Estimation of $P(Y < X)$ for the 3-Parameter Generalized Exponential Distribution

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Abstract

This paper considers the estimation of $R = P(Y < X)$ when X and Y are distributed as two independent 3-parameter generalized exponential (GE) random variables with different shape parameters but having the same location and scale parameters. A modified maximum likelihood method and a Bayesian technique are used to estimate R on the basis of independent complete samples. The Bayes estimator can not be obtained in explicit form, and therefore it has been implemented using the importance sampling procedure. Analysis of a simulated data and a real life data have been presented for illustrative purposes.

Keywords: Generalized exponential distribution; Profile likelihood; Bayesian estimation; Importance sampling, HPD intervals; Modified maximum likelihood estimation, Parametric Bootstrap confidence intervals.

1 Introduction

The three-parameter GE distribution has the following cumulative distribution function (CDF)

$$F(x; \alpha, \lambda, \theta) = (1 - e^{-\lambda(x-\theta)})^{\alpha}; \qquad x > \theta, \quad \alpha, \lambda > 0, \tag{1}$$

and the corresponding probability density function (PDF)

$$f(x;\alpha,\lambda,\theta) = \alpha\lambda(1-e^{-\lambda(x-\theta)})^{\alpha-1}e^{-\lambda(x-\theta)}; \qquad x > \theta, \quad \alpha,\lambda > 0. \tag{2}$$

Here α and θ are the shape and location parameters, respectively, and $\lambda > 0$ is the reciprocal of a scale parameter. From now on a three-parameter GE distribution with the density function (2) will be denoted by $GE(\alpha, \lambda, \theta)$.

The three-parameter generalized exponential (GE) distribution has been studied by Gupta and Kundu (1999). Gupta and Kundu (1999) studied the theoretical properties of the distribution and compared them to the well studied properties of the gamma and Weibull distributions. In Gupta and Kundu (2001a), they compared it to the two-parameter Gamma and Weibull distribution and in Gupta and Kundu (2001b) they discussed different methods of estimation. Raqab and Ahsanullah (2001) and Raqab (2002) studied the properties of order and record statistics from the two-parameter GE distribution and their inferences, respectively.

Due to the convenient structure of its distribution function, the GE distribution can be used quite effectively in analyzing many lifetime data. It is observed that the hazard function of the GE distribution can be increasing, decreasing or constant depending on the shape parameter α. For any λ, the hazard function is nondecreasing if $\alpha > 1$, it is decreasing if $\alpha < 1$ and for $\alpha = 1$, it is constant. The GE has a unimodal and right skewed density function. For fixed scale parameter, as the shape parameter increases the skewness gradually decreases. If the data come from a right tailed distribution, then the GE can be used quite effectively for analyzing them. It is observed that in many situations, the GE distribution provides a better fit than gamma or Weibull distribution.

In this paper we consider the problem of estimating $R = P(Y < X)$, under the assumption that $Y \sim GE(\alpha, \lambda, \theta)$, $X \sim GE(\beta, \lambda, \theta)$, and that X and Y are independent. Here $\sim$ means follows or has the distribution. The problem of estimating R arises in mechanical reliability systems. Suppose X represents the strength of a component with a stress Y, then R can be considered as a measure of system performance. The system becomes out of control if the system stress exceeds its strength. Since R represents a relation between the stress and strength of a system, it is popularly known as the stress-strength parameter of that system. Based on complete X-sample and Y-sample, we would like to draw statistical inference on R.

Several authors have studied the problem of estimating R. Church and Harris (1970) derived the MLE of R when X and Y are independently normally distributed. Downtown (1973) obtained the uniformly minimum variance unbiased estimates (UMVUE) under the same model. The MLE of R, when X and Y have bivariate exponential distributions has been considered by Awad *et al.* (1981). Awad and Gharraf (1986) provided a simulation study to compare three estimates of R when X and Y are independent but not identically distributed Burr random variables. Ahmad, Fakhry and Jaheen (1997) and Surles and Padgett (2001, 1998) provided estimates for R when X and Y are Burr Type X distribution. Very recently, Kundu and Gupta (2005) and Raqab and Kundu (2005) considered the estimation of R for two-parameter GE distribution and scaled Burr Type X distribution respectively. A book length treatment of the different methods can be found in Kotz *et al.* (2003).

It is observed that the usual maximum likelihood estimators of the unknown parameters may not exist. In Section 2, we propose a modified maximum likelihood estimator of R. In Section 3, we show how importance sampling is used to get the Bayes estimates of the model parameters α, β, λ, θ and the stress-strength parameter R. For illustrative purposes, two examples have been provided in Section 4. Finally some experimental results are presented in Section 5.

