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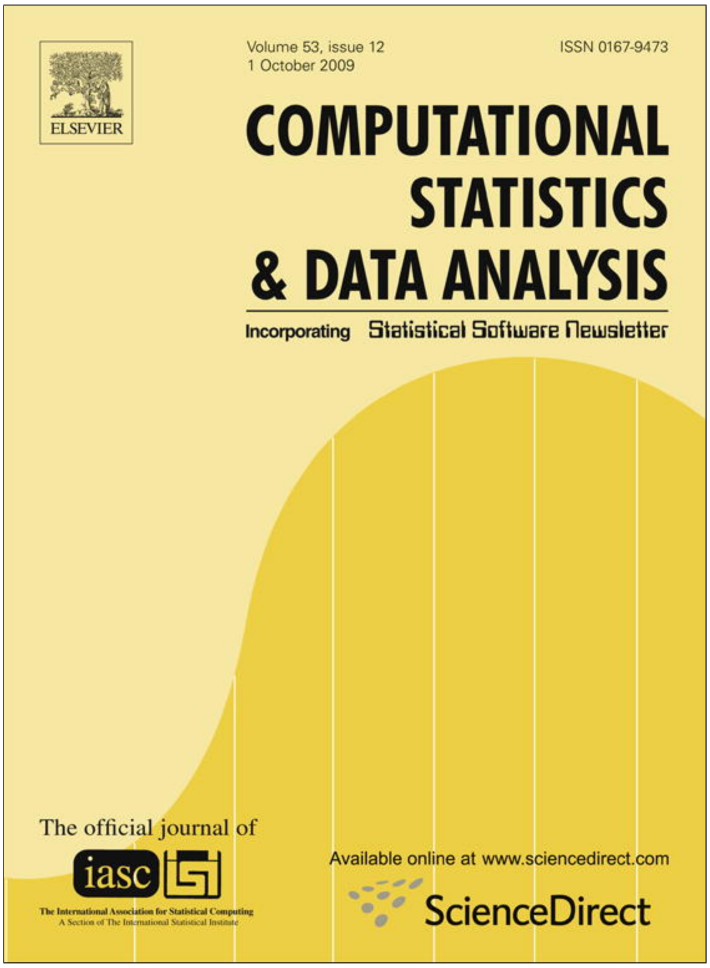

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On progressively censored competing risks data for Weibull distributions

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ABSTRACT

In survival analysis, or in reliability study, an investigator is often interested in the assessment of a specific risk in the presence of other risk factors. It is well known as the competing risks problem in statistical literature. Moreover, censoring is inevitable in any life testing or reliability study. In this paper, we consider a very general censoring scheme, namely a progressive censoring scheme. It is further assumed that the lifetime distribution of the individual causes are independent and Weibull-distributed with the same shape parameters but different scale parameters. We obtain the maximum likelihood and approximate maximum likelihood estimates of the unknown parameters. We also compute the observed Fisher information matrix using the missing information principles, and use them to compute the asymptotic confidence intervals. Monte Carlo simulations are performed to compare the performances of the different methods, and one data set is analyzed for illustrative purposes. We also discuss different optimality criteria, and selected optimal progressive censoring plans are presented.

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1. Introduction

In life testing experiments, an investigator is often interested in the assessment of a specific risk in the presence of other risk factors. In the statistical literature, it is well known as the competing risks model. In analyzing the competing risks data, ideally the data consist of a failure time and an indicator denoting the cause of failure. The causes of failure may be assumed to be dependent or independent. In this paper, we use the latent failure times model, as suggested by Cox (1959). In the latent failure times model, it is assumed that the failure times are independent. Although the assumption of independence seems very restrictive, in case of dependence, there are some identifiability issues of the underlying model. It is known (see for example Kalbfleish and Prentice (1980) or Crowder (2001)) that without the information of the covariates, it is not possible to test the assumption of independence of the failure time distributions just from the observed data.

In most of the life testing, experiment censoring is inevitable. In this paper, we develop an inference for the competing risk model under a very general censoring scheme, namely the progressive censoring scheme. The progressive censoring scheme has become very popular because of its wide scale applicability. For an exhaustive list of references and further details on progressive censoring, the readers are referred to the monograph by Balakrishnan and Aggarwala (2000) or the recent review article by Balakrishnan (2007).

In this paper, we consider the competing risks data under progressive type-II censoring. It can be defined as follows; consider n individuals in a study and assume that there are K causes of failure, which are known in advance. At the time of each failure, one or more surviving units may be removed from the study at random. It is assumed that the failed items are not replaced. The data from a progressive type-II censored sample is as follows;

$$(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{m:m:n}, \delta_m, R_m), \quad (1)$$

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where $X_{1:m:n} < \cdots < X_{m:m:n}$ denote the m observed failure times, $\delta_1, \ldots, \delta_m$ denote the m causes of failure and $R_1, \ldots, R_m$ denote the number of units removed from the study at the failure times $X_{1:m:n} < \cdots < X_{m:m:n}$.

