the shape parameter is known. Note that the same problem was discussed by Zhang and Meeker (2005) in the case of type II censored data. It can be easily shown that for any progressive censoring plan $P = \{R_1, \ldots, R_m\}$,

$$V_{posterior(P)}(\ln T_p) = \frac{1}{\alpha^2}\psi'(a+m) \qquad \text{and}$$

$$V_{posterior(P)}(\ln T_p) = \frac{1}{\alpha^2}\psi'(a+n). \tag{13}$$

Interestingly, in this case both $V_{posterior(P)}(\ln T_p)$ and $V_{posterior(P)}(\ln T_p)$ also are independent of p and the *data*; therefore, (11) and (12) become

$$C_1(P) = C_2(P) = \frac{\psi'(a+m)}{\psi'(a+n)}, \tag{14}$$

where ψ' is the trigamma function. This demonstrates that both $C_1(P)$ and $C_2(P)$ are independent of the known shape parameter and also $R_1, \ldots, R_m$. Therefore, if the shape parameter is known, then any progressive censoring plan P_1 provides the same information as any other progressive censoring plan P_2 as long as m and n are fixed. In this case it is reasonable to choose the particular censoring scheme that has the minimum expected experimental time. Therefore, when the shape parameter is known, the usual type II censoring scheme should be the natural choice. Now consider the more important case where the shape parameter is unknown.

5.3 Optimum Scheme: Shape Parameter Unknown

In this case, the main problem is computing (11) and (12). No explicit expressions are available when both parameters are unknown. Here we mainly discuss how to approximate and compute (11) and (12) using Monte Carlo simulations. In Appendix C we explain how to approximate $V_{posterior(P)}(\ln T_p)$ and $\int_0^1 V_{posterior(P)}(\ln T_p)\,dW(p)$. Now we explain how to approximate $E_{data}(V_{posterior(P)}(\ln T_p))$ or $E_{data}(\int_0^1 V_{posterior(P)}(\ln T_p)\,dW(p))$ based on Monte Carlo simulations. For a given sample size n, effective sample size m, and the censoring scheme $(R_1, \ldots, R_m)$, we explain the method for (11); (12) can be obtained in exactly the same manner.

Monte Carlo Approximation Method.

Step 1: Generate α and λ from the joint prior distribution of α and λ, assuming that they are proper.

Step 2: Generate a progressively censored sample $x_{(1)}, \ldots, x_{(m)}$ from the given censoring scheme and when the lifetime distribution is Weibull with parameters α and λ.

Step 3: Calculate the approximate value of $V_{posterior(P)}(\ln T_p)$ based on the method outlined in Appendix C.

Step 4: Repeat Steps 1–3, say N times.

Step 5: Compute the average value of the approximated $V_{posterior(P)}(\ln T_p)$, which will be an approximation of $E_{data}(V_{posterior(P)}(\ln T_p))$.

For illustrative purposes, we present the optimal sampling schemes based on four different objective functions for selected combinations of m and n in Tables 7 and 8. We consider, the $C_1(P)$ values for $p = .99$, $p = .6$, and $p = .5$ and the $C_2(P)$ values for $W(p) = 1$. We also present $\frac{E(T(P))}{E(T(C))}$. Here $E(T(C))$ and $E(T(P))$ denote the expected experimental time needed for complete sample and for the particular sampling scheme P. It gives an idea about the relative experimental time needed for a given sampling scheme. For comparison purposes, we have also presented the results based on type II censoring. The objective functions {1}, {2}, and {3} are based on criterion 1, when $p = .99$, .6, and .5. The objective function {4} is based on criterion 2 and when $W(p) = 1.0$. It is assumed that both α and λ have independent *Gamma*(40, 40) prior distributions.

In all cases, the censoring scheme opposite to the type II censoring scheme provides the maximum information (minimum expected posterior variance) if the objective function {1} is considered. Some heuristic justification can be provided for this finding. For the objective function {1}, when $p = .99$, we are interested in the tail behavior of the distribution. Among all of the sampling schemes, {1} clearly provides the maximum information about the tail behavior, as is quite apparent from the expected experimental time as well. Moreover, it always matches the optimal censoring scheme of Ng et al. (2004) obtained by maximizing the trace of the Fisher information matrix of the MLEs.

