ESTIMATING THE PARAMETERS OF BURST-TYPE SIGNALS

SWAGATA NANDI[1] AND DEBASIS KUNDU[2]

Abstract. In this paper, we study a model which exhibits burst-type features such as ECG signals, under certain condition. The model is proposed by Sharma and Sircar (2001) and we call it burst-type signals. It is a generalization of the fixed amplitude sinusoidal model. The amplitudes take a certain deterministic function. We assume that the error random variables are independent and identically distributed. The least square method is proposed to estimate the unknown parameters. We show that the least squares estimators are strongly consistent and find their asymptotic distribution as Gaussian. Some numerical results based on simulations results are reported for illustrative purposes.

1. Introduction

Estimation of parameters of a parametric model is of prime interest in almost all problems involved in statistical modelling. In signal detection, once the model in decided, the next step is the estimation of parameters from the given set of observations. The present article addresses the estimation problem of parameters in the following model.

$$y(t) = \sum_{i=1}^{q} A_i \exp[b_i\{1 - \cos(\alpha t + c_i)\}] \cos(\theta_i t + \phi_i) + e(t), \qquad t = 1, \ldots, N, \tag{1}$$

where for $i = 1, \ldots, q$, A_i is the amplitude of the carrier wave; b_i and c_i are the gain part and the phase part of the exponential modulation signal; θ_i is the carrier angular frequency, α is the modulation angular frequency and ϕ_i is the phase corresponding to the carrier angular frequency. The number of components present in the signal is denoted by q. The error random variables $\{e(t)\}$ are assumed to be independent and identically distributed (i.i.d.) with mean zero and finite variance. The model (1) is a sinusoidal model with time-dependent amplitude like $\sum_{i=1}^{q} s_i(t) \cos(\theta_i t + \phi_i) + e(t)$. Here $s_i(t)$ is taking the particular exponential

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Corresponding Author: Swagata Nandi, Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi - 110016, Ph:+91-11-41493931
Email: nandi@isid.ac.in

form $\exp[b_i\{1-\cos(\alpha t+c_i)\}]$ multiplied by a constant A_i. The modulation angular frequency α is assumed to be same through all components which ensures the presence of burst like signal.

The model (1) is proposed by Sharma and Sircar (2001). The authors used the complex-valued model corresponding to (1). In real life, we mostly deal with real-valued observations, hence it is suggested in Sharma and Sircar (2001) to estimate the imaginary part of each of them using Hilbert transform. Then, one has complex-valued data (as observed, estimated by Hilbert transform) and the techniques of complex model can be implemented. Sharma and Sircar (2001) used the proposed model in describing a segment of real electrocardiograph (ECG) signal. In an earlier article, Mukhopadhyay and Sircar (1996) proposed a similar kind of model to analyze an ECG signal. Actually it is the same model as (1) with a different representation of parameters. Also some of the parameters themselves are related with certain relationships. In both these papers the authors analyzed ECG data using some ad-hoc estimation procedures of the unknown parameters. Here we consider the particular form as the real model given in (1) (real in the sense of carrier part). The main aim of this paper is to study the least square estimators (LSEs) of the unknown parameters of the model (1) and derive their theoretical properties in a systematic manner.

Many real life data, for example, ECG signals exhibit burst-type features. Similar structures have been observed in the plot of data generated by equation (1) for different sets of values. For an example, see Fig. 6 in section 4. Model (1) was proposed to employ one or more amplitude modulated sinusoidal signals with the aim of modelling different features of an ECG output signal separately. Following Sharma and Sircar (2001), we term (1) as the burst-type signal. For another type of amplitude modulated sinusoidal model, the readers are referred to Sircar and Syali (1996) and Nandi, Iyer and Kundu (2004).

We discuss the problem of parameter estimation of the burst-type signal in presence of i.i.d. noise. We use the least square method for estimation and study the properties of the estimators. It is known that the constant amplitude multiple sinusoidal model does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981) for the LSEs to be consistent. Model (1), being a more complicated general model does not satisfy them. However, the special structure of the model allows us to establish the strong consistency and the large sample distribution of the LSEs of the unknown parameters. In case of using model 1 to real data, the main problem is to guess the initial estimates. We would like to mention that we do not address any computational problem in this paper. In this article, we mainly study the theoretical properties of the LSEs.

The paper is organized as follows. In section 2, we state the asymptotic properties of the LSEs for single burst-type signal ($q = 1$). The results for general q are discussed in section 3. Numerical results are presented in section 4 and we conclude the paper in section 5. All proofs are provided in Appendix A.

2. Asymptotic Distribution of LSEs for Single Burst-Type Signal

In this section, we consider the case when the number of signals $q = 1$ and write the model (1) as

$$y(t) = A \exp[b\{1 - \cos(\alpha t + c)\}] \cos(\theta t + \phi) + e(t), \quad t = 1, \ldots, N. \tag{2}$$

It is assumed that $|b| \leq J$, therefore, $e^{b\cos(\alpha t)} \leq e^{|b|} \leq e^J = K(\text{say})$ a finite constant, $\forall\ t$ and the frequencies $\alpha, \theta \in [0, \pi]$; the phases $c, \phi \in [-\pi, \pi]$; $A \in \mathcal{R}$ is a finite constant; and $e(t)$ is i.i.d. with mean zero and finite variance σ^2. Thus, $\{y(t)\}$ is a sequence of mean non-stationary random variables. We note that A is a linear parameter whereas other parameters are non-linear. The condition $|b| \leq J$ is not a serious restriction because A is unbounded. Now our problem is to estimate the unknown parameters, A, b, α, c, θ and ϕ from a given sample of size N.

Define the parameter vector $\boldsymbol{\eta} = (A, b, \alpha, c, \theta, \phi)$ and $\boldsymbol{\eta}^0$ denote the true value of $\boldsymbol{\eta}$. The LSE of $\boldsymbol{\eta}$, $\hat{\boldsymbol{\eta}}$ for the model (2) minimizes the following residual sum of squares

$$Q(\boldsymbol{\eta}) = \sum_{t=1}^{N} \Big[y(t) - A \exp[b\{1 - \cos(\alpha t + c)\}] \cos(\theta t + \phi) \Big]^2, \tag{3}$$

with respect to $\boldsymbol{\eta}$. Since the model is a partial non-linear regression model and the parameter space corresponds to the non-linear parameters is compact (stated in Theorem 2.1), therefore following the same approach as Lemma 2 of Jennrich (1969), the existence of the LSEs can be established. Also, here the LSE $\hat{\boldsymbol{\eta}}$ means the local minimum in the neighbourhood of the true parameter value $\boldsymbol{\eta}^0$. In the following we first state the consistency property of $\hat{\boldsymbol{\eta}}$ in the following Theorem 2.1.

Theorem 2.1. *Let $\boldsymbol{\eta}^0 = (A^0, b^0, \alpha^0, c^0, \theta^0, \phi^0)$, the true parameter value, be an interior point of the parameter space $\{(-\infty, \infty) \times [-\log(K), \log(K)] \times [0, \pi] \times [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi]\}$, where K is a large positive real number, so that $\exp\{|b^0|\} < K$. If the error random variables $e(t)$s are i.i.d., then $\hat{\boldsymbol{\eta}}$, the LSE of $\boldsymbol{\eta}^0$, is a strongly consistent estimator of $\boldsymbol{\eta}^0$.*

For the proof of Theorem 2.1, see in Appendix A.

