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Survival Models for Step-Stress Experiments with Lagged Effects

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Abstract: In this article, we consider models for experiments in which the stress levels are altered at intermediate stages during the exposure. These experiments, referred to as step-stress tests, belong to the class of accelerated models that are extensively used in reliability and life-testing applications. Models for step-stress tests have largely relied on the cumulative exposure model (CEM) discussed by Nelson. Unfortunately, the assumptions of the model are fairly restrictive and quite unreasonable for applications in survival analysis. In particular, under the CEM the hazard function has discontinuities at the points at which the stress levels are changed.

We introduce a new step-stress model where the hazard function is continuous. We consider a simple experiment with only two stress levels. The hazard function is assumed to be constant at the two stress levels, and linear in the intermediate period. This model allows for a lag period before the effects of the change in stress are observed. Using this formulation in terms of the hazard function, we obtain the maximum likelihood estimators of the unknown parameters. A simple least-squares type procedure is also proposed that yields closed form solutions for the underlying parameters. A Monte Carlo simulation study is performed to study the behavior of the estimators obtained by the two methods for different choices of sample sizes and parameter values. We analyze a real dataset and show that the model provides an excellent fit.

Keywords and phrases: Accelerated testing, step-stress model, cumulative exposure model, lagged effects, hazard function, Bootstrap method

1.1 Introduction

In many applications in the physical and biomedical sciences, experimental conditions may change during the exposure duration. For example, travellers to high mountainous

regions or mountain climbers frequently experience acute mountain sickness (AMS) or altitude sickness. To avoid/decrease symptoms, it is recommended that individuals acclimatize by slowly increasing their elevation. Tables for altitude climbers and deep-sea divers are available that specify the length of time at each altitude. This staged ascent/descent provides protection by 'delaying' or eliminating the onset of symptoms.

In a recent article by Greven et al. [6], the authors examine the effects of water contamination on fish, in particular on swimming performance. The underlying hypothesis was that fish exposed to toxins may exhibit a lower threshold for fatigue. Fatigue was induced by increasing the stress, in this case water velocity, at fixed time points during the experiment. Stress tests are also routinely used to assess cardiac function, with the speed and elevation of the treadmill being increased progressively during the experiment.

In the first example, the stress levels were increased with the hope that a gradual increase would significantly prolong the time to symptoms when compared to a direct ascent/descent to the final altitude. In the remaining two examples, the experiment was designed to reduce the time to fatigue by gradually increasing the stress levels.

All the examples describe what are commonly termed 'step stress tests' in the reliability literature. Step-stress experiments are a particular type of accelerated test routinely used in life-testing experiments. With an increased emphasis on quality and reliability, many products manufactured today have extremely long times to failure on the average. The products, however, must undergo rigorous testing to determine the effect of different stress factors on the reliability and performance. In accelerated testing, the items or units are subjected to higher stress levels than normal. This increased stress induces shorter failure times. Using a model relating the stress levels and failure distributions, it is often possible to determine the properties of the underlying failure distribution under normal operating conditions.

In a standard step-stress experiment, all individuals or items are subject to an initial stress level. The stress is gradually increased at pre-specified times during the exposure. The stress factor may refer to the dose of a drug, elevation of the treadmill, temperature, voltage, and pressure. It is surprising that there are very limited applications of step-stress tests in the survival literature: most references on accelerated testing are found in the reliability literature.

Key references in the broad area of accelerated testing include the books by Nelson [12], Meeker and Escobar [9], and Bagdonavicius and Nikulin [1]. Step-stress models have been studied quite extensively in the literature using the CEM formulation discussed earlier by Nelson [11], [12]. The reader may refer to the work of DeGroot and Goel [5], Miller and Nelson [10], Bai, Kim and Lee [2], Khamis and Higgins [8], Xiong [14], Xiong and Milliken [15], Gouno and Balakrishnan [7], and Balakrishnan et al. [4] for some other important developments on step-stress testing. Balakrishnan [3] has recently provided a synthesis of exact inferential results and optimal accelerated tests in the context of exponential step-stress models.

In Section 1.2, we will formally introduce the CEM and discuss its properties and limitations. We introduce a new model that accounts for the possibility that units will not show immediate effects of the stress change. We model this 'lag' effect using a piecewise continuous hazard function. The new model includes the CEM as a limiting case. We discuss both likelihood and least-squares type estimators and assess their

performance using Monte Carlo simulations. We use the model to analyze a real dataset and show that the model provides an excellent fit.

1.2 Model Description

The standard formulation for step-stress experiments uses the cumulative exposure model discussed by Nelson [12]. The CEM relates the survival distribution of the units at one stress level to the next level. A key assumption of the model is that the residual life of the experimental units depends only on the cumulative exposure they have experienced, with no memory of how this exposure was accumulated.

If we look at the assumptions of the model in terms of the hazard function, we notice that it translates to discontinuities at the points at which the stress levels are changed. In other words, the effect of the change in stress level is instantaneous. This is clearly not reasonable for most applications: the effect of a change in stress will produce an increase in the risk but it seems quite likely that one will observe a lag or latency period. The assumption of an instantaneous jump in the hazard function, though unrealistic leads to a more simple and tractable model.