Another interesting point observed from both Tables 7 and 8 is that there is a significant difference between the expected experimental time of the type II censoring scheme and that of the other sampling schemes, particularly if $n - m$ is not very small. Clearly, if we put a cost function on time, similar to the method proposed by Lam (1994) or Lin, Liang, and Huang (2002), then it is quite likely that in many situations the usual type II sampling scheme will be the optimum choice. That also justifies the overwhelming popularity of the type II sampling scheme in general.

Finally, we also checked the sensitivity of different optimal censoring schemes. We can see the effect on efficiency if we depart from the optimal censoring scheme even slightly; for example, when $n = 10$ and $m = 5$, with respect to criterion 1, the optimal censoring scheme is [5 0 0 0 0]. If we depart from it slightly, say [4 1 0 0 0], then the relative efficiency becomes .9777, and at [3 2 0 0 0], the relative efficiency becomes .9711; that is, they are near optimal. Similar behavior is observed for the other Criterion 2 as well. This demonstrates if we depart from the optimal sampling scheme only slightly, then the efficiency does not change much.

6. CONCLUSION

In this article we have considered the Bayesian inference of the Weibull parameters when the data are progressively censored. We found that when both parameters are unknown, the Bayes estimates cannot be obtained in explicit form. We used Lindley's approximations and the MCMC technique to compute the approximate Bayes estimates and the corresponding

Table 7. Optimal sampling schemes and type II sampling scheme for $n = 10$ and $m = 5$–9

m	Criterion	Optimal scheme	$C_1(P)$ $(p = .99)$ [1]	$C_1(P)$ $(p = .6)$ [2]	$C_1(P)$ $(p = .5)$ [3]	$C_2(P)$ $W(p) = 1$ [4]	$\frac{E(T(P))}{E(T(C))}$
5	[1]	[5 0 0 0 0]	2.0683	2.0501	1.9769	1.8861	.7234
	[2]	[3 1 0 1 0]	2.1821	1.6835	1.9344	1.8409	.6047
	[3]	[2 0 1 0 2]	2.3112	2.0647	1.6699	1.8116	.3209
	[4]	[0 0 0 2 3]	2.4364	2.0665	1.8895	1.6614	.2356
	Type II	[0 0 0 0 5]	2.4837	2.1034	1.9536	1.8305	.2041
6	[1]	[4 0 0 0 0 0]	1.6991	1.6835	1.6307	1.5924	.7854
	[2]	[0 2 1 0 1 0]	1.8415	1.3725	1.5973	1.5289	.6535
	[3]	[1 0 1 0 0 2]	1.9027	1.6914	1.3681	1.4964	.3682
	[4]	[0 0 0 0 1 3]	1.9864	1.6789	1.5763	1.3758	.2987
	Type II	[0 0 0 0 0 4]	2.0013	1.6839	1.5805	1.4960	.2676
7	[1]	[3 0 0 0 0 0 0]	1.4488	1.5123	1.4071	1.3825	.8460
	[2]	[0 0 1 1 0 1 0]	1.5578	1.4067	1.3518	1.3124	.6961
	[3]	[0 0 0 0 1 0 2]	1.6358	1.4070	1.3404	1.2933	.3951
	[4]	[0 0 0 0 1 0 2]	1.6358	1.4070	1.3404	1.2933	.3951
	Type II	[0 0 0 0 0 0 3]	1.6715	1.4263	1.3566	1.3070	.3720
8	[1]	[2 0 0 0 0 0 0 0]	1.2673	1.2631	1.2458	1.2324	.9069
	[2]	[0 0 0 0 0 1 0 1]	1.3738	1.2245	1.1854	1.1587	.5390
	[3]	[0 0 0 0 0 1 0 1]	1.3738	1.2245	1.1854	1.1587	.5390
	[4]	[0 0 0 0 0 1 0 1]	1.3738	1.2245	1.1854	1.1587	.5390
	Type II	[0 0 0 0 0 0 0 2]	1.3938	1.2323	1.1917	1.1640	.4719
9	[1]	[1 0 0 0 0 0 0 0 0]	1.1194	1.1047	1.1112	1.1041	.9453
	[2]	[0 0 0 1 0 0 0 0 0]	1.1378	1.1029	1.1020	1.0855	.9071
	[3]	[0 0 0 0 0 0 0 1 0]	1.1782	1.1090	1.0723	1.0754	.8259
	[4]	[0 0 0 0 0 0 0 0 1]	1.1848	1.1080	1.0825	1.0683	.6499
	Type II	[0 0 0 0 0 0 0 0 1]	1.1848	1.1080	1.0825	1.0683	.6499

credible intervals. We compared the Bayes estimates based on noninformative priors with the corresponding MLEs and found that their behaviors were very similar, as expected. But when we had informative priors, their performance was much better than that of the MLEs.