In rest of this section, we develop the joint asymptotic distribution of the LSEs of the unknown parameters for single component model (2). We use the usual Taylor series expansion. We denote the first derivative vector of $Q(\boldsymbol{\eta})$ as $Q'(\boldsymbol{\eta})$ which is of order 1×6 and the 6×6 matrix of second order derivatives as $Q''(\boldsymbol{\eta})$. Now expanding $Q'(\hat{\boldsymbol{\eta}})$ around $\boldsymbol{\eta}^0$ by multivariate Taylor series up to first order terms, we have

$$Q'(\hat{\boldsymbol{\eta}}) - Q'(\boldsymbol{\eta}^0) = (\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)Q''(\bar{\boldsymbol{\eta}}), \tag{4}$$

where $\bar{\boldsymbol{\eta}}$ is a point between $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}^0$. Now define a diagonal matrix of order six as follows:

$$\mathbf{D} = \text{diag}\{N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}\}. \tag{5}$$

Since $Q'(\hat{\boldsymbol{\eta}}) = 0$, (4) can be written as

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1} = -\left[\mathbf{Q}'(\boldsymbol{\eta}^0)\mathbf{D}\right]\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right]^{-1}. \tag{6}$$

We can write (6) because $\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]$ is an invertible matrix *a.e.* for large N. From Theorem 2.1, it follows that $\hat{\boldsymbol{\eta}}$ converges *a.s.* to $\boldsymbol{\eta}^0$ and since each element of $Q''(\boldsymbol{\eta})$ is a continuous function of $\boldsymbol{\theta}$, therefore,

$$\lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right] = \lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\boldsymbol{\theta}^0)\mathbf{D}\right] = 2\boldsymbol{\Sigma}(\boldsymbol{\eta}^0) \quad \text{(say)}. \tag{7}$$

In obtaining the exact form of the limit matrix $\boldsymbol{\Sigma}(\boldsymbol{\eta})$, let us write $\boldsymbol{\eta} = (A, \boldsymbol{\xi})$, where $\boldsymbol{\xi} = (b, \alpha, c, \theta, \phi)$. Then $\boldsymbol{\Sigma}(\boldsymbol{\eta}) = e^{2b^0}\boldsymbol{\Delta}(\boldsymbol{\eta}^0)$, where

$$\boldsymbol{\Delta}(\boldsymbol{\eta}) = \begin{bmatrix} \delta_1(0) & A\delta_5(0) & Ab\delta_6(1) & Ab\delta_6(0) & A\delta_7(1) & A\delta_7(0) \\ A\delta_5(0) & A^2\delta_2(0) & A^2b\delta_8(1) & A^2b\delta_8(0) & -A^2\delta_9(1) & -A^2\delta_9(0) \\ Ab\delta_6(1) & A^2b\delta_8(1) & A^2b^2\delta_3(2) & A^2b^2\delta_3(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) \\ Ab\delta_6(0) & A^2b\delta_8(0) & A^2b^2\delta_3(1) & A^2b^2\delta_3(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) \\ Ab\delta_7(1) & -A^2\delta_9(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) & A^2\delta_4(2) & A^2\delta_4(1) \\ A\delta_7(0) & -A^2\delta_9(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) & A^2\delta_4(1) & A^2\delta_4(0) \end{bmatrix}, \tag{8}$$

where $\delta_k(m) = \delta_k(\boldsymbol{\eta}, m)$, $m = 0, 1, 2$, $k = 1, \ldots, 10$ are defined in Appendix B. Now let us consider the random vector $Q'(\boldsymbol{\eta}^0)\mathbf{D}$ of order 1×6,

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} = \begin{bmatrix} -\frac{2}{\sqrt{N}} \sum_{t=1}^{N} e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\} \cos(\theta^0 t + \phi^0) \\ -\frac{2}{\sqrt{N}} A^0 \sum_{t=1}^{N} e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\}(1-\cos(\alpha^0 t + c^0)) \cos(\theta^0 t + \phi^0) \\ -\frac{2}{N^{\frac{3}{2}}} A^0 b^0 \sum_{t=1}^{N} t e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\} \sin(\alpha^0 t + c^0) \cos(\theta^0 t + \phi^0) \\ -\frac{2}{\sqrt{N}} A^0 b^0 \sum_{t=1}^{N} e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\} \sin(\alpha^0 t + c^0) \cos(\theta^0 t + \phi^0) \\ \frac{2}{N^{\frac{3}{2}}} A^0 \sum_{t=1}^{N} t e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\} \sin(\theta^0 t + \phi^0) \\ \frac{2}{\sqrt{N}} A^0 \sum_{t=1}^{N} e(t) \exp\{b^0(1-\cos(\alpha^0 t + c^0))\} \sin(\theta^0 t + \phi^0) \end{bmatrix}.$$

All the elements of $Q'(\boldsymbol{\eta}^0)\mathbf{D}$ satisfy the Lindeberg-Feller's condition, therefore it converges to a 6-variate normal distribution. Using the limits given in Appendix B, it follows that

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} \rightarrow \mathcal{N}_6\Big(\mathbf{0}, 4\sigma^2\boldsymbol{\Sigma}(\boldsymbol{\eta}^0)\Big). \tag{9}$$

Therefore, using (7) and (9) in (6), we have the asymptotic distribution as

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1} \rightarrow \mathcal{N}_{\mathbf{6}}\Big(\mathbf{0}, \boldsymbol{\sigma^2\Sigma^{-1}(\eta^0)}\Big). \tag{10}$$

Now using the inequality (25), we have shown in Appendix B, that $\delta_k(\boldsymbol{\xi}, p) = 0$ for $k = 6, \ldots, 10$ and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$, $p = 0, 1, 2, \ldots$. Thus

$$\boldsymbol{\Delta}(\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{\Delta}_1(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3(\boldsymbol{\eta}) \end{bmatrix},$$

where

$$\boldsymbol{\Delta}_1(\boldsymbol{\eta}) = \begin{bmatrix} \delta_1(0) & A\delta_1(0) \\ A\delta_1(0) & A^2\delta_2(0) \end{bmatrix}, \quad \boldsymbol{\Delta}_2(\boldsymbol{\eta}) = A^2 b^2 \begin{bmatrix} \delta_3(2) & \delta_3(1) \\ \delta_3(1) & \delta_3(0) \end{bmatrix}, \quad \boldsymbol{\Delta}_3(\boldsymbol{\eta}) = A^2 \begin{bmatrix} \delta_4(2) & \delta_4(1) \\ \delta_4(1) & \delta_4(0) \end{bmatrix}. \tag{11}$$

Thus, the asymptotic variance-covariance matrix of $(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1}$ is

$$\sigma^2\Sigma^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0}\boldsymbol{\Delta}^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0} \begin{bmatrix} \boldsymbol{\Delta}_1^{-1}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2^{-1}(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3^{-1}(\boldsymbol{\eta}) \end{bmatrix} \tag{12}$$

with

$$\mathbf{\Delta}_1^{-1}(\boldsymbol{\eta}) = \frac{1}{\delta_2(0) - \delta_1(0)} \begin{bmatrix} \frac{\delta_2(0)}{\delta_1(0)} & -\frac{1}{A} \\ -\frac{1}{A} & \frac{1}{A^2} \end{bmatrix}, \quad \mathbf{\Delta}_2^{-1}(\boldsymbol{\eta}) = \frac{1}{A^2 b^2 [\delta_3(2)\delta_3(0) - \delta_3(1)^2]} \begin{bmatrix} \delta_3(0) & -\delta_3(1) \\ -\delta_3(1) & \delta_3(2) \end{bmatrix},$$

and

$$\mathbf{\Delta}_3^{-1}(\boldsymbol{\eta}) = \frac{1}{A^2[\delta_4(2)\delta_4(0) - \delta_4(1)^2]} \begin{bmatrix} \delta_4(0) & -\delta_4(1) \\ -\delta_4(1) & \delta_4(2) \end{bmatrix}.$$

So (12) implies that the pairs of parameters (A, b), (α, c) and (θ, ϕ) are asymptotically independent to each other whereas the parameters in each pair are asymptotically dependent. In comparison to the constant amplitude frequency model, in present case, the amplitude is a deterministic function of the time variable t, which is equal to $A \exp\{b(1 - \cos(\alpha t + c))\}$. Thus, it depends on parameters A, b, α and c. For constant amplitude case least square estimator of amplitude is independent of that of frequency and phase. Along the same line, we observe that for model (2) that the estimators of amplitude parameters $\hat{A}, \hat{b}, \hat{\alpha}$ and $\hat{c}$ are independent of frequency and phase estimators, $\hat{\theta}$ and $\hat{\phi}$.