The main focus of this paper is the analysis of competing risks data when the data are progressively type-II censored under the latent failure times model. Recently Kundu et al. (2004) considered such a model when the lifetime distribution of the latent failure times are exponential. Unfortunately, the assumptions of exponential distributions are quite restrictive. Since the exponential distribution has a constant failure rate, it is well known that it has serious limitations in modeling lifetime data. In this paper, we consider the same latent failure time model formulations, and it is assumed that the latent failure times are independent Weibull-distributed random variables, with the same shape parameter but different scale parameters.

We obtain the maximum likelihood estimators (MLEs) of the unknown parameters. It is observed that the MLEs cannot be obtained in explicit form. It can be obtained by solving a one dimensional optimization problem. We propose a simple fixed point type algorithm to solve this optimization problem. Since the MLEs cannot be obtained in explicit form, we propose approximate maximum likelihood estimators (AMLEs), which have explicit expressions. It may be mentioned that the AMLEs have been proposed in the literature by Balasooriya et al. (2000) and Ng et al. (2004), also without the presence of competing risks. It is not obvious as to how to extend their results when competing risks are present.

We obtain the observed Fisher information matrix using the missing value principle of Louis (1982). The observed Fisher information matrix can be used to compute the confidence intervals of the unknown parameters. Monte Carlo Simulations are performed to compare the performances of the MLEs and AMLEs. One data analysis is performed for illustrative purposes.

Finding the optimal censoring scheme is an important problem and it has received considerable attention in the last few years. An optimal censoring scheme means that for fixed m and n the choice of $\{R_1, \ldots, R_m\}$, which provides the maximum information of the unknown parameters. In this paper, we present selected optimal censoring schemes based on different optimality criteria. Since finding the optimal censoring scheme is computationally very expensive, we propose a sub-optimal censoring scheme, which can be obtained very quickly.

The rest of the paper is organized as follows. In Section 2, we describe the model and present the notation used throughout the paper. The MLEs and AMLEs are obtained in Sections 3 and 4, respectively. The observed and expected Fisher information matrices are presented in Section 5. Simulation results are presented in Section 6. We analyze one data set in Section 7. The procedure to find the optimal censoring scheme is provided in Section 8.

2. Model description and notation

Without loss of generality, we assume that there are only two independent causes of failure, although all the methods developed here can be easily extended to the case $K > 2$. We assume the following notation

X_i: lifetime of system i.
X_{ji}: lifetime of the ith individual under cause j, $j = 1, 2$.
$F(\cdot)$: cumulative distribution function of X_i.
$F_j(\cdot)$: cumulative distribution function of X_{ji}.
$\bar{F}_j(\cdot)$: survival function of X_{ji}.
δ_i: indicator variable denoting the cause of failure of the ith individual.
m: number of complete failures observed before termination.
$x_{i:m:n}$: ith observed failure times; $i = 1, \ldots, m$.
R_i: number of units removed at the time of the ith failure; $R_i \geq 0$, $R_1 + \cdots + R_m = n$.
I_j: $\{i; \delta_i = j\}$, $j = 1, 2$.
$|I_j|$: cardinality of I_j, we assume $|I_j| = m_j$, $j = 1, 2$.
$\exp(\lambda)$: exponential random variable with the probability density function (PDF); $\lambda e^{-\lambda x}$; $x > 0$.
$\text{gamma}(\alpha, \lambda)$: gamma random variable with PDF; $\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$; $x > 0$.
$\text{Weibull}(\alpha, \lambda)$: Weibull random variable with PDF; $\alpha \lambda x^{\alpha-1} e^{-\lambda x^\alpha}$; $x > 0$.

We assume that (X_{1i}, X_{2i}), for $i = 1, \ldots, n$ are n independent identically distributed random variables. Further, it is assumed X_{1i} and X_{2i} are independent and they follow Weibull(α, λ_1) and Weibull (α, λ_2) respectively and $X_i = \min\{X_{1i}, X_{2i}\}$. Under the above assumptions, based on the sample (1), we want to draw inferences of the unknown parameters.

3. Maximum likelihood estimates

For the given $R_1, \ldots, R_m$, we have the following m observations;

$$\{(x_{i:m:n}, 1); i \in I_1\} \quad \& \quad \{(x_{i:m:n}, 2); i \in I_2\}. \tag{2}$$

Based on the above observations, the log-likelihood function without the additive constant can be written as follows;

$$\ln l(\alpha, \lambda_1, \lambda_2) = m \ln \alpha + m_1 \ln \lambda_1 + m_2 \ln \lambda_2 + (\alpha - 1) \sum_{i=1}^{m} \ln x_{i:m:n} - (\lambda_1 + \lambda_2) \sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha}. \tag{3}$$

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Taking the derivatives of (3) with respect to λ_1 and λ_2 and equating them to zeros we obtain the MLEs of λ_1 and λ_2 for fixed α as

$$\widehat{\lambda}_1(\alpha) = \frac{m_1}{\sum_{i=1}^{m}(R_i+1)x_{i:m:n}^{\alpha}} \quad \text{and} \quad \widehat{\lambda}_2(\alpha) = \frac{m_2}{\sum_{i=1}^{m}(R_i+1)x_{i:m:n}^{\alpha}}. \tag{4}$$

Substituting back $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$ in (3) and ignoring the additive constant we obtain the profile log-likelihood of α as

$$p(\alpha) = m\ln\alpha - m\ln\left(\sum_{i=1}^{m}(R_i+1)x_{i:m:n}^{\alpha}\right) + \alpha\sum_{i=1}^{m}\ln x_{i:m:n}, \tag{5}$$

and the MLE of α can be obtained by maximizing (5) with respect to α. A rigorous proof of it can be carried out along the same line, as in Cramer and Kamps (1996). The details are avoided. We have the following result regarding the shape of $p(\alpha)$.