We also have proposed different criteria for comparing two different sampling schemes based on their information con-

tents, and reported the optimal sampling schemes with respect to different criteria for some small values of n and m. Although the total number of possible schemes are finite, they can be quite large. For fixed m and n, a total $\binom{n-1}{m-1}$ possible progressive censoring schemes are available; for example, when $n = 25$ and $m = 12$, the possible number of censoring schemes is $\binom{24}{11} = 2,496,144$, which is quite large. Finding an algorithm

Table 8. Optimal sampling schemes and type II sampling scheme for $n = 12$ and $m = 9$–11

m	CN	Optimal scheme	$C_1(P)$ $(p = .99)$ [1]	$C_1(P)$ $(p = .6)$ [2]	$C_1(P)$ $(p = .5)$ [3]	$C_2(P)$ $W(p) = 1$ [4]	$\frac{E(T(P))}{E(T(C))}$
9	[1]	[3 0 0 0 0 0 0 0 0]	1.3346	1.3264	1.3043	1.2834	.8781
	[2]	[0 0 0 0 1 1 0 1 0]	1.4657	1.2702	1.2614	1.1865	.6798
	[3]	[0 0 0 0 0 0 0 0 3]	1.5140	1.2865	1.2144	1.1877	.3921
	[4]	[0 0 0 0 0 0 0 2 1]	1.5073	1.2824	1.2276	1.1853	.4640
	Type II	[0 0 0 0 0 0 0 0 3]	1.5140	1.2865	1.2144	1.1877	.3921
10	[1]	[2 0 0 0 0 0 0 0 0 0]	1.2002	1.1873	1.1717	1.1779	.9254
	[2]	[0 0 0 0 0 0 1 1 0 0]	1.2733	1.1496	1.1365	1.1288	.7869
	[3]	[0 0 0 0 0 0 0 1 0 1]	1.2992	1.1680	1.1201	1.1055	.5679
	[4]	[0 0 0 0 0 0 0 0 0 2]	1.3127	1.1598	1.1248	1.0928	.4938
	Type II	[0 0 0 0 0 0 0 0 0 2]	1.3127	1.1598	1.1248	1.0928	.4938
11	[1]	[1 0 0 0 0 0 0 0 0 0 0]	1.0908	1.0923	1.0871	1.0822	.9817
	[2]	[0 0 0 0 0 0 0 0 0 1 0]	1.1268	1.0537	1.0384	1.0319	.8188
	[3]	[0 0 0 0 0 0 0 0 0 1 0]	1.1268	1.0587	1.0384	1.0319	.8188
	[4]	[0 0 0 0 0 0 0 0 0 1 0]	1.1268	1.0587	1.0384	1.0319	.8188
	Type II	[0 0 0 0 0 0 0 0 0 0 1]	1.1474	1.0661	1.0507	1.0417	.6442

for choosing the optimum sampling scheme from the possible finite number of schemes, which can be quite large in practice, remains an important problem; more work is needed in this direction.

ACKNOWLEDGMENTS

APPENDIX A: LINDLEY'S APPROXIMATION

For the two-parameter case, using the notation $(\lambda_1, \lambda_2) = (\alpha, \lambda)$, Lindley's approximation can be written as

$$\hat{g} = g(\hat{\lambda}_1, \hat{\lambda}_2) + \frac{1}{2}[A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}]$$
$$+ p_1 A_{12} + p_2 A_{21}, \quad (A.1)$$

where

$$A = \sum_{i=1}^{2} \sum_{j=1}^{2} w_{ij}\tau_{ij},$$

$$l_{ij} = \frac{\partial^{i+j}L(\lambda_1, \lambda_2)}{\partial\lambda_1^i \partial\lambda_2^j}; \qquad i, j = 0, 1, 2, 3, \ i+j = 3; \quad (A.2)$$