We consider a more realistic step-stress model that accounts for these latency periods using a piecewise hazard model. This new model reduces to Nelson's model under certain limiting conditions.

1.2.1 Step-stress models with latency

Consider a simple experiment wherein the stress is changed only once during the exposure duration. We assume that the hazard function associated with the initial and elevated stress levels are constant. All n individuals or items are exposed to the same initial stress level x_1. Subjects are continuously monitored, and at some pre-specified time point τ_1, the level of the stress is increased to x_2. The effect of increasing the stress is not seen immediately: we assume that there is a known latency period δ before the effects are completely observed. In the interval $[\tau_1, \tau_2]$ where $\tau_2 = \tau_1 + \delta$, the hazard slowly increases.

The piecewise hazard function is assumed to have the following form:

$$h(t) = \begin{cases} \theta_1, & 0 < t < \tau_1; \\ a + bt, & \tau_1 \leq t < \tau_2 \\ \theta_2, & t \geq \tau_2. \end{cases} ; \tag{1.1}$$

where $\tau_2 > \tau_1$ and both time points are known. The parameters a and b in the model are chosen to ensure the hazard function $h(t)$ is continuous. Therefore, a and b satisfy

$$a + b\tau_1 = \theta_1, \tag{1.2}$$

$$a + b\tau_2 = \theta_2. \tag{1.3}$$

The hazard is constant in the intervals $[0, \tau_1]$ and $[\tau_2, \infty)$. In the interval (τ_1, τ_2), the hazard changes linearly. The parameter b measures how quickly the effects of the

increased stress are observable. In this model, we assume that both τ_1 and τ_2 are known. We can easily extend this model to the case when both these constants are unknown.

We call this the Cumulative Risk Model (CRM) model, to emphasize the accumulated effects of the stress on the lifetimes. Using the definition of the hazard function in (1.1), the survival function (SF), $S(t)$ may be written as

$$S(t) = \begin{cases} e^{-(a+b\tau_1)t}, & 0 < t < \tau_1; \\ e^{-at-\frac{b(t^2+\tau_1^2)}{2}}, & \tau_1 \le t < \tau_2; \\ e^{-(a+b\tau_2)t-\frac{b}{2}(\tau_1^2-\tau_2^2)}, & t \ge \tau_2. \end{cases}$$

The corresponding probability density function (PDF), $f(t)$, is

$$f(t) = \begin{cases} (a+b\tau_1)e^{-(a+b\tau_1)t}, & 0 < t < \tau_1; \\ (a+bt)e^{-at-\frac{b(t^2+\tau_1^2)}{2}}, & \tau_1 \le t < \tau_2; \\ (a+b\tau_2)e^{-(a+b\tau_2)t-\frac{b}{2}(\tau_1^2-\tau_2^2)}, & t \ge \tau_2. \end{cases}$$

The Cumulative Hazard (CH) function is

$$H(t) = \begin{cases} (a+b\tau_1)t & 0 < t < \tau_1; \\ (a+b\tau_1)\tau_1 + \frac{1}{2}(t-\tau_1)^2 b & \tau_1 \le t < \tau_2; \\ (a+b\tau_1)\tau_1 + \frac{1}{2}(\tau_2-\tau_1)^2 b + (t-\tau_2)(a+b\tau_2), & t \ge \tau_2. \end{cases} \tag{1.4}$$

If we assume that the hazard functions are constant at both stress levels x_1 and x_2, then the cumulative hazard function for Nelson's CEM is

$$H(t) = \begin{cases} \theta_1 t & 0 < t < \tau_1; \\ \theta_1\tau_1 + \theta_2(t-\tau_1) & t \ge \tau_1. \end{cases}$$

It is clear that, in (1.4), if we let $\delta = (\tau_2 - \tau_1) \to 0$ with $\delta b \to (\theta_2 - \theta_1)$, then the CRM converges to the above CEM formulation of Nelson.

To compare Nelson's CEM with this new model, we plot the cumulative hazard for the two models. A graph is provided in Figure 1.1 for $\tau_1 = 1, \tau_2 = 3$. The cumulative hazard function for both models is identical up to τ_1. At τ_1, the CEM is linear with a slope θ_2, while the CRM is quadratic. After τ_2, the CRM is linear as well.

In the next section, we derive the estimators of a and b for the CRM using the maximum likelihood approach.

1.3 Maximum Likelihood Estimators for the CRM

Let

$$t_1 < \ldots < t_{n_1} < \tau_1 < t_{n_1+1} < \ldots < t_{n_1+n_2} < \tau_2 < t_{n_1+n_2+1} < \ldots < t_n \tag{1.5}$$

be the ordered failure times. Here n_1, n_2 and $n_3 = n - (n_1 + n_2)$ denote the number of failures that occur before τ_1, between τ_1 and τ_2, and beyond τ_2, respectively.