Lemma 1. *$p(\alpha)$ is log-concave.*

Proof of Lemma 1. It is provided in the Appendix.

From Lemma 1 and using the fact that $p(\alpha) \to -\infty$ as $\alpha \to 0$ or as $\alpha \to \infty$, it follows that $p(\alpha)$ is unimodal and it has a unique maximum. It is obvious from Lemma 1 that the MLE of α can be obtained by maximizing (5) with respect to α and it is unique. Since $p(\alpha)$ is unimodal, most of the standard iterative process can be used for finding the MLE. We propose the following simple algorithm. If $\widehat{\alpha}$ is the MLE of α, then it is obvious from $p'(\alpha) = 0$ that $\widehat{\alpha}$ satisfies the following fixed point type equation;

$$h(\alpha) = \alpha, \tag{6}$$

where

$$h(\alpha) = \left[\frac{\sum_{i=1}(R_i+1)x_{i:m:n}^{\alpha}\ln x_{i:m:n}}{\sum_{i=1}(R_i+1)x_{i:m:n}^{\alpha}} - \frac{1}{m}\sum_{i=1}\ln x_{i:m:n}\right]^{-1}. \tag{7}$$

We propose a simple iterative scheme to solve for $\widehat{\alpha}$ from (6). Start with an initial guess of α, say $\alpha^{(0)}$, obtain $\alpha^{(1)} = h(\alpha^{(0)})$ and proceeding in this way obtain $\alpha^{(k+1)} = h(\alpha^{(k)})$. Stop the iterative process when $|\alpha^{(k+1)} - \alpha^{(k)}| < \epsilon$, some pre-assigned tolerance limit. Once we obtain $\widehat{\alpha}$, the MLEs of λ_1 and λ_2 can be obtained from (4) as $\widehat{\lambda}_1(\widehat{\alpha})$ and $\widehat{\lambda}_2(\widehat{\alpha})$ respectively. Since the MLEs are not in a compact form, we propose the approximate MLEs and they have explicit expressions.

Comment: It may be observed that $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ both have positive probabilities at 0. Note that $m_1 \sim Bin\left(m, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$ and $m_2 \sim Bin\left(m, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$. Therefore, for all m, $P(m_1 = 0) > 0$ and $P(m_2 = 0) > 0$, but $P(m_1 = 0) \to 0$ and $P(m_2 = 0) \to 0$ as $m \to \infty$.

4. Approximate maximum likelihood estimators

If the random variable X follows Weibull (α, λ), then the PDF of $Y = \ln X$ has the extreme value distribution with the PDF

$$f_Y(y; \mu, \sigma) = \frac{1}{\sigma}\,e^{\frac{y-\mu}{\sigma} - e^{\frac{y-\mu}{\sigma}}}; \quad -\infty < y < \infty, \tag{8}$$

where $\mu = -\frac{1}{\alpha}\ln\lambda$ and $\sigma = \frac{1}{\alpha}$. The density function, as described by (8), is known as the density function of an extreme value distribution, with location and scale parameters as μ and σ, respectively, and it will be denoted as EV(μ, σ). Note that the Weibull (α, λ) and EV (μ, σ) are equivalent models in the sense that the procedure developed under one model can also be used for the other model. Although they are equivalent models, sometimes it is easier to work with the model (8) as μ and σ appear as the location and scale parameters.

We have assumed that X_{1i} and X_{2i} are independent Weibull random variables with parameters (α, λ_1) and (α, λ_2) respectively. Therefore, $Y_i = \min\{X_{1i}, X_{2i}\}$ also has Weibull (α, λ) distribution, where $\lambda = \lambda_1 + \lambda_2$.

Now, ignoring the cause of failures, and following the same approach as Balasooriya and Balakrishnan (2000) or Kundu (2007), the approximate maximum likelihood estimators (AMLEs) of μ and λ can be obtained as follows.

$$\widetilde{\mu} = A - B\widetilde{\sigma}, \tag{9}$$

and $\widehat{\sigma}$ can be obtained as the positive solution of

$$m\sigma^2 + D\sigma - E = 0. \tag{10}$$

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Here

$$A = \frac{\sum_{i=1}^{m}(R_i+1)\beta_i y_{i:m:n}}{\sum_{i=1}^{m}(R_i+1)\beta_i}$$

$$B = \frac{\sum_{i=1}^{m}\alpha_i - \sum_{i=1}^{m}R_i(1-\alpha_i)}{\sum_{i=1}^{m}(R_i+1)\beta_i}$$

$$D = \sum_{i=1}^{m}\alpha_i(y_{i:m:n}-A) - \sum_{i=1}^{m}R_i(1-\alpha_i)(y_{i:m:n}-A) - 2B\left(\sum_{i=1}^{m}(R_i+1)\beta_i(y_{i:m:n}-A)\right)$$

$$E = \sum_{i=1}^{m}(R_i+1)\beta_i(y_{i:m:n}-A)^2,$$

here

$$y_{i:m:n} = \ln x_{i:m:n}, \qquad p_i = \frac{i}{n+1}, \qquad q_i = 1-p_i, \qquad \alpha_1 = 1+\ln q_i(1-\ln(-\ln q_i)), \qquad \beta_1 = -\ln q_i.$$