Remark 1. The rate of convergence of each of A, b, c and ϕ is $O_p(N^{-1/2})$ whereas for the carrier angular frequency θ as well as for the modulating frequency α, the rate of convergence is $O_p(N^{-3/2})$. In comparison to constant amplitude sinusoidal model, α is a parameter of the amplitude. In this case it is observed that its LSE of the modulating frequency α has the same convergence rate as that of the carrier angular frequencies θ_i, $i = 1, \ldots, q$ as α, multiplied by t is appearing as the argument of a cosine function in the time varying amplitude of the model.

3. Theoretical Properties of LSEs for general q

In this section, we provide the asymptotic results of the LSEs for model (1). Let us write $\boldsymbol{\psi}_k = (A_k, b_k, c_k, \theta_k, \phi_k)$, $k = 1, \ldots, q$ and $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q, \alpha)$ be the parameter vector. Then the LSE of $\boldsymbol{\psi}$ is obtained by minimizing the residual sum of squares which can be defined similarly as (3). Let $\hat{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}^0$ denote the least squares estimator and the true value of $\boldsymbol{\psi}$. The consistency of $\hat{\boldsymbol{\psi}}$ follows similarly as the consistency of $\hat{\boldsymbol{\eta}}$, considering the parameter vector as $\boldsymbol{\psi}$ instead of $\boldsymbol{\eta}$. We state the asymptotic distribution of $\hat{\boldsymbol{\psi}}$ here. The proof involves routine calculations and the use of multiple Taylor series and a central limit theorem similarly as in section 2.

For asymptotic distribution of $\hat{\boldsymbol{\psi}}$, following the notation used in previous section, we write $\boldsymbol{\xi}_j = (b_j, \alpha, c_j, \theta_j, \phi_j)$, $j = 1, \ldots, q$; $\delta_k(\boldsymbol{\xi}_j, p) = \delta_k^j(p)$, $k = 1(1)4$, $j = 1(1)q$, $p = 0, 1, 2, \ldots$.

Now let $\mathbf{D}_q$ be a diagonal matrix of order $(5q+1)$, defined as follows;

$$\mathbf{D}_q = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & N^{-\frac{3}{2}} \end{bmatrix},$$

where $\mathbf{D}_1 = \text{diag}\{N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}\}$. Then the asymptotic distribution of $\hat{\boldsymbol{\psi}}$ is

$$(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}^0)\mathbf{D}_q^{-1} \xrightarrow{d} \mathcal{N}_{5q+1}\Big(\mathbf{0}, \sigma^2 \mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)\Big), \tag{13}$$

$$\mathbf{G}_q(\boldsymbol{\psi}) = \begin{bmatrix} e^{2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_1) \\ \mathbf{0} & e^{2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2) & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & e^{2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q) & \mathbf{w}(\boldsymbol{\psi}_q) \\ \mathbf{w}'(\boldsymbol{\psi}_1) & \mathbf{w}'(\boldsymbol{\psi}_2) & \cdots & \mathbf{w}'(\boldsymbol{\psi}_q) & f^* \end{bmatrix}.$$

Here $f^* = \sum_{j=1}^{q} e^{2b_j} A_j^2 b_j^2 \delta_3^j(2)$ and $\mathbf{w}'(\boldsymbol{\psi}_j) = \begin{pmatrix} 0 & 0 & e^{2b_j} A_j^2 b_j^2 \delta_3^j(1) & 0 & 0 \end{pmatrix}$. The sub-matrix $\boldsymbol{\Gamma}(\boldsymbol{\psi}_j)$ is obtained by deleting third row and third column of matrix $\boldsymbol{\Delta}(\boldsymbol{\eta})$ and replacing (A, b, c, θ, ϕ) by $(A_j, b_j, c_j, \theta_j, \phi_j)$. Thus,

$$\boldsymbol{\Gamma}(\boldsymbol{\psi}_j) = \begin{bmatrix} \delta_1^j(0) & A_j\delta_1^j(0) & 0 & 0 & 0 \\ A_j\delta_1^j(0) & A_j^2\delta_2^j(0) & 0 & 0 & 0 \\ 0 & 0 & A_j^2 b_j^2 \delta_3^j(0) & 0 & 0 \\ 0 & 0 & 0 & A_j^2\delta_4^j(2) & A_j^2\delta_4^j(1) \\ 0 & 0 & 0 & A_j^2\delta_4^j(1) & A_j^2\delta_4^j(0) \end{bmatrix}.$$

The inverse matrix $\mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)$ is given by

$$\mathbf{G}_q^{-1}(\boldsymbol{\psi}) =$$

$$\begin{pmatrix} e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)^{-1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \cdots & \mathbf{F}_{1q} & e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) \\ \mathbf{F}_{21} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)^{-1} + \mathbf{F}_{22} & \cdots & \mathbf{F}_{2q} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{F}_{q1} & \mathbf{F}_{q2} & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)^{-1} + \mathbf{F}_{qq} & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) \\ e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) & 1/d^* \end{pmatrix}$$

where $d^* = \sum_{j=1}^{q} e^{2b_j} A_j^2 b_j^2 \left[\delta_3^j(2) - \frac{\delta_3^j(1)^2}{\delta_3^j(0)}\right]$ and $\mathbf{F}_{jk}$, $j, k = 1, \ldots, q$ is a 5×5 symmetric matrix whose all elements are zero except (3,3) element which is equal to $\frac{1}{d^*}\frac{\delta_3^j(1)\delta_3^k(1)}{\delta_3^j(0)\delta_k^k(0)}$.

In the previous section, we have observed that for $q = 1$, the estimator of α only asymptotically depends on the estimator of c_1, but for $q > 1$, the estimator of α depends on all c_j, $j = 1, \ldots, q$ for large N. This is expected also as in the model, the modulating angular frequency is same for each component.

4. Numerical Experiments

In this section, for illustration, we present some numerical results based on simulations. We consider the model (1) with $q = 4$. Data are generated using the following values;

$$\begin{aligned}
A_1 &= 5.70166706 \times 10^{-5}, & A_2 &= 3.3049426 \times 10^{-25}, \\
A_3 &= 1.002 \times 10^{-3}, & A_4 &= 3.7575 \times 10^{-4} \\
b_1 = 4.989495798,\ b_2 &= 28.886554622, & b_3 &= 2.62605042,\ b_4 = 2.62605042,
\end{aligned}$$

$$\begin{aligned}
c_1 = .1904,\ c_2 &= 2.05632, & c_3 &= 5.9024,\ c_4 = 3.2368, \\
\theta_1 = .07616,\ \theta_2 &= .26656, & \theta_3 &= .03808,\ \theta_4 = .03808, \\
\phi_1 = 1.166198163,\ \phi_2 &= 18.007071552 & \phi_3 &= 10.928948246\ \phi_4 = 6.378392654 \\
\alpha &= .03808.
\end{aligned} \tag{14}$$

