Inference for P(Y<X) in Exponentiated Gumbel Distribution

C. S. KAKADE, D. T. SHIRKE AND D. KUNDU

Abstract

This paper deals with inference results in R=P(Y<X) when X and Y are independently but not identically exponentiated Gumbel distributed random variables with same scale parameter but different shape parameters. The maximum likelihood estimator of R and its asymptotic distribution is obtained. Asymptotic distribution is used to construct asymptotic confidence interval and bootstrap confidence interval. Assuming that the common scale parameter is known, the maximum likelihood estimator for R is given. Exact and asymptotic confidence intervals for the same are also discussed. Testing of the reliability based on asymptotic distribution of the maximum likelihood estimator is discussed. Simulation study to investigate performance of the confidence intervals and tests has been carried out.

Keywords: Exponentiated Gumbel distribution, Maximum likelihood estimator, Fisher information matrix, Pivotal quantity.

1. Introduction

In literature, exponentiated family of distributions defined in two ways. If $G(x/\underline{\theta})$ is cumulative distribution function (c.d.f.) of a baseline distribution then by adding one more parameter (say α), the c.d.f. of exponentiated baseline distribution is $F(x/\underline{\theta},\alpha)$ given by

(a) $F(x/\underline{\theta},\alpha)= [G(x/\underline{\theta})]^{\alpha}$, $\alpha>0$, $\underline{\theta} \in \Theta$ and $x\in R$,

(b) $F(x/\underline{\theta},\alpha)= 1-[1- G(x/\underline{\theta})]^{\alpha}$, $\alpha>0$, $\underline{\theta} \in \Theta$ and $x\in R$.

Gupta *et al.* (1998) introduced the exponentiated exponential (EE) distribution as a generalization of the standard exponential distribution. The two parameters EE distribution

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Corresponding Author Email: cskvng@yahoo.co.in.

associated with definition (a) above, have been studied in detail by Gupta and Kundu (2001) which is a sub-model of the exponentiated Weibull distribution, introduced by Mudholkar and Shrivastava (1993). Nadarajah (2005) introduced exponentiated Gumbel (EG) distribution using (b) above. Some of its application areas in climate modeling include global warming problem, flood frequency analysis, offshore modeling, rainfall modeling and wind speed modeling.

The c.d.f. of the Gumbel distribution is

$$G(x;\sigma) = \exp\left(-e^{\frac{-x}{\sigma}}\right), \ -\infty<x<\infty \ ; \sigma>0.$$

By introducing a shape parameter $\alpha > 0$ and using definition (a) above, the c.d.f. of the exponentiated Gumbel distribution is

$$F(x;\alpha,\sigma) = (G(x,\sigma))^{\alpha} = \left(\exp\left(-e^{\frac{-x}{\sigma}}\right)\right)^{\alpha}, \ \alpha, \sigma>0 \ ; -\infty<x<\infty, \tag{1.1}$$

which is simply the α^{th} power of c.d.f. of the Gumbel distribution.

The probability density function (p.d.f.) corresponding to (1.1) is

$$f(x;\alpha,\sigma) = \frac{\alpha}{\sigma}\left(\exp\left(-e^{\frac{-x}{\sigma}}\right)\right)^{\alpha} e^{\frac{-x}{\sigma}}, \ \alpha, \sigma>0 \ ; -\infty<x<\infty. \tag{1.2}$$

We shall write $X \sim EG(\alpha,\sigma)$ to denote an absolutely continuous random variable X having the exponentiated Gumbel distribution with shape and scale parameters α and σ respectively whose p.d.f. is given by (1.2).

In stress-strength model, the stress (Y) and the strength (X) are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval. Due to the practical point of view of reliability stress-strength model, the estimation problem of $R=P(Y<X)$ has attracted the attention of many authors. Recently Kundu and Gupta (2005) have considered estimation of $P(Y<X)$, when X and Y are independent generalized exponential random variables and Raqab and Kundu (2005) considered the case when X and Y are independent generalized Rayleigh random variables.