We propose the following AMLEs of α, λ_1 and λ_2 as follows. First obtain $\widetilde{\sigma} > 0$ as the solution of (10), then $\widetilde{\alpha} = \frac{1}{\widetilde{\sigma}}$ is a AMLE of α. Now compute $\widetilde{\lambda}_1 = \widehat{\lambda}_1(\widetilde{\alpha})$ and $\widetilde{\lambda}_2 = \widehat{\lambda}_2(\widetilde{\alpha})$ from (4) as the AMLEs of λ_1 and λ_2 respectively.

5. Fisher information matrix

In this section, we compute the observed and expected Fisher information matrices using the idea of missing information principle of Louis (1982), see also Ng et al. (2002). The observed Fisher information matrix can be used to construct the asymptotic confidence intervals, where the expected Fisher information matrix can be used for constructing optimal censoring plans.

5.1. Observed information matrix

In this section, we compute the observed Fisher information matrix given the observation $\{(x_{1:m:n}, \delta_1), \ldots, (x_{m:m:n}, \delta_m)\}$. The idea of the missing information principle of Louis (1982) can be expressed as follows;

Observed information = Complete information − Missing information.

Let us use the following notations: $\theta = (\alpha, \lambda_1, \lambda_2)$, X = the observed data, W = the complete data, $I_W(\theta)$ = the complete information, $I_X(\theta)$ = the observed information and $I_{W|X}(\theta)$ = the missing information, see Ng et al. (2002) also in this connection. From Louis (1982) it follows that

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta).$$

The complete information $I_W(\theta)$ is given be

$$I_W(\theta) = -E_W\left[\frac{\partial^2 l_c(W:\theta)}{\partial\theta^2}\right],$$

where $l_c(W:\theta)$ is the log-likelihood function of the complete data.

If the Fisher information matrix in one observation which is censored at the kth failure time $x_{k:m:n}$ is $I_{W|X}^{(k)}(\theta)$, then

$$I_{W|X}(\theta) = \sum_{k=1}^{m} R_j I_{W|X}^{(k)}(\theta),$$

see for example Ng et al. (2002).

Both the matrices $I_W(\theta)$ and $I_{W|X}^{(k)}(\theta)$ are of the order 3×3. We present all the elements of both the matrices. Let us denote $I_W(\theta) = (a_{ij})$; $i, j = 1, 2, 3$, and $I_{W|X}^{(k)}(\theta) = (b_{ij}^{(k)})$, $i, j = 1, 2, 3$. After some calculations, it can be observed that

$$a_{11} = \frac{n}{\alpha^2}\left[\frac{1}{6}\pi^2 + 1 - \gamma - (\ln(\lambda_1+\lambda_2))^2\right]$$

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$$a_{22} = \frac{n}{\lambda_1(\lambda_1+\lambda_2)}, \qquad a_{33} = \frac{n}{\lambda_2(\lambda_1+\lambda_2)}$$
$$a_{12} = a_{21} = a_{13} = a_{31} = \frac{n}{\alpha(\lambda_1+\lambda_2)}\left[1-\gamma-\ln(\lambda_1+\lambda_2)\right]$$
$$a_{23} = a_{32} = 0,$$

here $\gamma = 0.57721$ is the Euler's constant. Similarly, as in Ng et al. (2002), it can be easily observed that

$$b_{11}^{(k)} = \frac{1}{\alpha^2}\left[1 + \frac{(\lambda_1+\lambda_2)^2}{(1-e^{-(\lambda+\lambda_2)x_{k:m:n}^{\alpha}})}\int_{x_{k:m:n}^{\alpha}}^{\infty} y(\ln y)^2 e^{-(\lambda_1+\lambda_2)y}dy + (\lambda_1+\lambda_2)x_{k:m:n}^{\alpha}(\ln x_{k:m:n}^{\alpha})^2\right]$$
$$b_{22}^{(k)} = b_{33}^{(k)} = b_{23}^{(k)} = b_{32}^{(k)} = \frac{1}{(\lambda_1+\lambda_2)^2}$$
$$b_{12}^{(k)} = b_{21}^{(k)} = b_{13}^{(k)} = b_{31}^{(k)} = \frac{(\lambda_1+\lambda_2)}{\alpha(1-e^{-(\lambda+\lambda_2)x_{k:m:n}^{\alpha}})}\int_{x_{k:m:n}^{\alpha}}^{\infty} y(\ln y)e^{-(\lambda_1+\lambda_2)y}dy.$$

5.2. Expected Fisher information matrix

In this subsection, we provide the expected Fisher information matrix for progressively censored competing risks data using the similar idea as in Ng et al. (2002). It may be noted that, because of the structure of the log-likelihood function (3), to compute the expected Fisher information we just need to know the probability density function of $X_{j:m:n}$ for $j = 1, \ldots, m$ and the distribution of m_1 and m_2, as provided in Section 3.