2 Modified Maximum Likelihood Estimation

Let $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ and $\mathbf{X} = \{X_1, \ldots, X_m\}$ be independent random samples from $GE(\alpha, \lambda, \theta)$ and $GE(\beta, \lambda, \theta)$ respectively. Our problem is to estimate R from the given samples. We denote the ordered Y_i's and ordered X_j's as $\{Y_{(1)} < \ldots < Y_{(n)}\}$ and $\{X_{(1)} < \ldots < X_{(m)}\}$. Based on the observations $\mathbf{Y}$ and $\mathbf{X}$, the likelihood function of α, β, $\lambda$and θ is

$$l(\alpha, \beta, \lambda, \theta|\mathbf{y}, \mathbf{x}) \propto \alpha^n \beta^m \lambda^{m+n} e^{-\lambda\left(\sum_{i=1}^n (y_{(i)}-\theta) + \sum_{j=1}^m (x_{(j)}-\theta)\right)} e^{D(\lambda,\theta)} I_{\theta<z},$$

where

$$D(\lambda, \theta) = \left[(\alpha-1)\sum_{i=1}^n \ln\left(1-e^{-\lambda(y_{(i)}-\theta)}\right)\right] + \left[(\beta-1)\sum_{j=1}^m \ln\left(1-e^{-\lambda(x_{(j)}-\theta)}\right)\right],$$

$z = \min(y_{(1)}, x_{(1)})$ and $I_{\theta<z} = 1$, if $\theta < z$ and 0 otherwise.

Now observe that for fixed $\alpha < 1$, $\beta < 1$ and $\lambda > 0$ as θ approaches z, the likelihood function $l(\alpha, \beta, \lambda, \theta|\mathbf{y}, \mathbf{x})$ gradually increases to ∞. It implies that the maximum likelihood estimators (MLEs) of α, β, λ do not exist. Hence, we propose modified MLEs of the unknown parameters. Note that the likelihood function becomes maximum for $\theta = z$, therefore, we propose the modified MLE of θ, $\tilde{\theta} = z$. Consider two cases separately: Case 1: $y_{(1)} < x_{(1)}$ and Case 2: $x_{(1)} \leq y_{(1)}$. For Case 1, we propose the modified likelihood function of α, β, λ given the data vectors $\mathbf{Y}$ and $\mathbf{X}$ as

$$l_{mod}(\alpha, \beta, \lambda | \mathbf{y}, \mathbf{x}) \propto \alpha^{n-1} \beta^{m} \lambda^{m+n-1} e^{-\lambda\left(\sum_{i=2}^{n}(y_{(i)}-y_{(1)})+\sum_{j=1}^{m}(x_{(j)}-y_{(1)})\right)} e^{D_{mod}(\lambda,\theta)}, \tag{3}$$

where

$$D_{mod}(\lambda, \theta) = \left[(\alpha-1)\sum_{i=2}^{n} \ln\left(1-e^{-\lambda(y_{(i)}-y_{(1)})}\right)\right] + \left[(\beta-1)\sum_{j=1}^{m} \ln\left(1-e^{-\lambda(x_{(j)}-y_{(1)})}\right)\right].$$

We need to maximize (3) to obtain the modified MLEs of α, β and λ. Using a similar approach as Kundu and Gupta (2005), it can be easily seen that the modified MLEs of α and β can be obtained as

$$\tilde{\alpha} = -\frac{n-1}{\sum_{i=2}^{n} \ln\left(1-e^{-\tilde{\lambda}(y_{(i)}-y_{(1)})}\right)}, \tag{4}$$

and

$$\tilde{\beta} = -\frac{m}{\sum_{j=1}^{m} \ln\left(1-e^{-\tilde{\lambda}(x_{(j)}-y_{(1)})}\right)}. \tag{5}$$

Here, $\tilde{\lambda}$, the modified MLE of λ, can be obtained as a solution of the non-linear equation of the form

$$h(\lambda) = \lambda,$$

where

$$\begin{aligned} h(\lambda) \quad = \quad (m+n-1)\Bigg[& \frac{m}{\sum_{j=1}^{m} \ln\left(1-e^{-\lambda(x_{(j)}-y_{(1)})}\right)} \times \sum_{j=1}^{m} \frac{(x_{(j)}-y_{(1)})e^{-\lambda(x_{(j)}-y_{(1)})}}{1-e^{-\lambda(x_{(j)}-y_{(1)})}} + \\ & \frac{n-1}{\sum_{i=2}^{n} \ln\left(1-e^{-\lambda(y_{(i)}-y_{(1)})}\right)} \times \sum_{i=2}^{n} \frac{(y_{(i)}-y_{(1)})e^{-\lambda(y_{(i)}-y_{(1)})}}{1-e^{-\lambda(y_{(i)}-y_{(1)})}} + \\ & \sum_{j=1}^{m} \frac{(x_{(j)}-y_{(1)})}{1-e^{-\lambda(x_{(j)}-y_{(1)})}} + \sum_{i=2}^{n} \frac{(y_{(i)}-y_{(1)})}{1-e^{-\lambda(y_{(1)}-y_{(1)})}}\Bigg], \end{aligned}$$

see Kundu and Gupta (2005) for details. Once the modified MLEs of α and β are obtained, the modified MLE of R is proposed as

$$\tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}}.$$

Similarly, we can obtain the modified MLE of R for Case 2.

Although, we obtain the modified MLE of R, it is very difficult to compute the exact distribution of $\tilde{R}$. Therefore, constructing the exact confidence interval of R becomes quite difficult and it is not pursued here. We suggest to use the parametric Bootstrap confidence interval as proposed by Efron (1982). In the next section we present the Bayesian estimation of R.