The main aim of this paper is to discuss the inference of $R=P(Y<X)$, when X and Y are two independent but not identically random variables belonging to exponentiated Gumbel

distribution with two parameters. In Section 2, the point and interval estimation of reliability R will be obtained using maximum likelihood method and bootstrap technique. The asymptotic distribution of maximum likelihood estimator (MLE) of R and its asymptotic confidence interval will be obtained. Also we shall discuss the exact distribution of the MLE of R and its confidence interval, when scale parameter is known. Ordering properties and testing of hypothesis on the shape parameters will be discussed in Section 3. Simulation study will be carried out to assess the performance of asymptotic, exact and percentile bootstrap confidence intervals in Section 4. Comparison of asymptotic and exact tests will be made with respect to power of the tests.

2. Point and interval estimation of R

Let X and Y are two independent exponentiated Gumbel random variables with shape, scale parameters α, σ and β, σ respectively. Therefore

$$R=P(Y<X) = \int_{-\infty}^{\infty}\int_{-\infty}^{x}\frac{\alpha}{\sigma}e^{\frac{-x}{\sigma}}\left(\exp(-e^{\frac{-x}{\sigma}})\right)^{\alpha}\frac{\beta}{\sigma}e^{\frac{-y}{\sigma}}\left(\exp(-e^{\frac{-y}{\sigma}})\right)^{\beta}dy\,dx = \frac{\alpha}{\alpha+\beta}. \tag{2.1}$$

In the following, we shall discuss estimation of R using maximum likelihood method.

Case 1: When scale parameter is unknown.

Suppose $x_1,x_2,\ldots,x_n$ be a random sample from EG(α, σ) and $y_1,y_2,\ldots,y_m$ be a random sample from EG(β,σ). Therefore the log-likelihood function L of α, β and σ for the observed sample is

$$L= n\ln\alpha-\alpha\sum_{i=1}^{n}e^{\frac{-x_i}{\sigma}}+m\ln\beta-\beta\sum_{j=1}^{m}e^{\frac{-y_j}{\sigma}}-(m+n)\ln\sigma-\frac{\left(\sum_{i=1}^{n}x_i+\sum_{j=1}^{m}y_j\right)}{\sigma}. \tag{2.2}$$

Differentiating partially with respect to α, β and σ, setting the results equal to zero we get three nonlinear equations.

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}e^{\frac{-x_i}{\sigma}}=0 \quad , \quad \frac{\partial L}{\partial\beta}=\frac{m}{\beta}-\sum_{j=1}^{m}e^{\frac{-y_j}{\sigma}}=0 \text{ and}$$

$$\frac{\partial L}{\partial\sigma}=\frac{-\alpha}{\sigma^2}\sum_{i=1}^{n}x_i\,e^{\frac{-x_i}{\sigma}}-\frac{\beta}{\sigma^2}\sum_{j=1}^{m}y_j\,e^{\frac{-y_j}{\sigma}}-\frac{(m+n)}{\sigma}+\frac{(\sum x_i+\sum y_j)}{\sigma^2}=0\ .$$

After solving these equations simultaneously yields the MLEs, $\hat{\alpha}, \hat{\beta}$ and $\hat{\sigma}$.

Hence the MLE of R namely, $\hat{R}_1$ is given by $\hat{R}_1 = \dfrac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$. (2.3)

MLE, $\hat{R}_1$ can also be obtained by using following Remark (2.1).

Remark (2.1): If $U = e^{-x}$ and $V = e^{-y}$ where X and Y are two independent exponentiated Gumbel random variables with shape, scale parameters α, σ and β, σ respectively then U and V are independent Weibull random variables with shape parameter 1/σ and scale parameter α and β respectively.