The probability density function of $X_{j:m:n}$ for $j = 1, \ldots, m$ is

$$f_{X_{j:m:n}}(x) = \alpha(\lambda_1+\lambda_2)c_{j-1}\sum_{i=1}^{j} a_{i,j}e^{-(\lambda_1+\lambda_2)r_i x^{\alpha}}x^{\alpha-1}; \quad x > 0, \tag{11}$$

where

$$r_j = m - j + 1 + \sum_{i=j}^{m} R_i, \qquad c_{j-1} = \prod_{i=1}^{j} r_i, \qquad a_{1,1} = 1, \qquad a_{i,j} = \prod_{k=1,k\neq i}^{j}\frac{1}{r_k - r_i}.$$

Based on (11), the expected Fisher information matrix can be obtained. If we denote the 3×3 matrix I as follows

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33}\end{bmatrix} = -E\begin{bmatrix} \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\alpha^2} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\alpha\partial\lambda_1} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\alpha\partial\lambda_2} \\ \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_1\partial\alpha} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_1^2} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_1\partial\lambda_2} \\ \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_2\partial\alpha} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 \ln l(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_2^2}\end{bmatrix}, \tag{12}$$

then

$$I_{11} = \frac{m}{\alpha^2} + \frac{(\lambda_1+\lambda_2)}{\alpha^2}\sum_{j=1}^{m}(R_j+1)h_{1j}$$
$$I_{22} = \frac{1}{\lambda_1(\lambda_1+\lambda_2)}, \qquad I_{33} = \frac{1}{\lambda_2(\lambda_1+\lambda_2)}$$
$$I_{23} = I_{32} = 0$$
$$I_{12} = I_{21} = I_{13} = I_{31} = \frac{1}{\alpha}\sum_{j=1}^{m}(R_j+1)h_{2j},$$

where

$$h_{1j} = \frac{c_{j-1}}{(\lambda_1+\lambda_2)}\sum_{i=1}^{j}\frac{a_{i,j}}{r_i^2}\left[\frac{\pi^2}{6} - 2\gamma + \gamma^2 - 2(1-\gamma)\ln\{r_i(\lambda_1+\lambda_2)\} + (\ln\{r_i(\lambda_1+\lambda_2)\})^2\right]$$
$$h_{2j} = \frac{c_{j-1}}{(\lambda_1+\lambda_2)}\sum_{i=1}^{j}\frac{a_{i,j}}{r_i^2}\left[1-\gamma-\ln(r_i(\lambda_1+\lambda_2))\right],$$

see also Balakrishnan et al. (2008) in this connection for some of the similar derivations.

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Table 1
The average relative estimates of α, the average estimates of λ_1, λ_2 and their mean squared errors (within brackets) for different censoring schemes are reported. $\alpha = 2$, $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$.

Scheme	Method	α	λ_1	λ_2
[1]	MLE	1.1553 (0.1474)	0.6770 (0.1689)	0.4571 (0.1347)
	AMLE	1.1724 (0.1569)	0.6800 (0.1933)	0.4589 (0.1424)
[2]	MLE	1.1433 (0.1056)	0.7266 (0.1517)	0.4859 (0.0876)
	AMLE	1.1552 (0.1117)	0.7333 (0.1616)	0.4902 (0.0921)
[3]	MLE	1.1512 (0.1132)	0.7869 (0.2596)	0.5264 (0.1450)
	AMLE	1.1614 (0.1190)	0.7969 (0.2812)	0.5329 (0.1561)
[4]	MLE	1.0543 (0.0337)	0.6361 (0.0297)	0.4177 (0.0157)
	AMLE	1.0605 (0.0348)	0.6366 (0.0300)	0.4180 (0.0159)
[5]	MLE	1.1075 (0.0382)	0.6117 (0.0282)	0.4017 (0.0154)
	AMLE	1.0095 (0.0204)	0.6310 (0.0267)	0.4148 (0.0150)
[6]	MLE	1.0528 (0.0303)	0.6306 (0.0282)	0.4142 (0.0153)
	AMLE	1.0544 (0.0307)	0.6307 (0.0283)	0.4143 (0.0153)
[7]	MLE	1.1254 (0.0419)	0.6134 (0.0619)	0.4089 (0.0597)
	AMLE	1.0456 (0.0240)	0.6208 (0.0571)	0.4139 (0.0610)

6. Numerical experiments

In this section, we present some simulation results to compare the performance of the different methods proposed in the previous section. We mainly compare the performances of the MLEs, AMLEs in terms of their biases and the mean squared errors (MSEs). We also compare the average lengths and the coverage percentages of the asymptotic confidence intervals, based on MLEs and AMLEs. All the computations are performed at the Indian Institute of Technology Kanpur using a Pentium IV processor.