3 Bayesian Estimation

3.1 The Prior Distribution

A natural choice for the prior of α, β, λ and θ would be to assume that the four quantities are independent and that

$$\alpha \sim G(a_0,\, b_0),\ \beta \sim G(a_1,\, b_1) \ \text{ and } \quad \lambda \sim G(a_2,\, b_2),$$

where $G(a, b)$ denotes the gamma distribution with mean $\frac{a}{b}$. Moreover,

$$\pi(\theta) = \xi[1-\exp(-\xi\theta_0)]^{-1}\exp\{\xi(\theta-\theta_0)\}I_{(0<\theta<\theta_0)} = \pi_\theta(\xi,\theta_0)$$

i.e. θ has a truncated positive exponential distribution with parameters ξ and θ_0. The hyper parameters $a_0,\ b_0,\ a_1,\ b_1,\ a_2,\ b_2, \xi$ and θ_0 are chosen to reflect prior knowledge about β, α, λ and θ. For example, $a_0,\ b_0, a_1, b_1,\ a_2,$ and b_2 can be assessed to match the experimenter's notion of the average and precision of his prior for α, β and λ. While ξ and θ_0 could be obtained by using the experimenter's prior median and 95^{th} percentile for θ.

3.2 Posterior Distribution and Expectations

We now write the extended likelihood function of $\alpha, \beta, \lambda, \theta$ given $\mathbf{X}, \mathbf{Y}$ as follows:

$$\begin{aligned} L(\alpha,\beta,\lambda,\theta|\mathbf{x},\mathbf{y}) \ \propto\ & \beta^m\alpha^n\lambda^{m+n}\, Exp\{-[m\lambda(\bar{x}-\theta)-(\beta-1)D_1(\lambda,\theta)]\} \\ & Exp\{-[n\lambda(\bar{y}-\theta)-(\alpha-1)D_2(\lambda,\theta)]\}\ I_{(\theta<z)} \end{aligned}$$

where

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i, \bar{y} = \frac{1}{n}\sum_{i=1}^{m} y_i, D_1(\lambda,\theta) = \sum_{i=1}^{m} -\log(1-e^{-\lambda(x_i-\theta)}), \quad D_2(\lambda,\theta) = \sum_{i=1}^{n} -\log(1-e^{-\lambda(y_i-\theta)}),$$

From the extended likelihood function of α, β, λ and θ for the given samples $\mathbf{X}$ and $\mathbf{Y}$ and the above priors, it can be checked that the joint posterior density of α, β, λ and θ can be expressed as

$$\begin{aligned}
\pi(\alpha,\beta,\lambda,\theta|\mathbf{x},\mathbf{y}) \quad \propto \quad & g_\alpha[n+a_0, b_0+D_2(\lambda,\theta)] \times g_\beta[m+a_1, b_1+D_1(\lambda,\theta)] \times \\
& g_\lambda[m+n+a_2, b_2+m(\bar{x}-z_0)+n(\bar{y}-z_0)] \times \\
& \pi_\theta((n+m)\lambda+\xi, z_0) \times W(\lambda,\theta),
\end{aligned}$$

where

$$W(\lambda,\theta) = [(n+m)\lambda+\xi]^{-1}\, e^{D_1(\lambda,\theta)+D_2(\lambda,\theta)} \times (b_0+D_2(\lambda,\theta))^{-(n+a_0)}(b_1+D_1(\lambda,\theta))^{-(m+a_1)},$$

$z_0 = \min(z,\theta_0)$, $g_\eta(a,b)$ denotes a gamma density for η with parameters a and b and π_θ is a truncated positive exponential distribution for θ with parameters $(n+m)\lambda+\xi$ and z_0. The marginal posterior density of λ and θ is

$$\begin{aligned}
\pi(\lambda,\theta|\mathbf{x},\mathbf{y}) \quad \propto \quad & g_\lambda[m+n+a_2, b_2+m(\bar{x}-z_0)+n(\bar{y}-z_0)] \times \\
& \pi_\theta((n+m)\lambda+\xi, z_0) \times W(\lambda,\theta).
\end{aligned}$$

Also, the marginal posterior densities of α and β given λ and θ, are respectively,

$$\alpha|\lambda,\theta,\mathbf{x},\mathbf{y} \propto g_\alpha[n+a_0, b_0+D_2(\lambda,\theta)],$$

$$\beta|\lambda,\theta,\mathbf{x},\mathbf{y} \propto g_\beta[m+a_1, b_1+D_1(\lambda,\theta)].$$

As a consequence of that, the posterior expectation of any continuous function of $\alpha, \beta, \lambda, \theta$, say, $V(\alpha,\beta,\lambda,\theta)$ can be expressed as

$$\begin{aligned}
E(V(\alpha,\beta,\lambda,\theta)|\mathbf{x},\mathbf{y}) \quad = \quad & B^{-1}\int\int\int\int V(\alpha,\beta,\lambda,\theta)W(\lambda,\theta)\pi_\theta((n+m)\lambda+\xi, z_0). \\
& g_\lambda[m+n+a_2, b_2+m(\bar{x}-z_0)+n(\bar{y}-z_0)] \\
& g_\alpha[n+a_0, b_0+D_2(\lambda,\theta)].g_\beta[m+a_1, b_1+D_1(\lambda,\theta)]\; d\alpha\, d\beta\, d\lambda\, d\theta
\end{aligned}$$

where

$$B = \int_0^{z_0}\int_0^{\infty} W(\lambda,\theta) g_\lambda[m+n+a_2, b_2+m(\bar{x}-z_0)+n(\bar{y}-z_0)]\pi_\theta((n+m)\lambda+\xi, z_0)\, d\lambda\, d\theta.$$

To evaluate the above posterior expectation we use the following general importance sampling procedure:

Step 1: Generate λ from gamma distribution $G[m+n+a_2, b_2+m(\overline{x}-z_0)+n(\overline{y}-z_0)]$

Step 2: For this λ, generate θ from the truncated positive exponential distribution $\pi_\theta((n+m)\lambda+\xi, z_0)$

Step 3: For the values of λ and θ obtained in Steps 1-2, we generate α from the gamma distribution $G(n+a_0, b_0+D_2(\lambda,\theta))$ and β from the gamma distribution $G(m+a_1, b_1+D_1(\lambda,\theta))$.