The Fisher information matrix of (α,β,σ) is

$$I(\alpha,\beta,\sigma) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\beta^2}\right) & E\left(\frac{\partial^2 L}{\partial\beta\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\sigma\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) \end{bmatrix}.$$

Moreover $E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = -\frac{n}{\alpha^2}$, $E\left(\frac{\partial^2 L}{\partial\beta^2}\right) = -\frac{m}{\beta^2}$, $E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) = E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) = 0$,

$$E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) = \frac{\alpha}{\sigma\, 2^{\alpha}}\sum_{i=1}^{n} E(\ln U_i) = E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right), E\left(\frac{\partial^2 L}{\partial\beta\partial\sigma}\right) = \frac{\beta}{\sigma\, 2^{\beta}}\sum_{j=1}^{m} E(\ln V_j) = E\left(\frac{\partial^2 L}{\partial\sigma\partial\beta}\right)$$

$$E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) = \frac{n}{\sigma^2} - \frac{2}{\sigma^2}\sum_{i=1}^{n} E(\ln W_i) - \frac{\alpha^2}{\sigma^2\, 2^{\alpha-1}}\sum_{i=1}^{n} E(\ln U_i) - \frac{\alpha^2}{\sigma^2\, 2^{\alpha}}\sum_{i=1}^{n} E(\ln U_i)^2$$

$$+\frac{m}{\sigma^2} - \frac{2}{\sigma^2}\sum_{j=1}^{m} E(\ln Z_j) - \frac{\beta^2}{\sigma^2\, 2^{\beta-1}}\sum_{j=1}^{m} E(\ln V_j) - \frac{\beta^2}{\sigma^2\, 2^{\beta}}\sum_{j=1}^{m} E(\ln V_j)^2,$$

where U_i has gamma(2,α) distribution with mean$\left(\frac{2}{\alpha}\right)$, V_j has gamma(2,β) distribution with mean$\left(\frac{2}{\beta}\right)$, W_i has exponential distribution with mean$\left(\frac{1}{\alpha}\right)$ and Z_j has exponential distribution with mean $\left(\frac{1}{\beta}\right)$.

Theorem (2.1): *As* $m,n \to \infty$ *and* $\frac{m}{n} \to p$ *then*

$$\left(\sqrt{n}(\hat{\alpha}-\alpha), \sqrt{m}\left(\hat{\beta}-\beta\right), \sqrt{n}(\hat{\sigma}-\sigma)\right) \to N_3(0, I^{-1}(\alpha,\beta,\sigma)),$$

where $I^{-1}(\alpha,\beta,\theta) = \begin{bmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ *is the inverse of the Fisher information matrix.*

Proof : Proof follows from asymptotic properties of MLEs under regularity conditions and multivariate central limit theorem.

Theorem (2.2): As $m,n \to \infty$ and $\frac{n}{m} \to p$ then $\sqrt{n}(\hat{R}_1 - R) \to N(0,B)$, where

$$B = \frac{1}{(\alpha+\beta)^4}\left(\beta^2\sigma_{11} + \alpha^2\sigma_{22}\right).$$

Proof : Proof follows from invariance property of consistent asymptotically normal (CAN) estimator under continuous transformation, and omitted for brevity.

Using Theorem (2.2), we can obtain asymptotic confidence interval of R is

$$\left(\hat{R}_1 - Z_{1-\delta/2}\frac{\sqrt{\hat{B}}}{\sqrt{n}}, \quad \hat{R}_1 + Z_{1-\delta/2}\frac{\sqrt{\hat{B}}}{\sqrt{n}}\right) \tag{2.4}$$

Remark (2.2): To estimate variance B, the empirical Fisher information matrix and MLE of α, β and σ may be used. However simulation study due to Kundu and Gupta (2005) for exponentiated exponential distribution indicates that confidence interval defined in (2.4) has comparatively low coverage probability. They have suggested bootstrap method to get a better confidence interval with respect to coverage probability. In the following, we shall discuss in brief similar approach.

Bootstrap confidence interval

In this subsection, we propose a percentile bootstrap method (Efron, 1982) for constructing confidence interval of R which is as follows.