These parameter values have been obtained from Muthopadhyay and Sircar (1996). We consider the case when carrier frequencies are harmonics of the modulation angular frequency i.e. θ_i's are integer multiples of α. As ECG signal is periodic, θ_i has to be some integer multiple of α. We use the sample size $N = 100$ for simulations study. The error random variables are independent and identically distributed $\mathcal{N}(0, \sigma^2)$. We have reported results for $\sigma^2 = .00001$ and .0001. We generate the data using (1) and the true parameter values as mentioned above. The LSEs of different parameters are estimated by minimizing the residual sum of squares given in (3). The minimization has been carried out by using the downhill simplex method and for that purposes, routines given in Press et al. (1987) have been used. The true parameter values are taken as the initial estimates. Though with $q = 4$, the parameter set contains 21 parameters and the optimization is taken place in quite a higher dimensional space, the final results are quite satisfactory. We replicate the procedure of data generation and estimation of parameters 1000 times, then calculate the average estimate (AVEST) and the mean squared error (MSE) of each parameter. We summarize results in Figures 1 and 2. In Figure 1, we present the true values (using red points), average estimates (using green and blue points when $\sigma^2 = .00001$ and .0001 respectively). There are 25 subplots in Figure 1 and j^{th} row corresponds to the parameters of j^{th} component, $j = 1, \ldots, 4$. In section 3, we have developed the asymptotic distribution of LSEs when $q > 1$, which may be used for interval estimation for finite samples. As the limiting distribution in section 3 involves several limiting quantities, for samples of moderate size, we wish to see how the percentile bootstrap (boot-p) method - a well-used form of parametric bootstrap, works. In Nandi, Iyer and Kundu (2002) and Kundu and Nandi (2006), similar bootstrapping method has been used for interval estimation in case of sinusoidal frequency model and real chirp signal model respectively. In each replication of our experiment, we generate 1000 bootstrap resamples using the estimated parameters and then the bootstrap confidence intervals using the bootstrap quantiles at 95% nominal level. Thus, from the replicated experiment, we have 1000 intervals for each parameter. Then we estimate the 95% boot-p coverage probability by calculating the proportion covering the true value of the parameter. As they are very close to the nominal value except c_1 and A_3, we do not report them in the paper. In addition, we also obtain the average lengths of the boot-p confidence intervals and they are reported in Figure 2 along with the root mean squared errors (RMSE). In each sub-figure, for $\sigma^2 = .00001$ and .0001, we present the RMSEs (green points and connected by green line) and average length of the boot-p confidence intervals (red line and connected by red line). The results for the modulation frequency α is provided in Figure 3. We observe that the average estimates are quite good, which is reflected in Figure 1. The biases are quite small as the estimates (green and blue) are quite close to the true value (red). As the error variance increases, the biases increase as the vertical distance of the green point from red point increases in case of blue point. The RMSEs and average lengths of the boot-p confidence intervals are reasonably small. Their dependence on the magnitude of the constant amplitude A^0 is quite clear. The asymptotic variances of all parameters, except A, are reciprocally proportional to ${A^0}^2$ which is also visible in RMSEs to some extent. As the error variance increases the RMSEs and average lengths increase which is reflected from the fact that all lines in Figure 2 have positive slopes. The order of the asymptotic variance is reflected in the RMSE and the length of the interval in each case.

Apart from the replicated experiment, we consider the same model in data analysis format. For that, we generate the data of sample size $N = 500$ and the error variance $\sigma^2 = .05$. The generated data are plotted when no noise is present, in Fig. 5. The corrupted version of the same data set with $\sigma^2 = .05$ are presented in Fig. 6. Now we estimate the parameters by minimizing the residual sum of square and plugging them in the model, we estimate the signal. It is plotted in Fig. 7. Now if the level of noise increases to $\sigma^2 = 1.0$, then the form of the original signal (Fig. 5) is totally distorted. We wanted to see, whether in this case it is possible to extract the signal. The signal with completely destroyed form is plotted in Fig.

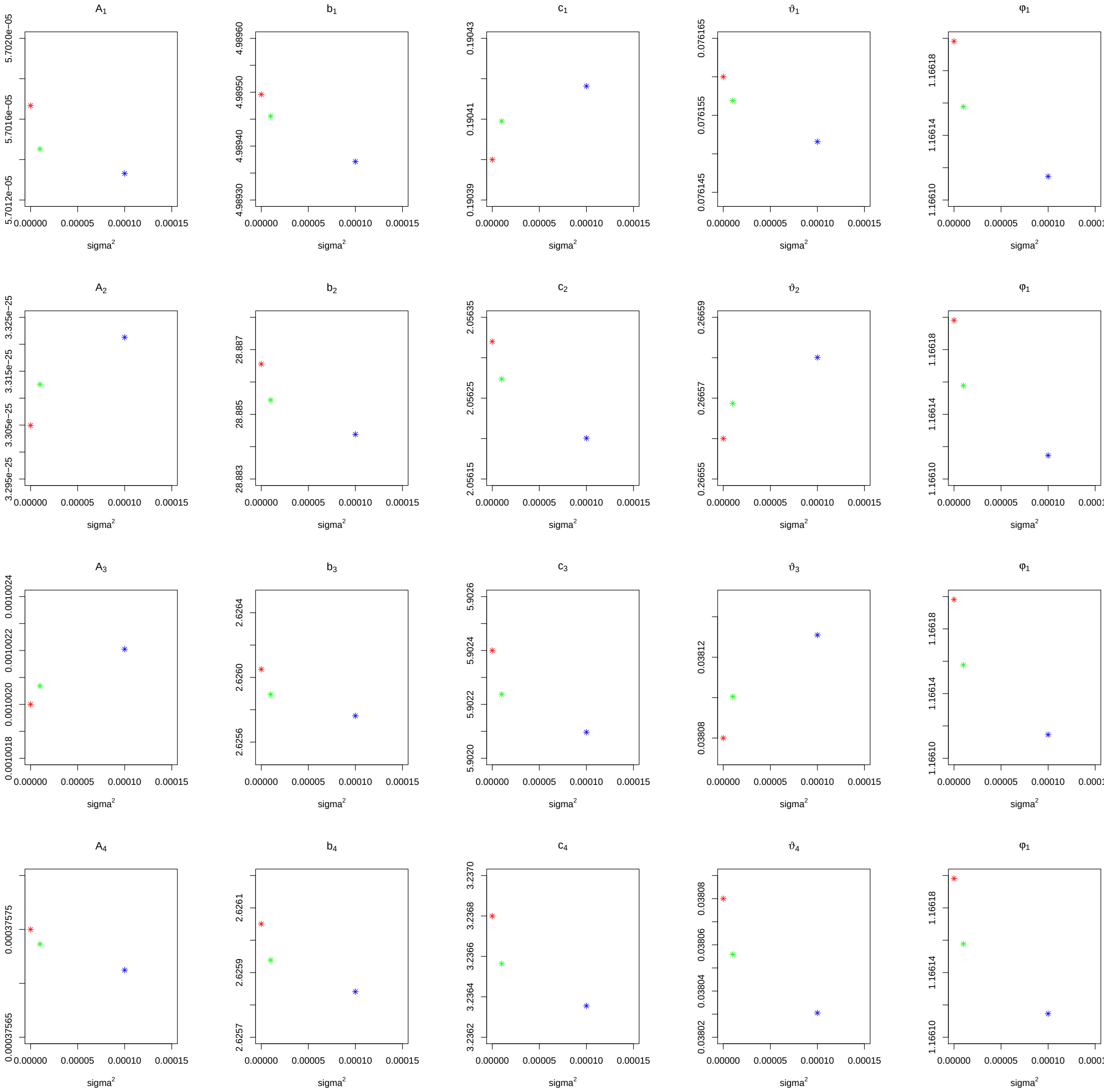

FIGURE 1. True values (red) and average estimates when $\sigma^2 = .00001$ (green) and $\sigma^2 = .0001$ (blue) in i^{th} row corresponding to i^{th} component, e.g in first row, A_1, b_1, c_1, θ_1 and ϕ_1 and so on. The results of α is given in Fig 3.