Step 4: From Steps 1-3, compute

$$E(V(\alpha,\beta,\lambda,\theta)|\mathbf{x},\mathbf{y}) = \frac{E^*[V(\alpha,\beta,\lambda,\theta)\, W(\lambda,\theta)]}{E^*[W(\lambda,\theta)]}, \tag{6}$$

by averaging the numerator and denominator with respect to these simulations.

3.3 HPD Intervals

Chen and Shao (1999) developed a Monte Carlo method for using importance sampling to compute HPD (highest probability density) intervals for the parameters of interest or any function of them. In this section, we use their proposed approach to find HPD intervals for the model parameters as well as the stress-strength function R .

To find an HPD interval for $\eta = v(\lambda,\theta,\alpha,\beta)$, we denote the sample from the importance sampling distribution as $(\lambda_i,\theta_i,\alpha_i,\beta_i, i=1,...m)$. Then we reorder this sample such that $\eta_{(1)} \le \eta_{(2)} \le ...\eta_{(m)}$ and then denote it as $(\lambda_{(i)},\theta_{(i)},\alpha_{(i)},\beta_{(i)}, i=1,...m)$. Note here that only the $\eta'_{(i)}s$ are ordered. We then compute

$$w_i = \frac{W(\lambda_{(i)},\theta_{(i)})}{\sum_{j=1}^{m} W(\lambda_{(j)},\theta_{(j)})}$$

For m sufficiently large, the $100(1-\alpha)\%$ HPD interval for η is chosen as the shortest of the intervals $I_j(m), j=1,2,...,m-[(1-\alpha)m]$, where

$$I_j(m) = \left(\widehat{\eta}^{(\frac{j}{m})}, \widehat{\eta}^{(\frac{j+[(1-\alpha)m]}{m})}\right)$$

and where $\widehat{\eta}^{(\alpha)}$ is an estimate of the α^{th}quantile of η and is given by:

$$\widehat{\eta}^{(\alpha)} = \begin{cases} \eta_{(1)} & \text{if } \alpha = 0 \\ \eta_{(i)} & \text{if } \sum\limits_{j=1}^{i-1} w_j \leq \alpha \leq \sum\limits_{j=1}^{i} w_j \end{cases}$$

HPD intervals for the four parameters, as well as the stress-strength function are obtained in this way.

4 Data Analysis

In this section, we illustrate the procedures by presenting a complete analysis for a simulated data set and a real life one. We estimate the unknown parameters using the modified maximum likelihood method and the Bayesian method via importance sampling.

4.1 Example 1 (Simulated Data)

The following two independent samples were generated from the generalized exponential distribution. The first sample $(X_1, X_2, ..., X_{20})$ from a $GE(2.5, 0.5, 1.0)$ and the second sample $(Y_1, Y_2, ..., Y_{20})$ from a $GE(1.5, 0.5, 1.0)$. The observations are as follows:

Sample 1: 1.34 3.00 3.44 10.14 3.52 2.47 1.74 5.18 2.72 3.41 2.67 9.24 3.41 4.77 9.94 4.69 2.48 3.83 2.19 3.75

Sample 2: 1.09 2.25 2.63 9.12 2.71 1.82 1.30 4.25 2.02 2.61 1.98 8.23 2.61 3.85 8.93 3.78 1.83 2.99 1.61 2.91

The true value of $R = 0.625$. We have already seen that the MLEs of the unknown parameters do not exist. We have used the modified MLE of θ, $\tilde{\theta} = \min\{x_{(1)}, y_{(1)}\} = 1.09$. The modified profile log-likelihood function *i.e.*, $\ln l_{mod}(\tilde{\alpha}(\lambda), \tilde{\beta}(\lambda), \lambda|\mathbf{y}, \mathbf{x})$ is provided in Figure 1. From Figure 1 it is clear that the modified profile log-likelihood function is a unimodal function. It implies that the proposed iterative procedure can be applied with a reasonable starting value. We have started the iteration with the initial guess of $\lambda = 1$ and the iteration stopped in 7 steps, when the absolute difference between two iterates becomes less that 10^{-6}. We obtained $\tilde{\lambda} = 0.4912$, $\tilde{\alpha} = 1.4035$ and $\tilde{\beta} = 1.9739$. Finally we obtained $\tilde{R} = 0.5845$. The 95% parametric Bootstrap confidence interval of R based on 1000 bootstrap samples, is $(0.4082, 0.7390)$.