Step-1: Generate random samples $x_1, x_2, \ldots, x_n$ from EG(α,σ) and $y_1, y_2, \ldots, y_m$ from EG(β,σ) and compute maximum likelihood estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{\sigma}$.

Step-2: Using $\hat{\alpha}$ and $\hat{\sigma}$ generate a bootstrap sample $x_1^*, x_2^*, \ldots, x_n^*$ from EG($\hat{\alpha}$, $\hat{\sigma}$) and similarly using $\hat{\beta}$ and $\hat{\sigma}$ generate a bootstrap sample $y_1^*, y_2^*, \ldots, y_m^*$ from EG($\hat{\beta}$, $\hat{\sigma}$). Based on these bootstrap samples compute bootstrap estimate of R, say $\hat{R}^*$ where

$\hat{R}^* = \dfrac{\hat{\alpha}^*}{\hat{\alpha}^* + \hat{\beta}^*}$, where $\hat{\alpha}^*$ and $\hat{\beta}^*$ are the MLEs of α and β obtained from the corresponding bootstrap samples.

Step-3: Repeat step-2 NBOOT times (usually NBOOT=1000).

Step-4: Compute cumulative distribution function of $\hat{R}^*$, say H(x), where

$H(x) = P(\hat{R}^* \leq x)$ and $\hat{R}_{Boot-p}(x) = H^{-1}(x)$ for a given x. The approximate 100(1-γ)% bootstrap confidence interval of R is $\left(\hat{R}_{Boot-p}(\gamma/2), \hat{R}_{Boot-p}(1-\gamma/2)\right)$. (2.5)

The performance of bootstrap confidence intervals of R for exponentiated Gumbel distributions will be provided in Section 4.

Case 2: When scale parameter σ is known.

Without loss of generality, we can assume that $\sigma = 1$. Suppose $x_1, x_2, \ldots, x_n$ be a random sample from EG(α,1) and $y_1, y_2, \ldots, y_m$ be a random sample from EG(β,1) and based on these samples we want to estimate R. Based on the above samples, it is clear that, the MLE of R namely $\hat{R}_2$ is given by

$$\hat{R}_2 = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\beta}_1}, \text{ where } \hat{\alpha}_1 = \frac{n}{\sum_{i=1}^{n} \exp(-x_i)} \text{ and } \hat{\beta}_1 = \frac{m}{\sum_{j=1}^{m} \exp(-y_j)}.$$

Note: MLE $\hat{R}_2$ can also be obtained by using Remark (2.1). That is when σ = 1, U and V are independent exponential random variables with means $\left(\frac{1}{\alpha}\right)$ and $\left(\frac{1}{\beta}\right)$ respectively.

Lemma (2.1): *The p.d.f. of* $\hat{R}_2$ *is given by*

$$f_{\hat{R}_2}(r) = \frac{\Gamma(m+n)}{\Gamma m\, \Gamma n}\left(\frac{n}{m}\right)^n \left(\frac{\alpha}{\beta}\right)^{n-1} \frac{\left(\frac{1-r}{r}\right)^{n-1}}{\left(1+\frac{n\alpha}{m\beta}\left(\frac{1-r}{r}\right)\right)^{m+n}} \qquad ,0<r<1 \ . \tag{2.6}$$

Proof : Omitted for brevity.

The shortest length confidence interval is given by following Lemma.

Lemma (2.2) :*The shortest length 100(1-γ)% confidence interval of R is*

$$\left(\left(1+F_{2m,2n;(1-\gamma/2)}\left(\frac{1}{\hat{R}_2}-1\right)\right)^{-1}, \left(1+F_{2m,2n;(\gamma/2)}\left(\frac{1}{\hat{R}_2}-1\right)\right)^{-1}\right). \tag{2.7}$$

Proof : Omitted for brevity.

In the following, we shall obtain an asymptotic confidence interval of R.