The likelihood function as a function of a and b is given by:

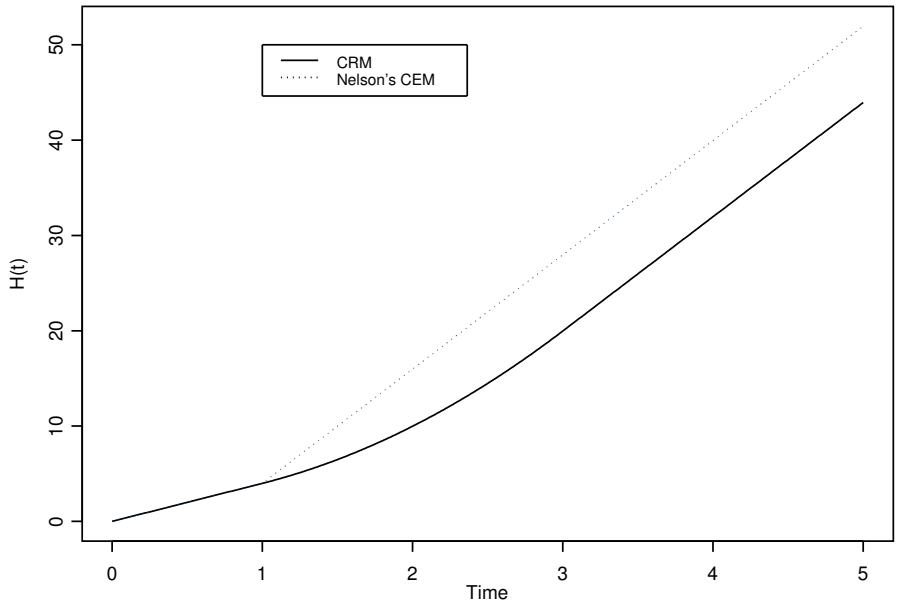

Figure 1.1. Cumulative Hazard Functions of Cumulative Exposure and Cumulative Risk Models

$$l(a,b) = (a+b\tau_1)^{n_1} e^{-(a+b\tau_1)\sum_{i\in I_1} t_i} \times \prod_{i\in I_2} (a+bt_i) e^{-\sum_{i\in I_2}(at_i+\frac{bt_i^2}{2})} e^{-\frac{n_2 b\tau_1^2}{2}}$$
$$\times (a+b\tau_2)^{n_3} e^{-(a+b\tau_2)\sum_{i\in I_3} t_i} \times e^{-\frac{n_3 b}{2}(\tau_1^2-\tau_2^2)}, \tag{1.6}$$

where $I_1 = \{1,\ldots,n_1\}$, $I_2 = \{n_1+1,\ldots,n_1+n_2\}$ and $I_3 = \{n_1+n_2+1,\ldots,n\}$. We use the convention that $\sum_{i\in I_j}(.) = 0$ if I_j is an empty set. Note that if $n_1 > 0, n_2 > 0, n_3 > 0$, then $(n_1, n_2, t_{n_1+1},\ldots,t_{n_1+n_2}, \sum_{i\in I_1} t_i, \sum_{i\in I_2} t_i)$ is a complete sufficient statistic.

The following observations are clear from the form of the likelihood function in (1.6):

- If $n_2 > 0$, then the MLE's of a and b always exist.
- If $n_2 = 0$, then the MLE's of a and b exist if both $n_1 > 0$ and $n_3 > 0$.
- If $n_2 = 0$ and either one of the two remaining counts is zero, then, the MLE's of a and b do not exist.

The derivation of the maximum likelihood estimators is provided in the appendix, along with the expressions for the observed Fisher Information. We note that the MLE of b can be obtained as the solution of the non-linear equation

$$-1 + \frac{n_1}{n-bK+b\tau_1 T} + \frac{n_3}{n-bK+bT\tau_2} + \sum_{i\in I_2} \frac{1}{n-bK+bTt_i} = 0; \tag{1.7}$$

here, $T = \sum_{i=1}^{n} t_i$ and

$$K = \tau_1 \sum_{i \in I_1} t_i + \frac{1}{2} \sum_{i \in I_2} t_i^2 + \frac{1}{2} n_2 \tau_1^2 + \tau_2 \sum_{i \in I_3} t_i - \frac{1}{2} n_3 (\tau_2^2 - \tau_1^2). \tag{1.8}$$

Once the MLE of b, $\hat{b}$, is obtained as the solution of (1.7), then the MLE of a, say $\hat{a}$, is simply

$$\hat{a} = \frac{n - \hat{b}K}{T}. \tag{1.9}$$

Remark: Suppose $n_2 = 0$. In this case, we can easily see from (1.7) that

$$\hat{b} = \frac{nK - T(n_1\tau_1 + n_3\tau_2)}{(K - T\tau_1)(K - T\tau_2)} \tag{1.10}$$

Remark: Suppose the ratio $\frac{\theta_2}{\theta_1}$ is known. The case of the known ratio is often intuitive in a reliability setting. We may have a physical model that relates the stress levels and the corresponding mean failure times. This is equivalent to saying that the ratio $\frac{a}{b}$ is a known quantity, say α. In this case,

$$\hat{b} = \frac{n}{K + \alpha T} \qquad \text{and} \qquad \hat{a} = \alpha \hat{b}. \tag{1.11}$$

1.4 Least Squares Estimators

In this section, we propose an alternative method for estimating a and b that does not require solving a non-linear equation. This method is based on least squares and yields estimators that are in explicit form. Since the cumulative hazard function for the CRM is simple, it is natural to obtain a and b by minimizing the least squares distance between the empirical cumulative hazard function and the fitted hazard function with respect to the unknown parameters.