8. Now we estimate the LSEs and the estimated signal is plotted in Fig. 9. In both cases, the LSEs are able to estimate quite satisfactorily and the estimated graphs (Figs. 7 and 9) match well with the no-noise signal. In simulated experiments, we have used percentile bootstrap method for interval estimation. In single data example, we used the same method. Using 1000 bootstrap resamples, we estimate the 95% confidence intervals of each parameter in case of $\sigma^2 = .05$ and $\sigma^2 = 1.0$. They (blue) are reported in Figure 4 along with their point

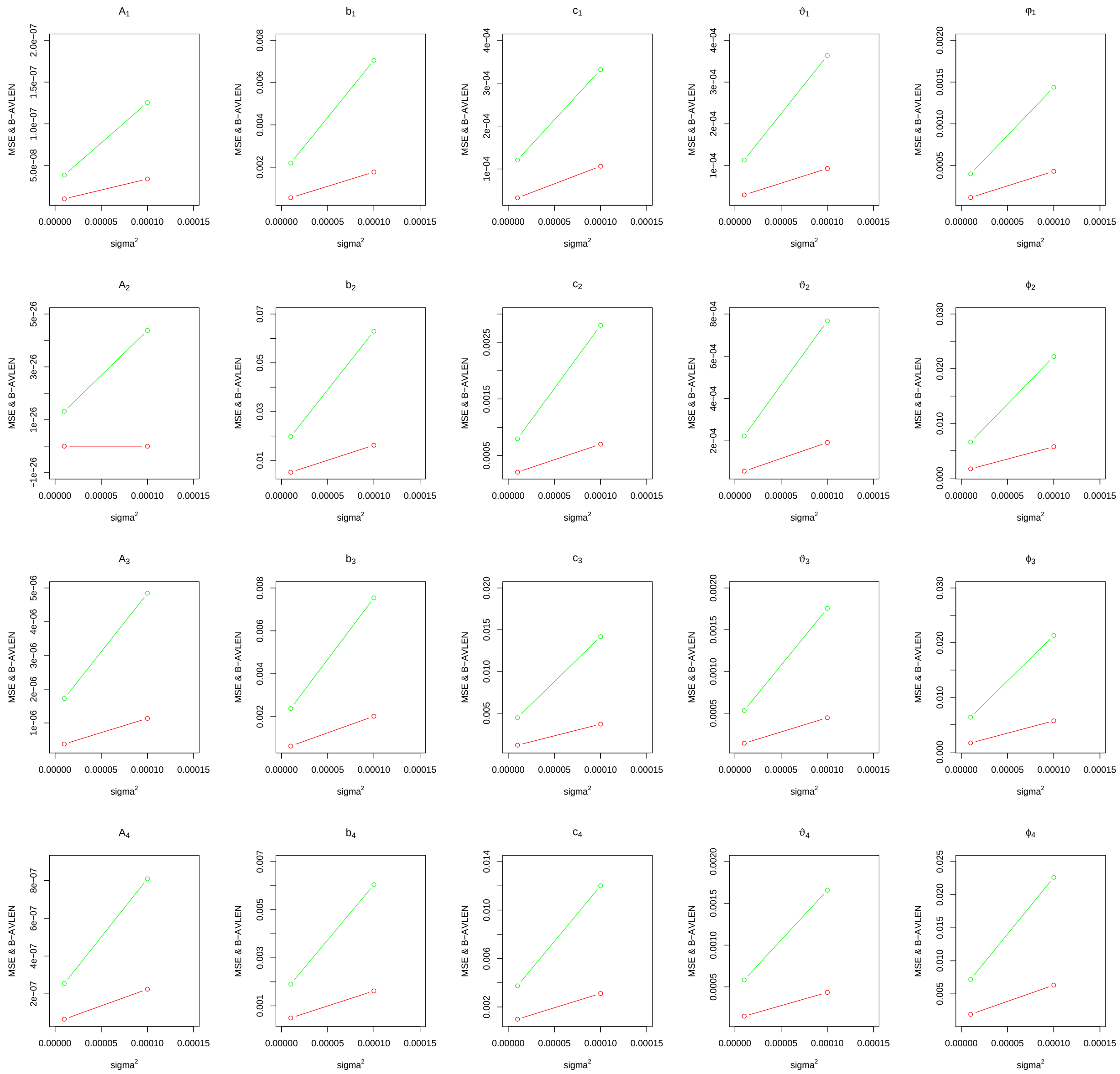

FIGURE 2. Root mean squared errors of LSEs of parameters when $\sigma^2 = .00001$ and $\sigma^2 = .0001$ (red, joined with a line) and average lengths of boot-p confidence intervals (green line). The results corresponding to first component in (a)-(e), second in (f)-(j), third in (k)-(o) and fourth in (p)-(u) are provided.

estimates (green) and true values (red). We see that bias is negligible in most of the cases when $\sigma^2 = .05$. Though, it is not true, in each parameter estimate with larger noise level, the estimated signal is quite good. The boot-p confidence intervals in each case include the true parameter value except c_4 and the order of asymptotic variance is reflected in the length of the interval. We would like to mention that for larger noise level, biases are large in case

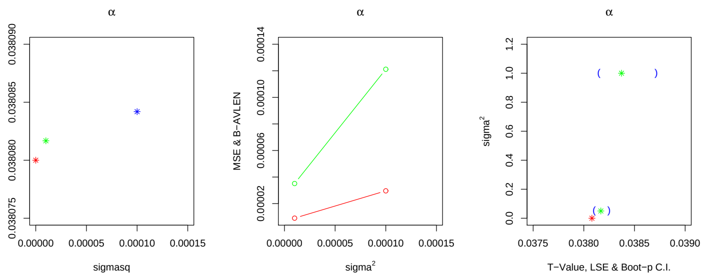

FIGURE 3. True value and average estimates of α same as 1 (left), root mean squared errors and average length of boot-p confidence interval of α same as 2 (middle) and True values, LSEs and 95% confidence intervals same as 4.

of some parameters, but the fit is quite good. The reason may lie on the presence of overall large absolute values of the amplitude function and the final effect is not visible.

5. Conclusion

In the present article, we study a comparatively new model. The model is proposed by Sharma and Sircar (2001) to analyze ECG data. We study the properties of the least square estimators to estimate the unknown parameters. The model is a particular functional amplitude sinusoidal model and the form is quite complicated specially when the number components is large. However, we observe that the LSEs satisfy the large sample properties, strong consistency and asymptotic normality, under the assumptions that errors are i.i.d. In numerical experiments, we find that one can use the percentile bootstrap method for interval estimation. We have used the true values as the starting estimates in simulations, but given a data set, it is extremely important to guess the initial estimates. We feel that the use of different combination of periodogram function may be used for this purpose. Further work is required in this direction and these computational aspects will be addressed elsewhere. Another point we would like to mention that we have assumed that the number of components is known. So the estimation of the number of burst components is of utmost important. LSEs for burst-type signals have the desirable theoretical properties. At the same time, obtaining LSEs is difficult; so the estimation of the number of signals as well as developing an efficient algorithm need to be addressed.