Lemma (2.3) : *The asymptotic 100(1-γ)% confidence interval of R is*

$$\left(\left(\hat{R}_2 - Z_{1-\gamma/2}\sqrt{\frac{m+n}{mn}}\quad \hat{R}_2(1-\hat{R}_2)\right), \left(\hat{R}_2 + Z_{1-\gamma/2}\sqrt{\frac{m+n}{mn}}\quad \hat{R}_2(1-\hat{R}_2)\right)\right) \ , \tag{2.8}$$

where $Z_{1-\gamma/2}$ *is the (1-γ/2)*th *quantile of the standard normal distribution.*

Proof : The MLE $\hat{R}_2$ is asymptotically normal with mean R and variance

$\sigma^2_{\hat{R}_2} = \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial R}{\partial \theta_i}\frac{\partial R}{\partial \theta_j} I^{-1}_{ij}$, where $(\theta_1,\theta_2) = (\alpha, \beta)$ and $(I^{-1}_{i,j})^{th}$ is the (i,j)th element of the inverse of the Fisher's information matrix I(α, β) about the parameters (α, β) and

$$I(\alpha,\beta) = -\begin{bmatrix} \frac{n}{\alpha^2} & 0 \\ 0 & \frac{m}{\beta^2} \end{bmatrix},$$ (See Rao (1965)). It can be seen that, $\sigma^2_{\hat{R}_2} = \left(\frac{m+n}{mn}\right) R^2 (1-R)^2$.

Therefore 100(1-γ)% asymptotic confidence interval of R can be obtained using standardized statistic as a pivotal quantity. We replace 'R' in the asymptotic variance by its MLE.

3. Testing of Hypothesis

Stochastic ordering of positive continuous random variable is an important tool to judge the comparative behavior. The exponentiated Gumbel distribution is ordered with respect to the 'likelihood ratio' ordering ($X \leq_{lr} Y$) as shown in the following theorem.

Theorem (3.1): *Let X~ EG(α,σ) and Y~ EG(β,σ). If α<β, then* $X \leq_{lr} Y$.

Proof: First note that for α, β>0,

$$\frac{f_X(x,\alpha,\sigma)}{f_Y(x,\beta,\sigma)} = \frac{\alpha}{\beta}\left[\exp\left(-e^{\frac{-x}{\sigma}}\right)\right]^{\alpha-\beta}.$$

Since, for α<β, $\frac{df_X(x,\alpha,\sigma)}{dxf_Y(x,\beta,\sigma)} = \frac{\alpha(\alpha-\beta)}{\sigma\,\beta}\left[\exp\left(-e^{\frac{-x}{\sigma}}\right)\right]^{\alpha-\beta} e^{\frac{-x}{\sigma}} < 0$,

then $\frac{f_X(x,\alpha,\sigma)}{f_Y(x,\beta,\sigma)}$ is decreasing in x. That is $X \leq_{lr} Y$.

Since α and β are unknown, it will be of interest to know whether α<β or not. We put this as a problem of hypothesis testing. We shall consider test for hypothesis $H_0:\alpha \leq \beta$ against $H_1 : \alpha > \beta$. Equivalently we shall test H_0: R ≤ 0.5 against H_1: R > 0.5.

An exact test of size γ for the above problem, using Lemma (2.2), rejects H_0 if

$$\left(\frac{\hat{R}_2}{1-\hat{R}_2}\right) > F_{2n,2m;1-\gamma} \quad , \tag{3.1}$$

where $F_{2n, 2m; 1-\gamma}$ is the $(1-\gamma)^{th}$ quantile of F distribution with (2n, 2m) d.f..

Using Lemma (2.3), an asymptotic test of size γ rejects the null hypothesis H_0, if

$$\left(\hat{R}_2 - \frac{1}{2}\right) > \sqrt{\frac{m+n}{16\, mn}} \quad Z_{1-\gamma} \qquad , \tag{3.2}$$

where $Z_{1-\gamma}$ is the $(1-\gamma)^{th}$ quantile of the standard normal distribution.