The estimators of a and b, say $\tilde{a}$ and $\tilde{b}$, can be obtained by minimizing

$$\sum_{i=1}^{n} \Big(H(t_i) - \hat{H}(t_i) \Big)^2 \tag{1.12}$$

with respect to the unknown parameters a and b. Here, $H(t_i)$ and $\hat{H}(t_i)$ are the cumulative hazard function and the estimated cumulative hazard function respectively. We use the standard non-parametric estimator given by

$$\hat{H}(t_i) = -\ln(\hat{S}(t_i)) = \ln n - \ln(n - i + 1) = c_i \quad \text{(say.)} \tag{1.13}$$

The cumulative hazard function $H(t)$ has a simple form given in (1.4) using which $\tilde{a}$ and $\tilde{b}$ can be obtained as follows:

$$\tilde{a} = \frac{(\sum_{i=1}^n v_i^2)(\sum_{i=1}^n c_i t_i) - (\sum_{i=1}^n t_i v_i)(\sum_{i=1}^n c_i v_i)}{(\sum_{i=1}^n t_i^2)(\sum_{i=1}^n v_i^2) - (\sum_{i=1}^n t_i)(\sum_{i=1}^n v_i)},$$
$$\tilde{b} = \frac{(\sum_{i=1}^n t_i^2)(\sum_{i=1}^n c_i v_i) - (\sum_{i=1}^n t_i v_i)(\sum_{i=1}^n c_i t_i)}{(\sum_{i=1}^n t_i^2)(\sum_{i=1}^n v_i^2) - (\sum_{i=1}^n t_i)(\sum_{i=1}^n v_i)},$$

where

$$v_i = \begin{cases} \tau_i t_i, & 1 \le i \le n_1; \\ \frac{1}{2}\left(\tau_1^2 + t_i^2\right), & n_1 \le i \le n_1 + n_2; \\ \frac{1}{2}\left(\tau_1^2 - \tau_2^2\right) + \tau_2 t_i, & i > n_3. \end{cases}$$

Remark: If $\frac{a}{b} = \alpha$ with α known, then

$$\tilde{b} = \frac{\alpha \sum_{i=1}^n c_i t_i + \sum_{i=1}^n c_i v_i}{\sum_{i=1}^n (\alpha t_i + v_i)^2}, \quad \text{and} \quad \tilde{a} = \alpha\, \tilde{b}. \tag{1.14}$$

1.5 Data Analysis

In this section, we analyze the data from Greven et al. [6]. In the paper, the authors consider the swimming performances of two groups of fish. The first group consisted of offspring where the parents were exposed to a contaminant. The second group was a control group with the same characteristics except for the lack of exposure to the chemical. The fish were placed in a chamber where the flow rate of the water was maintained at 15 cm/ sec. After 90 minutes, the flow rate was increased every 20 seconds by 5 cm/ sec. The time at which the fish could no longer maintain its' position in the chamber was recorded which is the fatigue time. We assumed for simplicity that there was only one stress level. The empirical cumulative hazard suggests that the first increase has a significant effect: the effect of subsequent increases is not evident.

For each dataset, we obtained the MLE's and the LSE's based on the cumulative risk model. Since τ_2 is unknown, we used a discrete optimization method to obtain an estimate from the data. The results of the two analyses are given in Tables 1.1 and 1.2. The tables also provide the approximate and bootstrap (parametric) confidence intervals for the two estimators.

Table 1.1. Estimates and Confidence Intervals for the Control Group

Methods	$\hat{a}$	$\hat{b}$	$\hat{\theta}_1$	$\hat{\theta}_2$
MLE	-0.23500	0.00218	0.00421	0.10424
($\hat{\tau}_2 = 156$)	(-0.48422, -0.11501)	(0.00103, 0.00443)	(0.00212, 0.00922)	(0.05712, 0.20786)
	(-0.42762,-0.04238)	(0.00043, 0.00393)	(0.00100, 0.00860)	(-0.08811, 0.29827)
LSE	-0.23469	0.00216	0.00325	0.06598
($\hat{\tau}_2 = 139$)	(-0.45200, -0.07468)	(0.00074, 0.00405)	(0.00044, 0.00710)	(0.02713, 0.12058)
	(-0.41597, -0.05341)	(0.00051, 0.00381)	(0.00060, 0.00522)	(-0.07433, 0.20542)