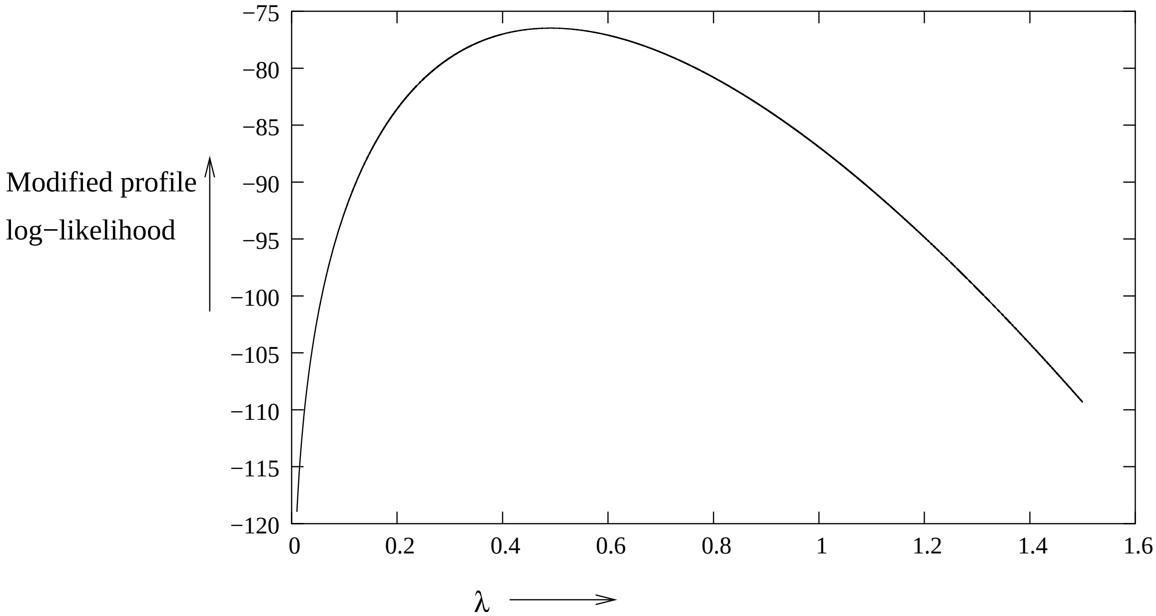

Figure 1: Modified profile log-likelihood function of the simulated data

To find the Bayes estimates, small values were given to the gamma hyper parameters to reflect vague prior information. Namely, we assumed that $a_i = b_i = 0.25$ for $i = 0, 1, 2$. We assumed also that $\theta_0 = 2.0$ and $\xi = 1.0$. Fifty thousand simulated values values of α, β, λ and θ were used to implement the importance sampling procedure. The numerator and denominator of (6) were averaged with respect to these simulations, where $V(\alpha, \beta, \lambda, \theta)$ is equal respectively to $\alpha, \beta, \lambda, \theta$ and the stress-strength parameter R. The Bayesian estimates of the unknown parameters are: $\widehat{\lambda}_B = 0.511, \widehat{\alpha}_B = 1.3775, \widehat{\beta_B} = 2.600, \widehat{\theta}_B = 0.944$, and $\widehat{R}_B = 0.639$. Moreover, the 95% HPD credible interval of R is $(0.4877, 0.7551)$.

To see which estimates provide better fit to the given data set. We compute the Kolmogorov-Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution functions based on the modified MLEs and Bayes estimators. The K-S distance between the empirical distribution function and the fitted distribution function for data set 1, based on modified MLEs (Bayes estimates) is 0.1485 (0.1911) and the corresponding p value is 0.7696 (0.4579). Similarly, for data set 2, the K-S distance based on modified MLEs (Bayes estimates) is 0.2040 (0.1493) and the corresponding p value is 0.3756 (0.7639). Interestingly, for data set 1, modified MLEs provide better fit than the Bayes estimates and for data set 2, it is the other way.

4.2 Example 2 (Real Life Data)

In this sub-section we present a data analysis of the strength data reported by Badar and Priest (1982). The data represent the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. It is already observed that the Weibull model does not work well in this case. Surles and Padgett (1998, 2001) and Raqab and Kundu (2005) observed that generalized Rayleigh works quite well for these strength data. For illustrative purposes we are also considering the same transformed data sets as it was considered by Raqab and Kundu (2005), the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge lengths with sample sizes 69 and 63 respectively. They are presented below:

Data Set I: 0.312 0.314 0.479 0.552 0.700 0.803 0.861 0.865 0.944 0.958 0.966 0.997 1.006 1.021 1.027 1.055 1.063 1.098 1.140 1.179 1.224 1.240 1.253 1.270 1.272 1.274 1.301 1.301 1.359 1.382 1.382 1.426 1.434 1.435 1.478 1.490 1.511 1.514 1.535 1.554 1.566 1.570 1.586 1.629 1.633 1.642 1.648 1.684 1.697 1.726 1.770 1.773 1.800 1.809 1.818 1.821 1.848 1.880 1.954 2.012 2.067 2.084 2.090 2.096 2.128 2.233 2.433 2.585 2.585

Data Set II: 0.101 0.332 0.403 0.428 0.457 0.550 0.561 0.596 0.597 0.645 0.654 0.674 0.718 0.722 0.725 0.732 0.775 0.814 0.816 0.818 0.824 0.859 0.875 0.938 0.940 1.056 1.117 1.128 1.137 1.137 1.177 1.196 1.230 1.325 1.339 1.345 1.420 1.423 1.435 1.443 1.464 1.472 1.494 1.532 1.546 1.577 1.608 1.635 1.693 1.701 1.737 1.754 1.762 1.828 2.052 2.071 2.086 2.171 2.224 2.227 2.425 2.595 3.220