As an independent interest, we can also obtain an asymptotic and exact test of the desired size for alternatives H'_1: R= 0.5 and H''_1: $R \neq 0.5$.

4. Simulation Study

We simulate data sets form exponentiated Gumbel distribution by taking α=2, β=3 and common scale parameter σ = 4 with sample sizes n = 25 and m = 25 and analyze as below.

X = 14.6640, 1.2459, 5.5493, 4.0778, 11.4213, 7.9851, 3.7448, -2.7624, 9.2762, 3.6138, 5.6635, 8.5948, 12.8055, 7.5417, 0.5668, 3.1846, 13.6024, 12.5514, 3.2345, 11.5139, -1.4155, 2.6093, 9.0762, -3.3482, 0.0522.

Y = 2.5252, 2.4750, 7.1310, 3.3411, 2.4762, -1.3283, 9.3188, 5.2400,14.9961, 5.4734, 4.9482, 11.5541, 6.1543, 2.5237, 8.0868, 11.3300, -1.0803, 8.2251, 4.5206, 11.1621, 5.8930, 8.6718, 5.0608, 3.7030, 2.3610.

The MLE's of parameters are $\hat{\alpha}$ = 2.0320, $\hat{\beta}$ = 2.7357 and $\hat{\sigma}$ = 4.1400 and hence MLE of R is 0.4262. We also obtain 95% confidence interval by using percentile bootstrap method as (0.3022, 0.5729).

We shall perform some simulation experiments using percentile bootstrap method when scale parameter σ is unknown to observe the behavior of the MLE and confidence intervals for various sample sizes and for various values of (α, β). We shall consider the sample sizes (n, m)= (10,10), (10, 20), (20, 20), (20, 40), (40, 40) and the parameter values α= 2, σ=4 and β = 2, 3, 4 and 8. Average bias and mean squared error (MSE) of R will be reported over 1000 replications for 1000 bootstrap samples. We shall compute 95% confidence intervals using (2.5) and estimate coverage percentage and average length of confidence interval. The results will be reported in Table 1.

Also we shall perform some simulation experiments when scale parameter σ is known (σ=1). We consider the sample sizes (n, m)= (10,10), (10, 20), (20, 20), (20, 40), (40, 40) and the parameter values α= 2 and β = 2, 3, 6 and 8. Average biases and MSE of R will be reported over 10000 replications. We shall compute 95% confidence intervals and estimate

Table 1: Bias, MSE, Confidence Length and Coverage Percentage of C. I.

Sample size	2	3	6	8
(10, 10)	-0.0058(0.0131) 0.4273(93.00)	-0.0005 (0.0124) 0.4139 (93.00)	-0.0096 (0.0077) 0.3286 (90.70)	-0.0054 (0.0061) 0.2899 (91.40)
(10, 20)	0.0125 (0.0109) 0.3748 (92.40)	0.0095 (0.0097) 0.3672 (0.9410)	0.0088 (0.0067) 0.3052 (93.10)	0.0011 (0.0050) 0.2643 (92.90)
(20, 20)	-0.0018 (0.0067) 0.3120 (93.70)	-0.0018 (0.0070) 0.3013 (92.80)	-0.0044 (0.0046) 0.2454 (91.50)	-0.0062 (0.0031) 0.2144 (92.30)
(20, 40)	0.0057 (0.0050) 0.2706 (94.00)	0.0067 (0.0050) 0.2630 (93.50)	0.0031 (0.0033) 0.2175 (93.90)	-0.0001 (0.0026) 0.1909 (93.40)
(40, 40)	0.0012 (0.0033) 0.2205 (94.20)	-0.0032 (0.0031) 0.2134 (94.40)	-0.0049 (0.0021) 0.1762 (93.90)	-0.0028 (0.0016) 0.1567 (93.60)

(The first row represents the average bias and MSE. Second row represents the average length, coverage percentage of the corresponding asymptotic bootstrap confidence interval.)

coverage percentages and average lengths of both asymptotic and exact confidence interval. The results will be reported in Table 2.