Table 1.2. Estimates and Confidence Intervals for the Test Group

Methods	$\hat{a}$	$\hat{b}$	$\hat{\theta}_1$	$\hat{\theta}_2$
MLE	-0.67057	0.00612	0.00248	0.03307
($\hat{\tau}_2 = 115$)	(-1.29324, -0.30603)	(0.00282, 0.01177)	(0.00127, 0.00684)	(0.01824, 0.06235)
	(-1.14949, -0.19165)	(0.00177, 0.01047)	(0.00053, 0.00473)	(-0.11280, 0.17926)
LSE	-1.34638	0.01226	0.00214	0.02666
($\hat{\tau}_2 = 112$)	(-2.55898, -0.54740)	(0.00497, 0.02276)	(0.00003, 0.00491)	(0.01350, 0.04754)
	(-2.29977, -0.39299)	(0.00359, 0.02093)	(0.00046, 0.00398)	(-0.15589, 0.20937)

Data for the Control Group: 83.50, 91.00, 91.00, 97.00, 107.00, 109.50, 114.00, 115.41, 128.61, 133.53, 138.58, 140.00, 152.08, 155.10.
Data for the Test Group : 91.00, 93.00, 94.00, 98.20, 115.81, 116.00, 116.50, 117.25, 126.75, 127.50, 154.33, 159.50, 164.00, 184.14, 188.33.

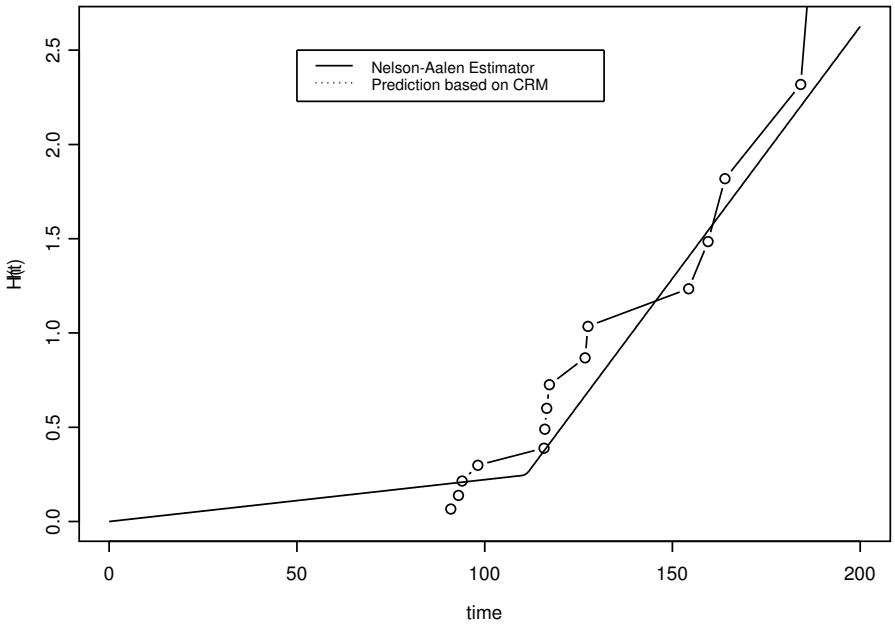

Figure 1.2. Empirical and Predicted CH Functions: Test Group

We computed the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the estimated CDF based on the CRM. For the control group, the KS distance using the MLEs (LSEs) was 0.3302 (0.2639), with corresponding p values of 0.0736 (0.2422). For the test group, the KS distance using the MLEs (LSEs) was 0.2128 (0.1946) with corresponding p values of 0.4564 (0.5726). These results reveal

that the CRM model does provide an excellent fit to the data. Figure 1.2 provides an overlay plot of the empirical cumulative hazard function and the predicted CH function based on the least squares fit of the CRM. The model is clearly able to isolate the 'change point'. We can say that there is approximately a 2 minute delay before the effects of the flow rate increase are really observable. This observation also provides some rationale for considering the case of a single stress level. The overlay plot does not show the quadratic nature of the cumulative hazard function: if we magnify the region from 90 minutes to 95 minutes, that trend is visible.

1.6 Simulation Results

To assess the performance of the MLE's and the least squares type estimators, a simulation study was conducted. We considered different sample sizes and different values for the parameters. For each combination of sample size and parameter values, the bias and the square root of the mean squared errors (RMSE's) are reported for both estimators in Tables 1.3–1.10. All the simulations are performed using RAN2 random deviate generator of Press et al. [13].

Each pair of tables uses the same parameter values: the second table in the pair assumes a fixed ratio. All the results are based on 1000 replications. For one simulation with $n = 25, \tau_1 = 100, \tau_2 = 150, \theta_1 = 0.01, \theta_2 = 0.02$, we also provide the sampling distribution of the MLE's of a, b, θ_1 and θ_2. These distributions are presented in Figures 1.3–1.6 and these figures display that the distributions are skewed and not close to normal. We would, therefore, recommend against using confidence intervals based on normality unless the sample sizes are large.