In this case the modified MLE of θ is 0.101. We have also plotted the modified profile log-likelihood function of λ in Figure 2. We have started the iterative process with the initial guess of λ as 1 and it converges in nine iterations. We obtained $\tilde{\lambda}$ = 1.8303, $\tilde{\alpha}$ = 6.5469 and $\tilde{\beta}$ = 4.3586. Finally we obtained $\tilde{R}$ = 0.3997 and the corresponding 95% parametric Bootstrap confidence interval is $(0.3092, 0.4912)$. To find the Bayes estimates, we have assumed the same prior as the previous example. Based on these priors, we obtain the Bayes estimates of the different parameters as: $\widehat{\lambda}_B = 1.169, \widehat{\alpha}_B = 2.979, \widehat{\beta_B} = 1.886, \widehat{\theta}_B = 0.0996$, and $\widehat{R}_B = 0.387$. Moreover, the 95% HPD credible interval of R is $(0.3586, 0.4664)$.

In this case also, we consider the goodness of fit of the three-parameter GE model to those

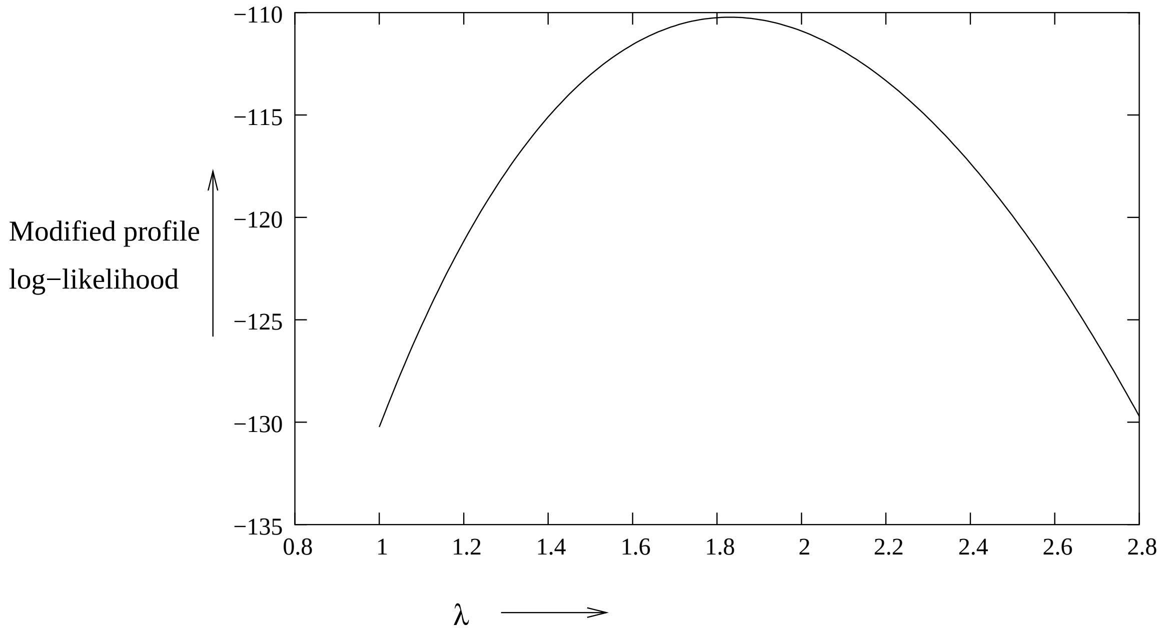

Figure 2: Modified profile log-likelihood function of the strength data

data sets based on modified MLEs and Bayes estimates. For Data Set 1, the K-S distance between the empirical distribution function and the fitted GE distributions based on modified MLEs (Bayes estimate) is 0.1179 (0.1952) and the corresponding p-value is 0.2923 (0.0104). Similarly, for Data Set II the K-S distance based on modified MLEs (Bayes estimates) is 0.0933 (0.1062) with the p-value as 0.6431 (0.4766). Therefore, in this case it is clear that the modified MLEs provide a better fit than the Bayes estimates. We have plotted the empirical survival functions and the corresponding fitted survival functions in Figures 3 and 4. The figures also indicate that the modified MLEs provide better fit than the Bayes estimates at least for these data sets.

5 Numerical Experiments and Discussions

Since the modified maximum likelihood estimators have not been previously studied, we perform some numerical experiments to see how these estimators behave for small samples and for different parameter values. All these numerical experiments have been performed using the random number generator RAN2 of Press *et al.* (1991). We have considered different sample sizes namely $n = m = 5$, $n = m = 10$, $n = m = 15$, $n = m = 20$, α, $\beta = 0.5$, 1.0, 1.5 and 2.0. Without loss of generality, we have taken $\theta = 0$ and $\lambda = 1.0$. For a particular set of parameters