$$p_i = \frac{\partial p}{\partial\lambda_i}, \qquad w_i = \frac{\partial g}{\partial\lambda_i}, \qquad w_{ij} = \frac{\partial^2 g}{\partial\lambda_i \partial\lambda_j}, \quad (A.3)$$

$$p = \ln\pi(\lambda_1, \lambda_2), \qquad A_{ij} = w_i\tau_{ii} + w_j\tau_{ji};$$

and

$$B_{ij} = (w_i\tau_{ii} + w_j\tau_{ij})\tau_{ii},$$
$$C_{ij} = 3w_i\tau_{ii}\tau_{ij} + w_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2), \quad (A.4)$$

where $L(\cdot, \cdot)$ is the log-likelihood function of the observed data, $\pi(\lambda_1, \lambda_2)$ is the joint prior density function of (λ_1, λ_2), and τ_{ij} is the (i, j)th element of the inverse of the Fisher information matrix. Moreover, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the MLEs of λ_1 and λ_2, and all of the quantities are evaluated at $(\hat{\lambda}_1, \hat{\lambda}_2)$. In this case we have

$$L(\alpha, \lambda) = m\ln\alpha + m\ln\lambda + (\alpha - 1)\sum_{i=1}^{m}t_i - \lambda\sum_{i=1}^{m}(R_i + 1)t_i^{\alpha};$$

therefore, we obtain

$$l_{30} = \frac{2m}{\hat{\alpha}^3} - \hat{\lambda}\sum_{i=1}^{m}(R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^3, \qquad l_{03} = \frac{2m}{\hat{\lambda}^3},$$

$$l_{21} = -\sum_{i=1}^{m}(R_i + 1)t_i^{\hat{\alpha}}(\ln t_i)^3, \qquad l_{12} = 0,$$

and U, V, and W the same as defined in (10). Now when $g(\alpha, \lambda) = \alpha$, we have

$$w_1 = 1, \qquad w_2 = 0, \qquad \text{and} \qquad w_{ij} = 0, \qquad i, j = 1, 2;$$

therefore,

$$A = 0, \qquad B_{12} = \tau_{11}^2, \qquad B_{21} = \tau_{21}\tau_{22},$$

$$C_{12} = 3\tau_{11}\tau_{12}, \qquad C_{21} = (\tau_{22}\tau_{11} + 2\tau_{21}^2),$$
$$A_{12} = \tau_{11}, \quad \text{and} \qquad A_{21} = \tau_{12}.$$

Now the first part of Lindley's approximation follows by using

$$p_1 = \frac{c - 1}{\hat{\alpha}} - d \qquad \text{and} \qquad p_2 = \frac{a - 1}{\hat{\lambda}} - b.$$

For the second part, note that $g(\alpha, \lambda) = \lambda$; then

$$w_1 = 0, \qquad w_2 = 1, \qquad w_{ij} = 0, \qquad i, j = 1, 2,$$

and

$$A = 0, \qquad B_{12} = \tau_{12}\tau_{11}, \qquad B_{21} = \tau_{22}^2,$$

$$C_{12} = \tau_{11}\tau_{22} + 2\tau_{12}^2, \qquad C_{21} = 3\tau_{22}\tau_{21},$$

$$A_{12} = \tau_{21}, \qquad \text{and} \qquad A_{21} = \tau_{22},$$

and thus the second part follows immediately.

APPENDIX B: PROOF OF THEOREM 2

The conditional density of α and λ given the data is

$$l(\alpha, \lambda|data) \propto \pi_2(\alpha)\alpha^m\lambda^{a+m-1}\prod_{i=1}^{m}t_i^{\alpha-1}e^{-\lambda\sum_{i=1}^{m}(R_i+1)t_i^{\alpha}}. \quad (B.1)$$

Therefore,

$$l(\alpha|data) \propto \pi_2(\alpha)\alpha^m\prod_{i=1}^{m}t_i^{\alpha-1}\frac{1}{(b + \sum_{i=1}^{m}(R_i + 1)t_i^{\alpha})^{a+m}}. \quad (B.2)$$

We want to show that (B.2) is log-concave. Consider

$$\ln l(\alpha|data) = k + \ln\pi_2(\alpha) + m\ln\alpha + (\alpha - 1)\sum_{i=1}^{m}\ln t_i$$
$$- (a + m)\ln\left[b + \sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}\right]. \quad (B.3)$$