From the tables, we can observe that, as expected, the average bias and the MSE's decrease as the sample size increases. When the ratio $\frac{\theta_2}{\theta_1}$ is known the performance of the estimators are better. The RMSE for the estimator of $\hat{\theta}_2$ for a known ration of 4 is twice the RMSE of $\hat{\theta}_2$ for a known ration of 2. This can be seen from the form of the estimators. When the sample size is small, the LSE outperforms the MLE, but as the sample size increases, the MLE clearly is superior. For fixed τ_1, if τ_2 increases, the estimates of θ_2 are not as good, although the performance of the estimator of θ_1 does not change significantly. This is intuitive since, as τ_2 increases, the number of failures n_3 will decrease. For fixed τ_1, τ_2 and θ_1, when θ_2 increases the performances of the estimators deteriorate.

1.7 Conclusions

This new model for a step-stress experiment was motivated by the limitations of Nelson's CEM. Our formulation in terms of a piecewise hazard function provides a more realistic model for biomedical applications and reduces to the standard CEM of Nelson in the limiting case. We have developed both likelihood estimators as well as estimators

Table 1.3. $\tau_1 = 100$, $\tau_2 = 150$, $\theta_1 = 1/100$, $\theta_2 = 1/50$

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01139	0.04032	0.00285	0.03266
	LSE	0.01090	0.02989	0.00254	0.02340
50	MLE	0.01084	0.02633	0.00190	0.01232
	LSE	0.01090	0.02105	0.00193	0.00972
75	MLE	0.01066	0.02362	0.00147	0.00744
	LSE	0.01080	0.01983	0.00158	0.00634
100	MLE	0.01052	0.02219	0.00134	0.00546
	LSE	0.01067	0.01918	0.00143	0.00478

Table 1.4. Known Ratio

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01047	0.02093	0.00226	0.00452
	LSE	0.00914	0.01828	0.00228	0.00456
50	MLE	0.01025	0.02050	0.00151	0.00302
	LSE	0.00941	0.01882	0.00162	0.00323
75	MLE	0.01016	0.02032	0.00120	0.00240
	LSE	0.00941	0.01882	0.00132	0.00265
100	MLE	0.01013	0.02026	0.001102	0.00205
	LSE	0.00960	0.01919	0.00113	0.00227

Table 1.5. $\tau_1 = 100$, $\tau_2 = 150$, $\theta_1 = 1/100$, $\theta_2 = 1/25$

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01086	0.06454	0.00273	0.03688
	LSE	0.01029	0.05022	0.00256	0.02597
50	MLE	0.01040	0.05234	0.00194	0.02020
	LSE	0.01028	0.04411	0.00198	0.01578
75	MLE	0.01031	0.04959	0.00151	0.01569
	LSE	0.01029	0.04337	0.00161	0.01262
100	MLE	0.01021	0.04728	0.00128	0.01256
	LSE	0.01023	0.04242	0.00137	0.01078

Table 1.6. Known Ratio

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01041	0.04164	0.00227	0.00910
	LSE	0.00907	0.03630	0.00231	0.00923
50	MLE	0.01018	0.04074	0.00149	0.00594
	LSE	0.00934	0.03737	0.00162	0.00648
75	MLE	0.01011	0.04045	0.00115	0.00460
	LSE	0.00946	0.03783	0.00131	0.00523
100	MLE	0.01005	0.04021	0.00100	0.00398
	LSE	0.00952	0.03810	0.00115	0.00461

Table 1.7. $\tau_1 = 100$, $\tau_2 = 125$, $\theta_1 = 1/100$, $\theta_2 = 1/50$

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01088	0.03175	0.00264	0.01652
	LSE	0.01016	0.02565	0.00250	0.01101
50	MLE	0.01060	0.02669	0.00195	0.00949
	LSE	0.01029	0.02343	0.00194	0.00682
75	MLE	0.01035	0.02491	0.00151	0.00720
	LSE	0.01015	0.02266	0.00160	0.00547
100	MLE	0.01028	0.02386	0.00136	0.00558
	LSE	0.01013	0.02215	0.00143	0.00441

Table 1.8. Known Ratio

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01046	0.02091	0.00223	0.00447
	LSE	0.00906	0.01812	0.00224	0.00447
50	MLE	0.01026	0.02052	0.00151	0.00302
	LSE	0.00940	0.01879	0.00163	0.00326
75	MLE	0.01019	0.02038	0.00123	0.00245
	LSE	0.00954	0.01908	0.00136	0.00271
100	MLE	0.01015	0.02029	0.00105	0.00210
	LSE	0.00962	0.01923	0.00118	0.00236

Table 1.9. $\tau_1 = 100$, $\tau_2 = 125$, $\theta_1 = 1/100$, $\theta_2 = 1/25$

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01072	0.04723	0.00254	0.02006
	LSE	0.01012	0.03806	0.00234	0.01742
50	MLE	0.01063	0.04355	0.00198	0.01248
	LSE	0.01052	0.03670	0.00197	0.01158
75	MLE	0.01042	0.04268	0.00147	0.01025
	LSE	0.01045	0.03731	0.00158	0.01009
100	MLE	0.01029	0.04180	0.00132	0.00839
	LSE	0.01039	0.03720	0.00144	0.00899