Table 2: Bias, MSE, Confidence Length and Coverage Percentage of C. I.

Sample size	2	3	6	8
(10, 10)	-0.0003(0.0119) 0.4174(91.47) 0.4058(94.83)	0.0033(0.0110) 0.4027(91.86) 0.3935(95.22)	0.0087(0.0073) 0.3237(91.77) 0.3258(95.30)	0.0098(0.0056) 0.2810(91.50) 0.2876(94.93)
(10, 20)	0.0042(0.0090) 0.3659(92.60) 0.3581(94.70)	0.0093(0.0086) 0.3542(92.30) 0.3507(94.46)	0.0120(0.0057) 0.2851(93.52) 0.2927(94.72)	0.0105(0.0043) 0.2459(92.90) 0.2572(94.74)
(20, 20)	-0.0018(0.0060) 0.3024(93.45) 0.2977(95.02)	0.0016(0.0057) 0.2909(93.11) 0.2872(94.78)	0.0056(0.0037) 0.2313(93.15) 0.2323(94.61)	0.0045(0.0026) 0.1984(93.49) 0.2012(95.18)
(20, 40)	0.0025(0.0045) 0.2636(94.03) 0.2605(95.15)	0.0040(0.0043) 0.2539(93.83) 0.2527(94.98)	0.00629(0.0027) 0.2017994.220 0.2048(94.910)	0.0056(0.0021) 0.1732(94.10) 0.1776(94.92)
(40, 40)	-0.0009(0.0030) 0.2165(94.57) 0.2147(95.39)	0.0016(0.0028) 0.2082(94.25) 0.2068(95.15)	0.0018(0.0018) 0.1636(94.22) 0.1640(95.04)	0.0021(0.0013) 0.1402(94.40) 0.1413(95.23)

(The first row represents the average bias and the corresponding MSEs are reported within brackets. Second and third rows represent the average length and the corresponding coverage percentage of the asymptotic and exact confidence intervals respectively.)

We observe the following from the simulation study.

(i) Even for small sample sizes, the performances of the MLEs are quite satisfactory in terms of bias and MSE. When sample size increases, the MSE decreases. It verifies the consistency property of the MLE of R.

(ii) The confidence intervals based on the MLEs work quite well even when the sample size is very small, say (10, 10). The performance of the asymptotic confidence intervals with respect to length is comparable with the length of exact confidence intervals.

(iii) The coverage probability of asymptotic confidence interval is slightly below than the nominal when sample sizes are small. However, for large sample sizes it meets the desired level.

Through simulation study, comparison of power will be made for exact and asymptotic tests given in (3.1) and (3.2). The power will be determined by generating 1000 random samples of sizes (n, m)=(10,10), (10,20), (20,20), (20,40) and (40,40). The results for the tests at the significance level γ=0.01 and 0.05 will be presented in Table 3 and Table 4 respectively. P_1 and P_2 are referred to as powers based on exact and asymptotic test respectively.

Table 3 : Power of the test based on asymptotic and exact distribution of R, $\gamma = 0.01$.

R	(10, 10)		(10, 20)		(20, 20)		(20, 40)		(40, 40)	
	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2
0.5000	0.089	0.0060	0.0201	0.0103	0.0093	0.0076	0.0177	0.0123	0.0105	0.0094
0.5263	0.016	0.0119	0.0349	0.0187	0.0220	0.0199	0.0403	0.0301	0.0305	0.0279
0.5556	0.0297	0.0213	0.0660	0.0399	0.0422	0.0481	0.0851	0.0634	0.0873	0.0821
0.5882	0.0578	0.0399	0.1134	0.0727	0.1099	0.0981	0.1818	0.1428	0.2233	0.2103
0.6250	0.1057	0.0799	0.2031	0.1373	0.2272	0.2081	0.3683	0.3074	0.4787	0.4640
0.6667	0.2085	0.1593	0.3479	0.2572	0.4349	0.4083	0.6203	0.5560	0.7779	0.7667
0.7143	0.3666	0.3018	0.5675	0.4600	0.7010	0.6756	0.8765	0.8392	0.9583	0.9550
0.7692	0.6065	0.5391	0.8279	0.7448	0.9243	0.9141	0.9872	0.9808	0.9991	0.9988
0.8333	0.8793	0.8404	0.9787	0.9569	0.9964	0.9956	0.9999	0.9999	1	1
0.9091	0.9955	0.9929	0.9999	0.9997	1	1	1	1	1	1

Table 4 : Power of the test based on asymptotic and exact distribution of R, γ=0.05.