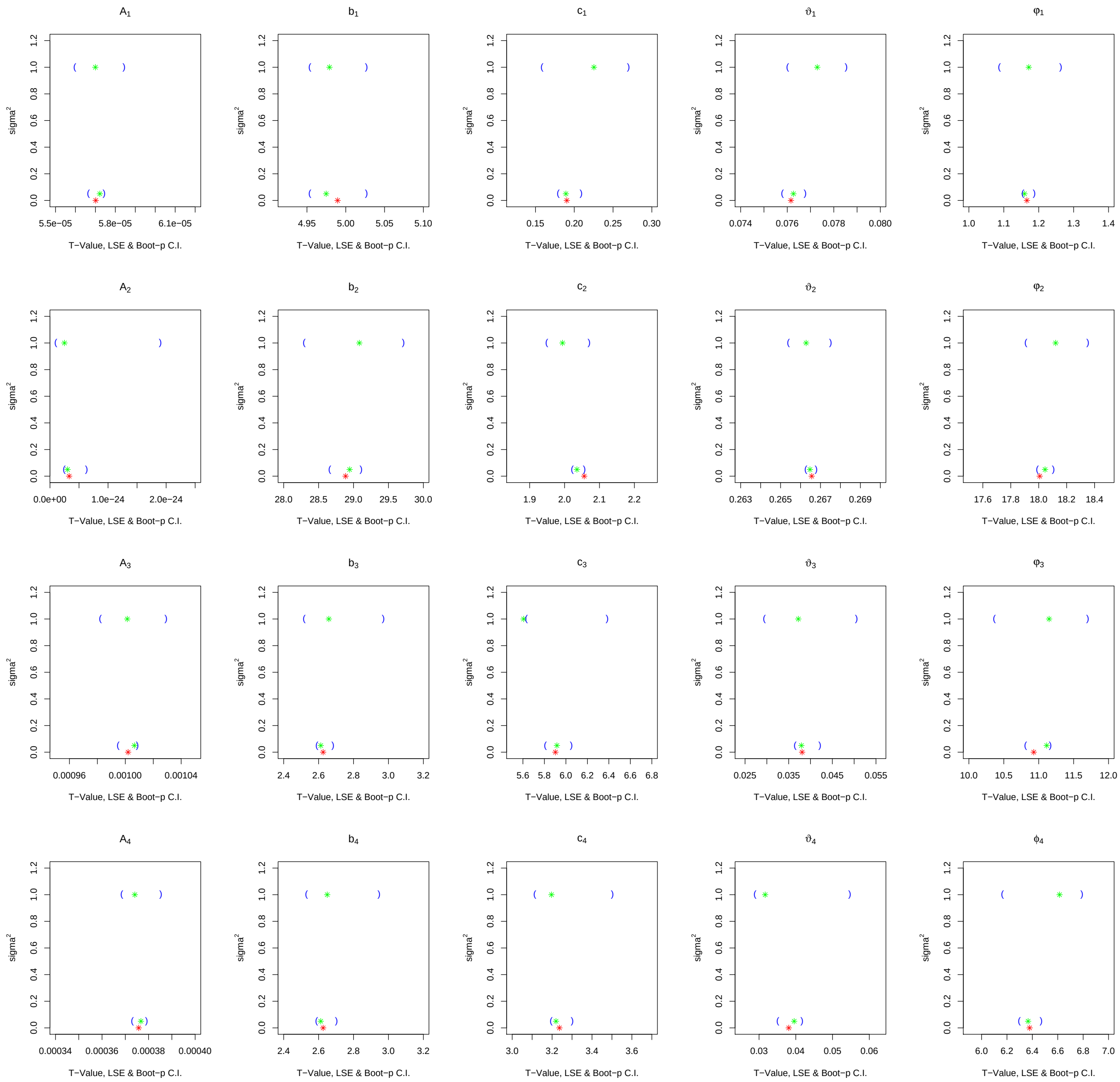

FIGURE 4. True values (red), LSEs at $\sigma^2 = .05$ and 1.0 (green) and 95% percentile bootstrap confidence intervals (blue) when the sample size $N = 500$.

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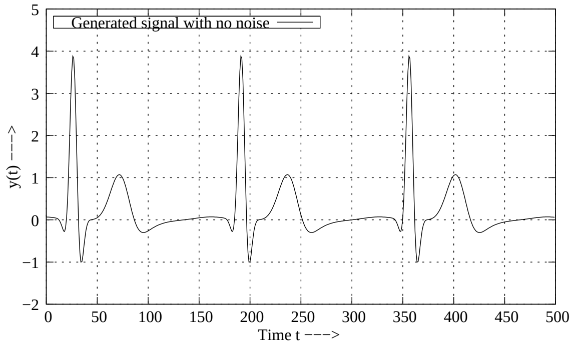

FIGURE 5. The signal using model (14) subjected to no noise.

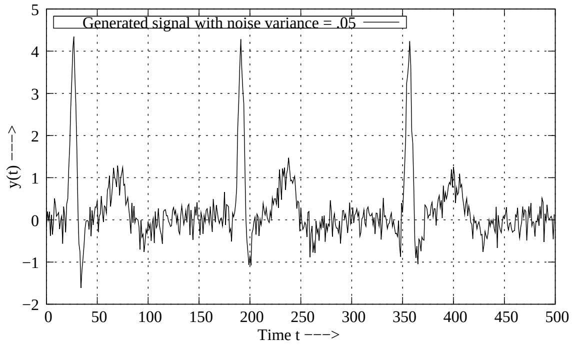

FIGURE 6. The same signal as in Fig. 5 corrupted by noise (variance=.05).

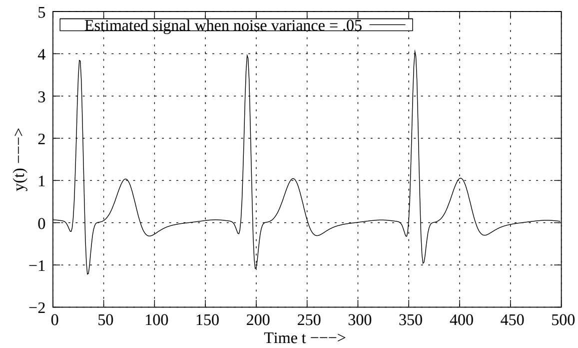

FIGURE 7. Estimated signal from the signal given in Fig 6.

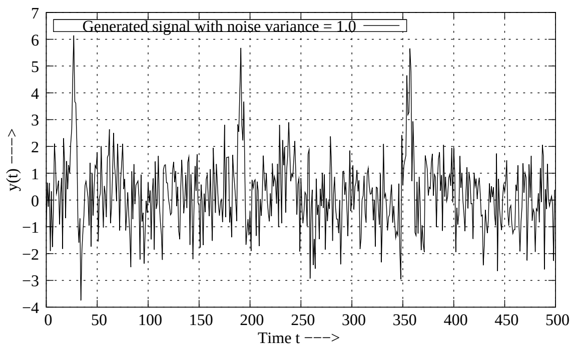

FIGURE 8. The same signal as in Fig. 5 corrupted by noise (Error variance =1.0).

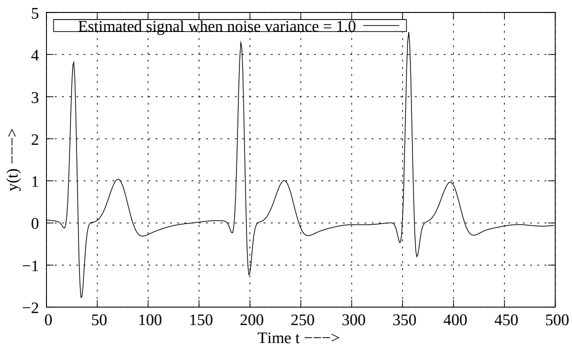

FIGURE 9. Estimated signal from the signal given in Fig 8.

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Appendix A

In Appendix A, we first provide the proofs of the results for model (1) (with $q = 1$), stated in section 2. The technique follows in the following proof is the same as Wu (1981). Lemmas 1 and 2 are required to prove theorem 2.1. Lemma 2 gives a sufficient condition for strong consistency of the LSEs and lemma 1 is required to verify the condition given in lemma 2 under the condition that the error random variables $e(t)$s are i.i.d. The methology adopted in the following, might be applicable in case of undamped periodic signal models.