Table 1.10. Known Ratio

n	Methods	$\hat{\theta}_1$	$\hat{\theta}_2$	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)
25	MLE	0.01039	0.04157	0.00226	0.00903
	LSE	0.00908	0.03631	0.00231	0.00923
50	MLE	0.01023	0.04090	0.00158	0.00633
	LSE	0.00936	0.03745	0.00171	0.00684
75	MLE	0.01016	0.04065	0.00126	0.00502
	LSE	0.00950	0.03798	0.00139	0.00556
100	MLE	0.01011	0.04043	0.00104	0.00416
	LSE	0.00956	0.03823	0.00120	0.00480

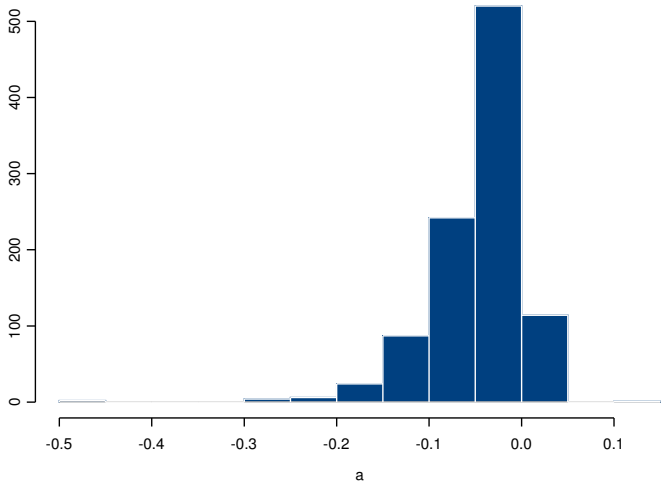

Figure 1.3. Sampling Distribution of the MLE of a

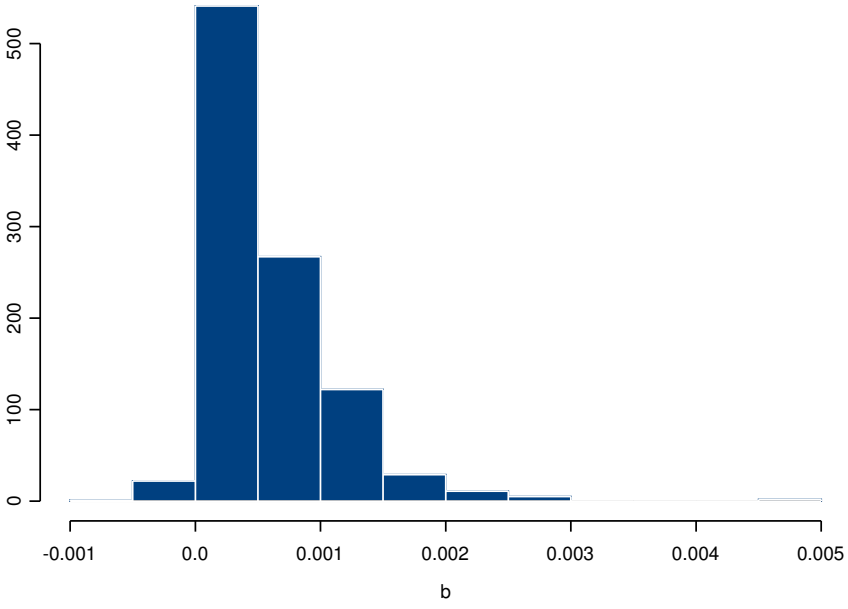

Figure 1.4. Sampling Distribution of the MLE of b

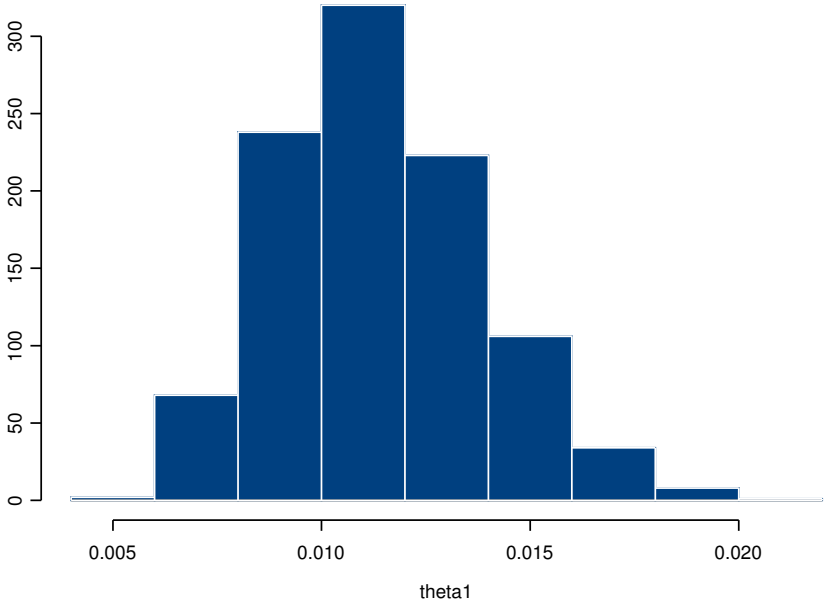

Figure 1.5. Sampling Distribution of the MLE of θ_1

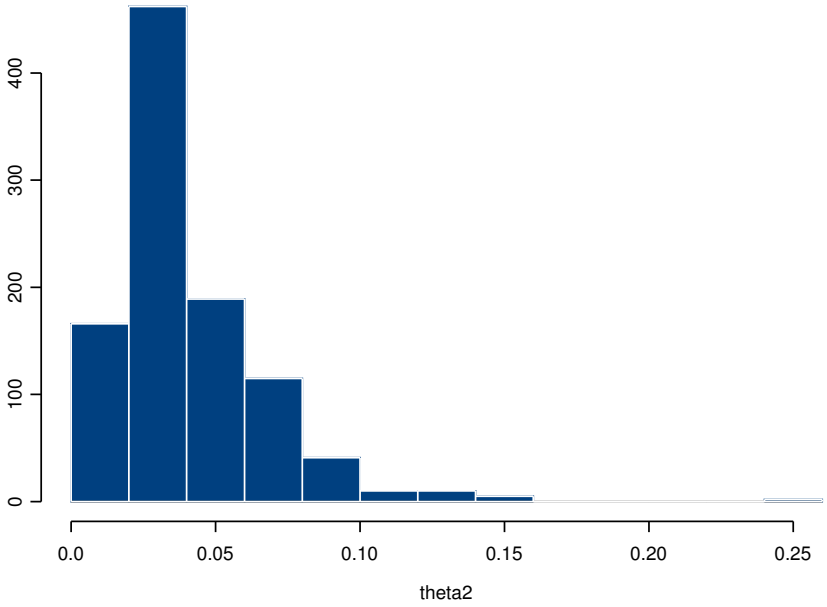

Figure 1.6. Sampling Distribution of the MLE of θ_2

based on least squares. The performance of these estimators have been evaluated using a simulation study. The least-squares estimators, obtained in closed form, perform extremely well for small samples. In addition to the simulation results, we have used a real dataset to fit the new model. The model is extremely flexible and provides an excellent fit to the fatigue data.

Appendix

The Likelihood Equations

Here we show that the MLE of b can be obtained as the solution of (1.7). Note that the log-likelihood function can be written from (6) as

$$L(a,b) = \ln l(a,b) = -aT - bK + n_1 \ln(a + b\tau_1) + \sum_{i \in I_2} \ln(a + bt_i) + n_3 \ln(a + b\tau_2).$$

Therefore, $\hat{a}$ and $\hat{b}$ can be obtained by solving the equations

$$\frac{\partial L}{\partial a} = 0 \quad \text{and} \quad \frac{\partial L}{\partial b} = 0. \tag{1.15}$$