R	(10, 10)		(10, 20)		(20, 20)		(20, 40)		(40, 40)	
	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2
0.5000	0.0503	0.0466	0.0733	0.0575	0.0474	0.0461	0.0657	0.0573	0.0476	0.0476
0.5263	0.0734	0.0691	0.1096	0.0886	0.0878	0.0857	0.1299	0.1148	0.1169	0.1161
0.5556	0.1258	0.1193	0.1799	0.1483	0.1751	0.1714	0.2297	0.2073	0.2588	0.2571
0.5882	0.1891	0.1789	0.2785	0.2408	0.2981	0.2933	0.4023	0.3709	0.4821	0.4798
0.6250	0.2939	0.2826	0.4220	0.3766	0.4858	0.4799	0.6225	0.5916	0.7303	0.7286
0.6667	0.4445	0.4311	0.6101	0.5647	0.7028	0.6977	0.8378	0.8157	0.9213	0.9208
0.7143	0.6398	0.6279	0.8048	0.7652	0.8840	0.8815	0.9674	0.9599	0.9924	0.9924
0.7692	0.8406	0.8301	0.9455	0.9314	0.9822	0.9816	0.9984	0.9977	1	1
0.8333	0.9686	0.9663	0.9969	0.9950	0.9993	0.9993	1	1	1	1
0.9091	0.9993	0.9993	1	1	1	1	1	1	1	1

From Table 3 and Table 4, we observe the following.

(i) Both tests perform well with respect to the power.

(ii) Power of the test based on exact test is slightly higher than that of asymptotic test.

(iii) Both the tests are consistent in the sense that as sample size increases, the power of the test shows improvement.

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References

1. Efron, B (1982), The Jackknife, the Bootstrap and other resampling plans, CBMS-NSF Regional Conference series in Applied Mathematics, Vol. 38, SIAM, Philadelphia, PA.

2. Gupta, R. C, Gupta, P., L. and Gupta, R.D. (1998), Modeling failure time data by Lehman alternatives, Communication Statistics-Theory and Methods, **27**, 887-904.

3. Kundu, D. and Gupta, R. D. (2005), Estimation of P(Y<X) for Generalized Exponential Distributions, Metrika, **61(3)**, 291-308.
4. Mudholkar, G. S. and Srivastava, D. (1993), The exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE Trans. Reliability*, Vol.42, No. 2, pp. 299-302.
5. Nadarajah, S. (2005), The exponentiated Gumbel distribution with climate application, Environmetrics, **17(1)**, 13-23.
6. Rao, C.R. (1965), Linear Statistical Inference and its Applications, New York: John Wiley and Sons.
7. Raqab, M.Z. and Kundu, D (2005), Comparison of different estimators of P(Y<X) for Scaled Burr Type X distribution, Communication in Statistics-Computations and Simulations, **34(2)**, 465-483.

About the Authors

C. S. Kakade is Professor of Statistics at the Br. Balasaheb Khardekar College, Vengurla, India.

D. T. Shirke is at the Shivaji University, Kolhapur, Maharashtra-416 004, India.

D. Kundu is at the Department of mathematics, IIT, Kanpur-208 016, India.

Corresponding Author

C.S. Kakade

Department of Statistics
Br. Balasaheb Khardekar College
Vengurla-416 516.
Dist. Sindhudurg, India
E-mail: `cskvng@yahoo.co.in`