Suppose that

$$g(\alpha) = b + \sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}; \quad (B.4)$$

then

$$g'(\alpha) = \sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}\ln t_i \qquad \text{and}$$
$$g''(\alpha) = \sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}(\ln t_i)^2. \quad (B.5)$$

Observe that

$$\left(\sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}(\ln t_i)^2\right)\left(\sum_{i=1}^{n}(R_i + 1)t_i^{\alpha}\right)$$
$$- \left(\sum_{i=1}^{m}(R_i + 1)t_i^{\alpha}\ln t_i\right)^2 \quad (B.6)$$
$$= \sum_{1\leq i<j\leq m}(R_i + 1)(R_j + 1)(\ln t_i - \ln t_j)^2$$
$$\geq 0. \quad (B.7)$$

Therefore, for $b \geq 0$,

$$g''(\alpha)g(\alpha) \geq (g'(\alpha))^2. \qquad \text{(B.8)}$$

From (B.8), it follows that if $\pi_2(\alpha)$ is log-concave, then for $b \geq 0$, the second derivative of the right side of (B.2) with respect to α is negative, and thus $l(\alpha|data)$ is log-concave.

APPENDIX C: POSTERIOR VARIANCE

Note that $V_{posterior(P)}(\ln T_p)$ and $\int_0^1 V_{posterior(P)}(\ln T_p)\,dW(p)$ cannot be computed analytically. Here we provide the working formulas for both of them. We again use Lindley's approximation technique to provide the approximate formulas for both. Because

$V_{posterior(P)}(\ln T_p)$

$$= E_{posterior(P)}(\ln T_p)^2 - (E_{posterior(P)}(\ln T_p))^2,$$

we use Lindley's approximation for both parts. For approximating $E_{posterior(P)}(\ln T_p)$, from (A.1), we need to provide

$$g(\alpha, \lambda) = \ln T_p(\alpha, \lambda)$$

$$= \frac{1}{\alpha}[\psi - \ln \lambda], \quad \text{where } \psi = \ln(-\ln(1-p)),$$

$$w_1 = -\frac{1}{\alpha^2}(\psi - \ln \lambda), \qquad w_2 = -\frac{1}{\alpha\lambda},$$

$$w_{11} = \frac{2}{\alpha^3}(\psi - \ln \lambda), \qquad w_{22} = \frac{1}{\alpha\lambda^2}, \qquad \text{and}$$

$$w_{12} = w_{21} = \frac{1}{\alpha^2\lambda}.$$

The rest of the quantities are the same as in (A.1), and thus the approximate value of $E_{posterior(P)}(\ln T_p)$ can be obtained.

Now to approximate $E_{posterior(P)}(\ln T_p)^2$, note that in this case

$$g(\alpha, \lambda) = (\ln T_p(\alpha, \lambda))^2$$

$$= \frac{1}{\alpha^2}[\psi - 2\psi \ln \lambda + (\ln \lambda)^2],$$

$$w_1 = -\frac{2}{\alpha^3}(\psi + (\ln \lambda)^2 - 2\psi \ln \lambda),$$

$$w_2 = \frac{2}{\alpha^2\lambda}(\ln \lambda - \psi),$$

$$w_{11} = \frac{6}{\alpha^4}(\psi + (\ln \lambda)^2 - 2\psi \ln \lambda),$$

$$w_{22} = \frac{2}{\alpha^2\lambda^2}(1 - \ln \lambda + \psi), \qquad \text{and}$$

$$w_{12} = w_{21} = -\frac{4}{\alpha^3\lambda}(\ln \lambda - \psi).$$

Now note that to approximate $\int_0^1 V_{posterior(P)}(\ln T_p)\,dW(p)$, we need to approximate

$$\int_0^1 E_{posterior(P)}(\ln T_p)^2\,dW(p) \qquad \text{and}$$

$$\int_0^1 E_{posterior(P)}(\ln T_p)\,dW(p).$$

These can be easily obtained by replacing ψ by $\int_0^1 \psi\,dW(p)$ in all of the foregoing expressions. For different weight functions $W(p)$, the integrations must be carried out numerically using, for example, for the uniform weight function,

$$\int_0^1 \psi\,dW(p) = \int_0^1 \psi\,dp = -\gamma,$$

where γ is Euler's constant and $\gamma = .57722$.

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