From (1.15), we obtain

$$a \times \frac{\partial L}{\partial a} + b \frac{\partial L}{\partial b} = 0. \tag{1.16}$$

After simplification, we obtain

$$a = \frac{n - bK}{T}. \tag{1.17}$$

Substituting this expression of a from (1.17) in $\frac{\partial L}{\partial a} = 0$, we obtain the equation in (1.7).

Fisher Information Matrix

In this subsection, we obtain the explicit expressions for elements of the observed Fisher information matrix. We can use the asymptotic normality property of the MLEs to construct approximate confidence intervals of a and b for large n.

Let $O(a,b) = O_{ij}(a,b), i,j = 1,2$, denote the observed Fisher information matrix of a and b. Then

$$\begin{aligned}
O_{11} &= \frac{n_1}{(\hat{a} + \hat{b}\tau_1)^2} + \sum_{i \in I_2} \frac{1}{(\hat{a} + \hat{b}t_i)^2} + \frac{n_3}{(\hat{a} + \hat{b}\tau_2)^2} \\
O_{22} &= \frac{n_1\tau_1^2}{(\hat{a} + \hat{b}\tau_1)^2} + \sum_{i \in I_2} \frac{t_i^2}{(\hat{a} + \hat{b}t_i)^2} + \frac{n_3\tau_2^2}{(\hat{a} + \hat{b}\tau_2)^2} \\
O_{12} &= \frac{n_1\tau_1}{(\hat{a} + \hat{b}\tau_1)^2} + \sum_{i \in I_2} \frac{t_i}{(\hat{a} + \hat{b}t_i)^2} + \frac{n_3\tau_2}{(\hat{a} + \hat{b}\tau_2)^2}.
\end{aligned}$$

The approximate variances of $\hat{a}$ and $\hat{b}$ can then be obtained through the observed information matrix as

$$V_a = \frac{O_{22}}{O_{11}O_{22} - O_{12}^2} \quad \text{and} \quad V_b = \frac{O_{11}}{O_{11}O_{22} - O_{12}^2}. \tag{1.18}$$

Using the observed asymptotic variances of $\hat{a}$ and $\hat{b}$, approximate confidence intervals can be easily constructed.

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