We used different sample sizes n, different effective sample sizes m and the following sets of parameters $\{\alpha = 2.0, \lambda_1 = 0.6, \lambda_2 = 0.4\}$ and $\{\alpha = 2.0, \lambda_1 = 0.8, \lambda_2 = 0.2\}$. In both cases, we considered the following sampling schemes:

- Scheme 1: $n = 15, m = 12, R_1 = \cdots = R_{11} = 0, R_{12} = 3;$
- Scheme 2: $n = 30, m = 15, R_1 = \cdots = R_{14} = 0, R_{15} = 15;$
- Scheme 3: $n = 40, m = 15, R_1 = \cdots = R_{14} = 0, R_{15} = 25;$
- Scheme 4: $n = 40, m = 30, R_1 = \cdots = R_{29} = 0, R_{30} = 10;$
- Scheme 5: $n = 40, m = 30, R_2 = \cdots = R_{30} = 0, R_1 = 10;$
- Scheme 6: $n = 40, m = 30, R_2 = \cdots = R_{29} = 0, R_1 = R_{30} = 5;$
- Scheme 7: $n = 40, m = 30, R_1 = \cdots = R_{14} = R_{17} = \cdots = R_{30} = 0, R_{15} = R_{16} = 5.$

Note that the sampling schemes [1], [2], [3] and [4] are the usual Type-II censoring schemes, *i.e.* $n - m$ items are removed at the time of the mth failure. The sampling scheme [5] is just the opposite of the Type-II censoring scheme and, in this case, $n - m$ items are removed at the time of the first failure itself. This particular progressive censoring scheme is known as first step-censoring and this is a particular case of one-step censoring introduced by Balakrishnan et al. (2008). It is well known that for fixed n and m, the expected experimental time of the Type-II censoring scheme is less than the corresponding first step-censoring scheme. In fact, for fixed n and m, the expected experimental time of any other censoring schemes are always between these two extremes. For example, the expected experimental time of the sampling scheme [6] and [7] will be between the schemes [4] and [5].

For generating the progressively censored Weibull samples, we use the algorithm suggested in Balakrishnan and Sandhu (1995). For each data point, we assigned the cause of failure as 1 or 2 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$ respectively. In each case, we calculate the MLEs and AMLEs and also the asymptotic confidence bounds based on the observed Fisher information matrix. We replicate the process 1000 times and compute the average biases, standard deviations of the different estimates. We also compute 95% confidence intervals using the observed Fisher information matrix, replacing the parameters by MLEs and AMLEs. We report the average confidence lengths and the coverage percentages over 1000 replications. The results are reported in Tables 1–4.

Some of the points are quite clear from Tables 1–4. It is observed that the MLEs and AMLEs behave almost in a similar manner in all the cases considered. Therefore, for all practical purposes, AMLEs can definitely be used in place of MLEs. Comparing Scheme [2] and Scheme [3], it is observed that for fixed m as n increases in case of Type-II censoring scheme, the biases, mean squared errors, and the average confidence lengths increase for all the estimates. Comparing Scheme [3] and Scheme [4], it is observed that for fixed n as m increases in case of Type-II censoring scheme, the biases, mean squared errors and the average confidence lengths decrease for all the estimates.

7. Data analysis

In this section, we re-analyze one data set which was originally analyzed by Hoel (1972) and later by Kundu et al. (2004). The data was obtained from a laboratory experiment in which male mice received a radiation dose of 300 roentgens at 35

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Table 2
The average 95% confidence lengths and the corresponding coverage percentages (within brackets) based on MLEs and AMLEs for different censoring schemes are reported. $\alpha = 2, \lambda_1 = 0.6$ and $\lambda_2 = 0.4$.

Scheme	Method	α	λ_1	λ_2
[1]	MLE	1.1452 (0.95)	1.0378 (0.93)	0.8296 (0.92)
	AMLE	1.1294 (0.95)	1.0296 (0.94)	0.8244 (0.92)
[2]	MLE	1.0838 (0.96)	1.1362 (0.95)	0.8810 (0.95)
	AMLE	1.0730 (0.96)	1.1217 (0.95)	0.8708 (0.95)
[3]	MLE	1.1141 (0.96)	1.4487 (0.95)	1.0872 (0.95)
	AMLE	1.1045 (0.96)	1.4220 (0.95)	1.0684 (0.95)
[4]	MLE	0.6640 (0.95)	0.5923 (0.95)	0.4760 (0.95)
	AMLE	0.6604 (0.96)	0.5919 (0.95)	0.4757 (0.95)
[5]	MLE	0.5548 (0.95)	0.5935 (0.96)	0.4758 (0.95)
	AMLE	0.5965 (0.94)	0.5817 (0.93)	0.4647 (0.94)
[6]	MLE	0.6282 (0.95)	0.5853 (0.95)	0.4710 (0.95)
	AMLE	0.6270 (0.95)	0.5851 (0.95)	0.4709 (0.95)
[7]	MLE	0.5621 (0.95)	0.5771 (0.95)	0.4681 (0.94)
	AMLE	0.5879 (0.94)	0.5726 (0.93)	0.4638 (0.93)

Table 3
The average relative estimates of α, the average estimates of λ_1, λ_2 and their mean squared errors (within brackets) for different censoring schemes are reported. $\alpha = 2, \lambda_1 = 0.8$ and $\lambda_2 = 0.2$.