Lemma 1. *Let $X(1), X(2), \ldots$ be i.i.d. random variables with mean zero and finite second moment and b is a real number such that $e^{|b|} \leq K$. Define $\Pi = (0, \pi) \times (0, \pi) \in \mathcal{R}^2$. Then as $n \to \infty$,*

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} X(t) \exp\{b\cos(\alpha t)\} \cos(\theta t) \stackrel{a.s.}{\to} 0, \quad as \quad N \to \infty \tag{15}$$

Proof of Lemma 1: Define

$$Z(t) = \begin{cases} X(t) & \text{if} \quad |X(t)| \leq t^{\frac{1}{2}} \\ 0 & \text{if} \quad |X(t)| > t^{\frac{1}{2}} \end{cases}$$

Then

$$\begin{aligned}
\sum_{t=1}^{\infty} P[Z(t) \neq X(t)] &= \sum_{t=1}^{\infty} P[|X(t)| > t^{\frac{1}{2}}] \\
&= \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq n \leq 2^t} P[|X(1)| > n^{\frac{1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t P[|X(1)| > 2^{\frac{t-1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t \sum_{j=t}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \\
&\leq \sum_{j=1}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \sum_{t=1}^{j} 2^t \\
&\leq c \sum_{j=1}^{\infty} 2^{j-1} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \ \leq \ cE|X(1)|^2 < \infty.
\end{aligned}$$

So, $P[Z(t) \neq X(t) \text{ i.o.}] = 0$ and $Z(t)$ and $X(t)$ are equivalent random variables. Thus

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} X(t) \exp\{b\cos(\alpha t)\}\cos(\theta t) \xrightarrow{a.s.} 0 \Leftrightarrow \sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} Z(t) \exp\{b\cos(\alpha t)\}\cos(\theta t) \xrightarrow{a.s.} 0. \tag{16}$$

Let $U_t = Z(t) - E(Z(t))$. Then

$$\sup_{(\alpha,\theta)\in\Pi} \left| \frac{1}{N} \sum_{t=1}^{N} Z(t) \exp\{b\cos(\alpha t)\}\cos(\theta t) \right| \leq e^{|b|} \frac{1}{N} \sum_{t=1}^{N} |Z(t)| \rightarrow 0.$$

Thus, it is enough to show that

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} U_t \exp\{b\cos(\alpha t)\}\cos(\theta t) \xrightarrow{a.s.} 0. \tag{17}$$

For any fixed $\epsilon > 0$, assume that $0 \leq h \leq \frac{1}{2N^{1/2}K}$. Then $|hU_t \cos(\theta t)e^{b\cos(\alpha t)}| \leq \frac{1}{2}$. Now, using $e^{|x|} \leq 2e^x$ and $e^x \leq 1 + x + 2x^2$ for $|x| \leq \frac{1}{2}$, we have

$$\begin{aligned} P\left[\left|\frac{1}{N}\sum_{t=1}^{N} U_t \cos(\theta t)e^b \cos(\alpha t)\right| \geq \epsilon\right] &\leq e^{-hN\epsilon}\prod_{t=1}^{N} E\left(\exp\{|hU_t\cos(\theta t)e^{b\cos(\alpha t)}|\}\right) \\ &\leq 2e^{-hN\epsilon}\prod_{t=1}^{N} E\left(\exp\{hU_t\cos(\theta t)e^{b\cos(\alpha t)}\}\right) \\ &\leq 2e^{-hN\epsilon}\prod_{t=1}^{N}(1+2h^2\sigma^2K^2) \\ &\leq 2e^{-hN\epsilon+2Nh^2\sigma^2K^2}. \end{aligned}$$

Take $h = \frac{1}{2N^{1/2}K}$ in the above inequality.

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N} U_t \cos(\theta t)e^b \cos(\alpha t)\right| \geq \epsilon\right] \leq 2e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon+\frac{1}{2}\sigma^2} \leq ce^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}.$$

Let $L = N^2$. Choose $(\alpha_1, \theta_1), \ldots, (\alpha_L, \theta_L)$ such that for each $(\alpha, \theta) \in \Pi$, we have a (α_j, θ_j) satisfying $|\alpha_j - \alpha| \leq \frac{\pi}{N^2}$ and $|\theta_j - \theta| \leq \frac{\pi}{N^2}$. Now let us consider

$$\begin{aligned} & \left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \\ = & \left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha t)} + \cos(\theta_j t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \\ \leq & |e^{b\cos(\alpha t)}|\,|\cos(\theta t) - \cos(\theta_j t)| + |\cos(\theta_j t)|\left|e^{b\cos(\alpha t)} - e^{b\cos(\alpha_j t)}\right| \\ \leq & Kt|\theta_j - \theta| + Kt|b||\alpha_j - \alpha|. \end{aligned}$$

So,

$$\begin{aligned} \left|\frac{1}{N}\sum_{t=1}^{N} U_t\left(\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right)\right| &\leq \frac{2}{N}\sum_{t=1}^{N} t^{\frac{1}{2}}Kt(|\theta_j - \theta| + |b||\alpha_j - \alpha|) \\ &\leq \frac{2}{N}\sum_{t=1}^{N} t^{\frac{1}{2}}Kt\frac{\pi}{N^2}(1+|b|) \\ &\leq 2K(1+|b|)\frac{\pi}{\sqrt{N}} \;\rightarrow\; 0, \quad \text{as} \quad N \rightarrow \infty. \end{aligned}$$

Therefore, for large N,

$$\begin{aligned} P\left[\sup_{\alpha,\theta}\left|\frac{1}{N}\sum_{t=1}^{N} U_t\cos(\theta t)e^{b\cos(\alpha t)}\right| \geq 2\epsilon\right] &\leq P\left[\max_{j\leq N^2}\left|\frac{1}{N}\sum_{t=1}^{N} U_t\cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \geq \epsilon\right] \\ &\leq cN^2e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} n^2 e^{-\frac{1}{2}n^{\frac{1}{2}}K^{-1}\epsilon} < \infty$, we have the following

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} X(t)\exp\{b\cos(\alpha t)\}\cos(\theta t) \overset{a.s.}{\to} 0, \tag{18}$$

as $N \to \infty$, using Borel Canteli Lemma. ∎

Lemma 2. *Let us denote the set*

$$S_{\epsilon,M} = \left\{\boldsymbol{\eta} : |\boldsymbol{\eta} - \boldsymbol{\eta}^0| > 6\epsilon, |A| \le M\right\}.$$

If for any $\epsilon > 0$ and for some $M < \infty$,

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\eta}\in S_{\epsilon,M}} \frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] > 0 \quad a.s.$$

then $\hat{\boldsymbol{\eta}}$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

Proof of Lemma 2: It is simple and can be proved by contradiction along the same lines as Wu (1981), so it is not provided here. ∎

Proof of Theorem 1:

In this proof, we denote $\hat{\boldsymbol{\eta}}$ by $\hat{\boldsymbol{\eta}}_N = (A_N, b_N, \alpha_N, c_N, \theta_N, \phi_N)$, to emphasize that $\hat{\boldsymbol{\eta}}$ depends on N. Let us assume that $\hat{\boldsymbol{\eta}}_N$ is not a consistent estimator for $\boldsymbol{\eta}^0$. Then either:
Case I: For all sub sequences $\{N_k\}$ of $\{N\}$, $|\hat{A}_{N_k}| \to \infty$. This implies

$$\frac{1}{N_k}\left[Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0)\right] \to \infty.$$