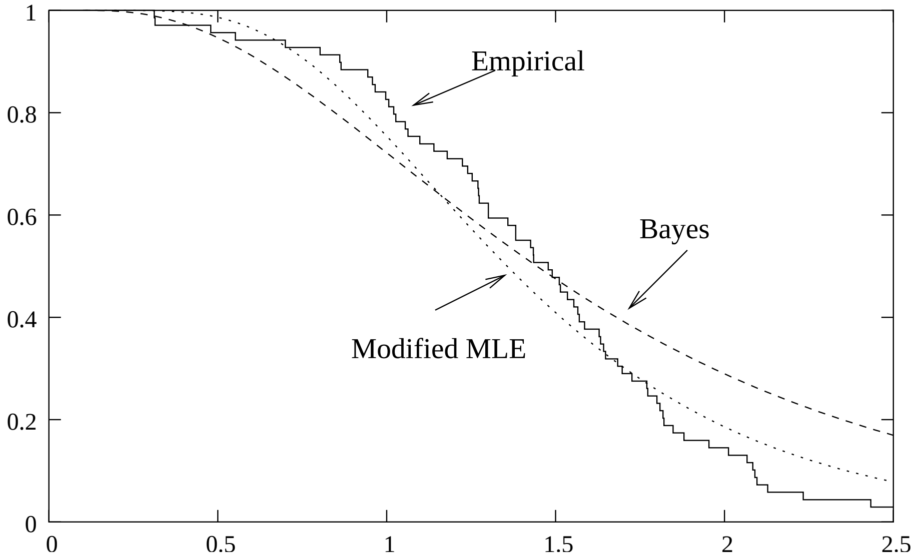

Figure 3: Empirical and fitted survival functions for Data Set I

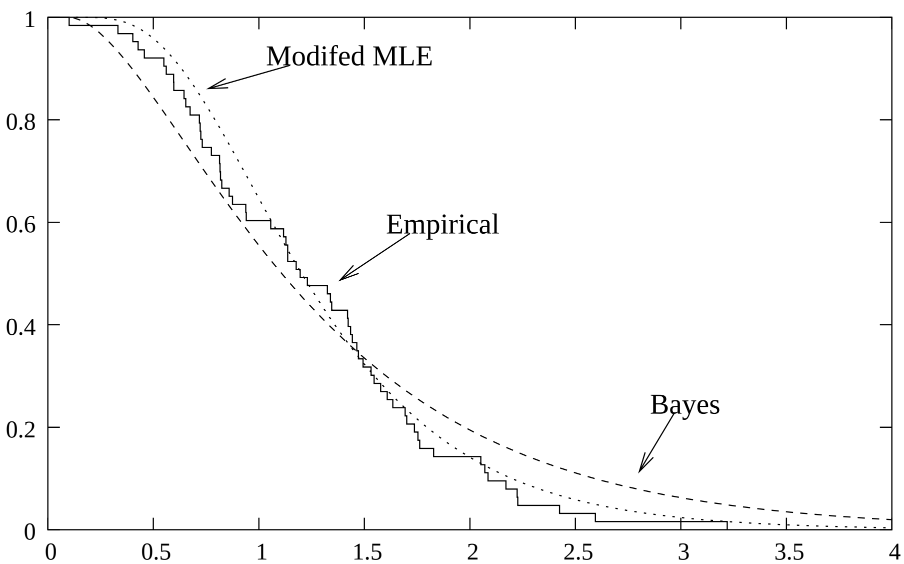

Figure 4: Empirical and fitted survival functions for Data Set II

and from a given generated sample we computed the modified maximum likelihood estimator of R and we replicated the process 1000 times. We computed the average biases and the mean squared errors in each case. The results are reported in Tables 1 to 4.

From the Tables 1 to 4 it is quite clear that the performances of the modified maximum likelihood estimators are quite satisfactory even for very small sample sizes, both in terms of biases and mean squared errors. Moreover, it is observed that as the sample size increases, the average biases and mean squared errors decrease for all sets of parameters considered here. Although we could not prove the consistency property of the modified maximum likelihood estimator of R, but the numerical results indicate indicate its validity. More work is needed in this direction.

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References

[1] Ahmad, K.E., Fakhry, M.E. and Jaheen, Z.F.(1997). Empirical Bayes Estimation of P(Y<X) and Characterizations of the Burr-Type X Model, *Journal of Statistical Planning and Inference* 64, 297-308.

[2] Awad, A.M., Azzam, M.M. and Hamdan, M.A. (1981). Some Inference Results in P(Y<X) in the Bivariate Exponential Model, *Communications in Statistics-Theory and Methods* 10, 2515-2524.

[3] Awad, A.M. and Gharraf, M.K. (1986). Estimation of P(Y<X) in the Burr Case: A Comparative Study, *Communications in Statistics-Simulation and Computation* 15, 389-402.

[4] Bain, L. J. and Engelhardt, M. (1991). Statistical Analysis of Reliability and Life Testing Models. 2nd Edition, Marcel & Decker. New York.

[5] Chen, M. H. and Shao, Q. M. (1999). Monte Carlo Estimation of Bayesian Credible Intervals and HPD Intervals, *J. Computational & Graphical Statistics.* 8(1), 69-92.

[6] Church, J.D. and Harris, B. (1970). The Estimation of Reliability from Stress Strength Relationships, *Technometrics* 12, 49-54.

[7] Downtown, F. (1973). The Estimation of P(Y<X) in the Normal Case, *Technometrics* 15, 551-558.