Scheme	Method	α	λ_1	λ_2
[1]	MLE	1.1554 (0.1453)	0.8851 (0.2194)	0.2450 (0.0289)
	AMLE	1.1726 (0.1549)	0.8890 (0.2591)	0.2459 (0.0308)
[2]	MLE	1.1416 (0.1062)	0.9592 (0.2237)	0.2513 (0.0343)
	AMLE	1.1534 (0.1123)	0.9678 (0.2385)	0.2535 (0.0358)
[3]	MLE	1.1493 (0.1138)	1.0386 (0.4097)	0.2717 (0.0516)
	AMLE	1.1595 (0.1197)	1.0517 (0.4473)	0.2751 (0.0549)
[4]	MLE	1.0545 (0.0337)	0.8423 (0.0397)	0.2113 (0.0077)
	AMLE	1.0606 (0.0348)	0.8429 (0.0402)	0.2115 (0.0078)
[5]	MLE	1.1076 (0.0382)	0.8099 (0.0384)	0.2032 (0.0074)
	AMLE	1.0096 (0.0204)	0.8357 (0.0351)	0.2099 (0.0075)
[6]	MLE	1.0529 (0.0303)	0.8351 (0.0374)	0.2095 (0.0076)
	AMLE	1.0546 (0.0307)	0.8353 (0.0376)	0.2096 (0.0076)
[7]	MLE	1.1254 (0.0418)	0.8196 (0.0391)	0.2025 (0.0076)
	AMLE	1.0456 (0.0240)	0.8296 (0.0357)	0.2051 (0.0075)

Table 4
The average 95% confidence lengths and the corresponding coverage percentages (within brackets) based on MLEs and AMLEs for different censoring schemes are reported. $\alpha = 2, \lambda_1 = 0.8$ and $\lambda_2 = 0.2$.

Scheme	Method	α	λ_1	λ_2
[1]	MLE	1.1456 (0.95)	1.2011 (0.95)	0.5901 (0.95)
	AMLE	1.1297 (0.95)	1.1910 (0.95)	0.5871 (0.95)
[2]	MLE	1.0820 (0.96)	1.3618 (0.97)	0.5906 (0.95)
	AMLE	1.0713 (0.96)	1.3440 (0.97)	0.5845 (0.95)
[3]	MLE	1.1122 (0.96)	1.7845 (0.96)	0.6957 (0.95)
	AMLE	1.1027 (0.96)	1.7500 (0.96)	0.6849 (0.95)
[4]	MLE	0.6640 (0.95)	0.6853 (0.95)	0.3337 (0.94)
	AMLE	0.6605 (0.96)	0.6847 (0.95)	0.3334 (0.94)
[5]	MLE	0.5550 (0.95)	0.6890 (0.95)	0.3323 (0.93)
	AMLE	0.5967 (0.92)	0.6776 (0.92)	0.3233 (0.92)
[6]	MLE	0.6283 (0.95)	0.6767 (0.95)	0.3303 (0.93)
	AMLE	0.6271 (0.96)	0.6766 (0.95)	0.3303 (0.93)
[7]	MLE	0.5621 (0.95)	0.6700 (0.95)	0.3247 (0.92)
	AMLE	0.5879 (0.93)	0.6657 (0.94)	0.3212 (0.92)

days to 42 days (5–6 weeks) of age. The cause of death for each mouse was determined by autopsy to be thymine lymphoma, reticulum cell sarcoma, or other causes. For the purpose of analysis, we consider reticulum cell sarcoma as cause 1 and combine the other causes of death as cause 2. There were $n = 77$ observations in the data. Kundu et al. (2004) generated a progressively Type-II censored sample from the original data with $m = 25$ and censoring scheme $R_1 = R_2 = \cdots = R_{24} = 2$, $R_{25} = 4$. The progressively Type-II censored sample used by Kundu et al. (2004) is (40, 2), (42, 2), (62, 2), (163, 2), (179, 2),

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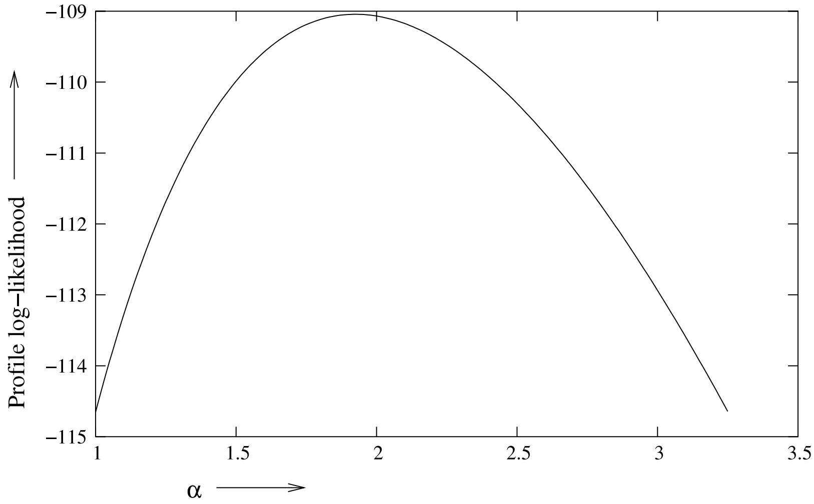

Fig. 1. Profile log-likelihood function (5).