But as $\hat{\boldsymbol{\eta}}_{N_k}$ is the LSE of $\boldsymbol{\eta}^0$ with sample size N_k, we have

$$Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0) < 0,$$

which leads to a contradiction. So $\hat{\boldsymbol{\eta}}_N$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

Case II: For at least one sub sequence $\{N_k\}$ of $\{N\}$, $\hat{\boldsymbol{\eta}}_{N_k} \in S_{\epsilon,M}$, for some $\epsilon > 0$ and for an $0 < M < \infty$. Now we write

$$\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] = f(\boldsymbol{\eta}) + g(\boldsymbol{\eta}),$$

where

$$\begin{aligned} f(\boldsymbol{\eta}) &= \frac{1}{N}\sum_{t=1}^{N}\Big[A^0\exp\{b^0(1-\cos(\alpha^0 t+c^0))\}\cos(\theta^0 t+\phi^0) \\ &\quad -A\exp\{b(1-\cos(\alpha t+c))\}\cos(\theta t+\phi)\Big]^2 \\ g(\boldsymbol{\eta}) &= \frac{2}{N}\sum_{t=1}^{N}e(t)\Big[A^0\exp\{b^0(1-\cos(\alpha^0 t+c^0))\}\cos(\theta^0 t+\phi^0) \\ &\quad -A\exp\{b(1-\cos(\alpha t+c))\}\cos(\theta t+\phi)\Big]. \end{aligned}$$

Using Lemma 1, we have

$$\lim_{N\to\infty}\sup_{\eta\in S_{\epsilon,M}} g(\boldsymbol{\eta}) = 0, \qquad a.s. \tag{19}$$

Define sets S_{ϵ}^i, $i = 1,\ldots,6$ as follows:

$$S_{\epsilon,M}^i = \left\{\boldsymbol{\eta} : |\eta_i - \eta_i^0| > \epsilon, |A| \le M\right\}, \tag{20}$$

where η_i, $i = 1,\ldots,6$ stands for the elements of $\boldsymbol{\eta}$, that is, A, b, α, c, θ and ϕ. Note that $S_{\epsilon,M} \subset \cup_{i=1}^{6} S_{\epsilon,M}^i = S$ (say). Therefore,

$$\liminf_{N\to\infty}\inf_{S_{\epsilon,M}}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] \ge \liminf_{N\to\infty}\inf_{S}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right]. \tag{21}$$

Next, our aim is to show that

$$\liminf_{N\to\infty}\inf_{S_{\epsilon,M}^i}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] = \liminf_{N\to\infty}\inf_{S_{\epsilon,M}^i} f(\boldsymbol{\eta}) > 0, \quad \text{a.s.} \tag{22}$$

for $i = 1,\ldots,6$ which implies (using (21)) that

$$\liminf_{N\to\infty}\inf_{S_{\epsilon,M}}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] > 0, \quad \text{a.s.} \tag{23}$$

So, for $i = 1$,

$$
\begin{aligned}
& \liminf_{N\to\infty} \inf_{S^1_{\epsilon,M}} f(\boldsymbol{\eta}) \\
= & \liminf_{N\to\infty} \inf_{|A-A^0|>\epsilon} \frac{1}{N}\sum_{t=1}^{N}\Big[A^0 \exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\cos(\theta^0 t+\phi^0) \\
& -A\exp\{b(1-\cos(\alpha t + c))\}\cos(\theta t+\phi)\Big]^2 \\
= & \lim_{N\to\infty} \inf_{|A-A^0|>\epsilon} \frac{1}{N}\sum_{t=1}^{N}(A-A^0)^2\exp\{2b^0(1-\cos(\alpha^0 t + c^0))\}\cos^2(\theta^0 t+\phi^0) \\
\geq & \; e^{2b^0}\epsilon^2 \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\exp\{-2b^0\cos(\alpha^0 t + c^0))\}\cos^2(\theta^0 t+\phi^0) \\
\geq & \; e^{2b^0}e^{-|2b^0|}\epsilon^2 \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta^0 t+\phi^0) \;=\; \frac{c_{b^0}\epsilon^2}{2} \;>\; 0 \quad \text{a.s.}
\end{aligned}
$$

where $c_b = 1$, if $b > 0$ and $c_b = e^{-|4b^0|}$. Using similar technique, the inequality (22) can be shown for other i also and that proves the theorem.

Appendix B

The following limits have been used to obtain the asymptotic distribution of the LSE $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}^0$.

$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\cos^2(\theta t+\phi)=\delta_1(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}(1-\cos(\alpha t+c))^2\cos^2(\theta t+\phi)=\delta_2(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin^2(\alpha t+c)\cos^2(\theta t+\phi)=\delta_3(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin^2(\theta t+\phi)=\delta_4(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}(1-\cos(\alpha t+c))\cos^2(\theta t+\phi)=\delta_5(\boldsymbol{\xi},p) \quad (24)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin(\alpha t+c)\cos^2(\theta t+\phi)=\delta_6(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin(\theta t+\phi)\cos(\theta t+\phi)=\delta_7(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin(\alpha t+c)\}(1-\cos(\alpha t+c))\cos^2(\theta t+\phi)=\delta_8(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}(1-\cos(\alpha t+c))\sin(\theta t+\phi)\cos(\theta t+\phi)=\delta_9(\boldsymbol{\xi},p)$$
$$\lim_{N\to\infty}\frac{1}{N^p+1}\sum_{t=1}^{N} t^p\exp\{-2b\cos(\alpha t+c)\}\sin(\alpha t+c)\sin(\theta t+\phi)\cos(\theta t+\phi)=\delta_{10}(\boldsymbol{\xi},p)$$

for $p=0,1,2.....$

Note that

$$\exp\{-2|b|\}\le\exp\{-2b\cos(\alpha t+c)\}\le\exp\{2|b|\}. \quad (25)$$

Using it in the first sequence listed above with $p=0$, we have

$$e^{-2|b|}\frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta t+\phi)\le\frac{1}{N}\sum_{t=1}^{N}\exp\{-2b\cos(\alpha t+c)\}\cos^2(\theta t+\phi)\le e^{2|b|}\frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta t+\phi). \quad (26)$$

Now taking limit as $\lim_{N\to\infty}$, we get

$$\frac{e^{-2|b|}}{2} \le \delta_1(\psi, 0) \le \frac{e^{2|b|}}{2}. \tag{27}$$

For notational simplicity, $\delta_k(\psi, p) = \delta_k(p)$, $k = 1, \ldots, 10$ has been used in obtaining the asymptotic distribution of the LSEs.

Using the inequality given in (25), in $\delta_6(\boldsymbol{\xi}, p)$, we have

$$\begin{aligned} \delta_6(\boldsymbol{\xi}, p) \quad &\le \Big\{\ge \quad \Big\} e^{|2b|}\Big\{e^{-|2b|}\Big\} \lim_{N\to\infty} \frac{1}{N^{p+1}} \sum_{t=1}^{N} t^p \sin(\alpha t + c)\cos^2(\theta t + \phi) \\ &\to \quad e^{|2b|}\Big\{e^{-|2b|}\Big\} \times 0. \end{aligned}$$

This implies that

$$\begin{aligned} & 0 \le \delta_6(\boldsymbol{\xi}, p) \le 0 \\ \Rightarrow \quad & \delta_6(\boldsymbol{\xi}, p) \to 0, \quad \text{for all} \quad p \quad \text{and} \quad \boldsymbol{\xi}. \end{aligned}$$

In a similar way, we find that $\delta_k(\boldsymbol{\xi}, p) \to 0$ for all p and $\boldsymbol{\xi}$ for $k = 7, \ldots, 10$ and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$.

[1]Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi - 110016, India, *E-mail*: nandi@isid.ac.in

[2]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India, *E-mail*: kundu@iitk.ac.in