[8] Efron, B. (1982). *The Jackknife, Bootstrap and other Re-sampling Plans*, Society for Industrial and Applied Mathematics, Philadelphia.

[9] Gupta, R. D. and Kundu, D. (1999). Generalized Exponential Distribution, *Austral. N. Z. Statist.* 41(2), 173-188.

[10] Gupta, R. D. and Kundu, D. (2001a). Exponentiated Exponential Distribution, An Alternative to Gamma and Weibull Distributions, *Biometrical J.* 43(1), 117-130.

[11] Gupta, R. D. and Kundu, D. (2001b). Generalized Exponential Distributions: Different Methods of Estimation, *J. Statist. Comput. Simulations* 69(4), 315-338.

[12] Gupta, R. D. and Kundu, D. (2003). Closeness of Gamma and Generalized Exponential Distribution, *Communications in Statistics - Theory and Methods* 32(4), 705-721.

[13] Kundu, D. and Gupta, R.D. (2005). Estimation of $P(Y < X)$ for Generalized Exponential Distribution, *Metrika*, vol. 61(3), 291-308.

[14] Kotz, S., Lumelskii, Y. and Pensky, M. (2003). *The Stress-Strength Model and its Generalizations*, World Scientific, New York.

[15] Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*, Wiley, New York.

[16] Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. (1991), Numerical Recipes: The Art of Scientific Computing, Cambridge University Press, Cambridge, U.K.

[17] Raqab, M. Z. (2002). Inferences for Generalized Exponential Distribution Based on Record Statistics, *Journal of Statistical Planning and Inference* 104(2), 339-350.

[18] Raqab, M. Z. and Ahsanullah, M. (2001). Estimation of Location and Scale Parameters of Generalized Exponential Distribution Based on Order Statistics, *Journal of Statistical Computation and Simulation* 69(2), 109-124.

[19] Raqab, M.Z. and Kundu, D. (2005). Comparison of different estimators of $P(Y < X)$ for a scaled Burr Type X distribution, *Communications in Statistics - Simulation and Computation*, vol. 34(2), 465-483.

[20] Surles, J.G. and Padgett, W. J. (1998). Inference for Reliability and Stress-Strength for a Scaled Burr-Type X Distribution, *Lifetime Data Analysis* 7, 187-200.

[21] Surles, J.G. and Padgett, W. J. (2001). Inference for $P(Y < X)$ in the Burr-Type X Model, *Journal of Applied Statistical Sciences* 7, 225-238.

Table 1: Biases, MSEs of $\tilde{R}$ when $\alpha = 0.5$

S.S. $\downarrow$ $\beta \rightarrow$	0.50	1.00	1.50	2.00
(5,5)	-0.0016(0.0307)	-0.0483(0.0340)	-0.0528(0.0311)	-0.0492(0.0272)
(10,10)	-0.0069(0.0140)	-0.0381(0.0139)	-0.0344(0.0117)	-0.0299(0.0095)
(15,15)	-0.0014(0.0084)	-0.0295(0.0092)	-0.0265(0.0075)	-0.0214(0.0058)
(20,20)	-0.0026(0.0060)	-0.0234(0.0065)	-0.0199(0.0051)	-0.0161(0.0039)

Table 2: Biases, MSEs of $\tilde{R}$ when $\alpha = 1.0$

S.S. $\downarrow$ $\beta \rightarrow$	0.50	1.00	1.50	2.00
(5,5)	0.0435(0.0319)	-0.0019(0.0343)	-0.0295(0.0368)	-0.0435(0.0364)
(10,10)	0.0275(0.0133)	-0.0072(0.0158)	-0.0263(0.0163)	-0.0324(0.0147)
(15,15)	0.0246(0.0089)	-0.0013(0.0094)	-0.0173(0.0103)	-0.0239(0.0097)
(20,20)	0.0193(0.0062)	-0.0029(0.0066)	-0.0145(0.0072)	-0.0180(0.0068)

Table 3: Biases, MSEs of $\tilde{R}$ when $\alpha = 1.5$

S.S. $\downarrow$ $\beta \rightarrow$	0.50	1.00	1.50	2.00
(5,5)	0.0450(0.0286)	0.0277(0.0345)	-0.0020(0.0361)	-0.0205(0.0393)
(10,10)	0.0260(0.0111)	0.0124(0.0151)	-0.0072(0.0168)	-0.0219(0.0167)
(15,15)	0.0223(0.0071)	0.0147(0.0099)	-0.0112(0.0100)	-0.0130(0.0104)
(20,20)	0.0169(0.0051)	0.0112(0.0069)	-0.0031(0.0071)	-0.0103(0.0075)

Table 4: Biases, MSEs of $\tilde{R}$ when $\alpha = 2.0$

S.S. $\downarrow$ $\beta \rightarrow$	0.50	1.00	1.50	2.00
(5,5)	0.0398(0.0233)	0.0382(0.0342)	0.0168(0.0356)	-0.0021(0.0371)
(10,10)	0.0234(0.0093)	0.0218(0.0143)	0.0051(0.0165)	-0.0072(0.0175)
(15,15)	0.0177(0.0055)	0.0189(0.0094)	0.0099(0.0106)	-0.0012(0.0104)
(20,20)	0.0133(0.0040)	0.0136(0.0065)	0.0071(0.0075)	-0.0032(0.0074)

The average biases and the corresponding MSEs, within brackets, are reported.