(206, 2), (222, 2), (228, 2), (252, 2), (259, 2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (517, 2), (517, 2), (524, 2), (525, 1), (558, 1), (536, 1), (605, 1), (612, 1), (620, 2), (621, 1).

There were $n_1 = 7$ deaths due to cause 1 and $n_2 = 18$ deaths due to cause 2. Progressive censoring in these kinds of experiments may be invaluable in obtaining information on growths of tumors in the mice. At the time of death of a particular mouse, other mice may be randomly selected and removed from the study. Autopsies on these mice may lead to information on the progression of the cancer over time.

Kundu et al. (2004) analyzed the above data set under the assumption that the lifetime distributions of the individual causes are independent and exponential distributed random variables. In this paper, it is assumed that the lifetime distributions of the individual causes are independent and Weibull distributed random variables with the same shape parameter but different scale parameters. Now we would like to compute the MLEs and AMLEs of the unknown parameters. Before going to compute the MLEs, we plot the profile log-likelihood function (5) in Fig. 1. From the Fig. 1 it is clear that the profile log-likelihood function is unimodal and the MLE of α is close to 2. We start the iteration to solve the fixed point type Eq. (6) with $\alpha = 2$, and obtain the $\widehat{\alpha} = 1.9246$, $\widehat{\lambda}_1 = 8.1102 \times 10^{-7}\widehat{\lambda}_1$ and $= 2.0855 \times 10^{-6}$. The corresponding 95% confidence intervals are (1.2709, 2.5783), $(1.6799 \times 10^{-7}, 1.4541 \times 10^{-6})$, $(1.0450 \times 10^{-6}, 3.1259 \times 10^{-6})$ respectively. We compute the AMLEs of α, λ_1 and λ_2 and they are as follows; $\widetilde{\alpha} = 1.9338$, $\widetilde{\lambda}_1 = 7.6603 \times 10^{-7}$ and 1.9698×10^{-6}. The corresponding 95% confidence intervals, obtained using the observed information matrix, are (1.2772, 2.5904) $(1.5867 \times 10^{-7}, 1.3734 \times 10^{-6})$, $(9.8707 \times 10^{-7}, 2.9525 \times 10^{-6})$, respectively. Therefore, both the MLEs and AMLEs suggest that the Weibull distribution should be used rather than the exponential distribution.

8. Optimum censoring scheme

So far, we have discussed the inferences of the unknown parameters for a given progressively censored scheme when the latent failure lifetimes are Weibull-distributed random variables. So the natural question arises as to how to choose the progressive censoring scheme. Should we choose a particular scheme just based on convenience or based on some scientific criterion? Recently, choosing the optimal censoring scheme in different related problems has received considerable attention, see for example Balakrishnan and Aggarwala (2000), Burkschat et al. (2006), Burkschat et al. (2007), Burkschat (2008), Balakrishnan et al. (2008), Zhang and Meeker (2005), Ng et al. (2004), Wang and Yu (2009), Kundu (2007), Kundu (2008) and Pradhan and Kundu (in press) and the references cited there.

In practice, it is very important to choose the *optimum* censoring scheme from a class of possible schemes. Here, possible schemes mean, for fixed sample size n and for fixed effective sample size m, the different choices of $\{R_1, \ldots, R_m\}$, such that

$$\sum_{i=1}^{m} R_i + m = n. \tag{13}$$

In most of the practical situations, the experimenter may not have any choice on m and n, but he/she can choose a particular $\{R_1, \ldots, R_m\}$ satisfying (13). Therefore, the problem is to choose that particular censoring scheme $\{R_1, \ldots, R_m\}$, for fixed m and n, which is *optimal* in the sense that it provides the maximum information of the unknown parameters. Now, when comparing two different progressive censoring schemes, say $P_1 = \{R_1^1, \ldots, R_m^1\}$ and $P_2 = \{R_1^2, \ldots, R_m^2\}$, where $\sum_{i=1}^{m} R_i^1 + m = n = \sum_{i=1}^{m} R_i^2 + m = n$, P_1 is said to be better than P_2 if P_1 provides more information about the unknown parameters than P_2.

Immediately the first question arises, as to how to define the 'information measure' of the unknown parameters of a particular censoring scheme. In this respect, comparison of the Fisher information matrices seems to be a reasonable choice.

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In presence of only one unknown parameter, the choice is quite clear. It simply boils down to comparing two real non-negative numbers. But if more than one parameter is unknown, then comparison of the two Fisher information matrices is not that trivial. Some of the existing choices are to compare the traces or the determinants of the two Fisher information matrices.

Alternatively, the information measures of the two different schemes can be compared by the precisions of the 100pth quantile estimators, *i.e.* to compare the variances of the corresponding estimators for different schemes. Similar ideas were used by Zhang and Meeker (2005) in the Bayesian set-up, and also by Ng et al. (2004) in the frequentest context. Interestingly, this information measure is independent of the scale parameter, but unfortunately it depends on 'p'. Zhang and Meeker (2005) and Ng et al. (2004) proposed some specific choices of 'p' based on some practical consideration, which may be $p = 0.95$ or $p = 0.99$, but they purely depend on the partitioner.

In this paper we consider the following information measure. The pth percentile points of the two latent life time distributions are

$$T_{p,1} = \left[-\frac{1}{\lambda_1}\ln(1-p)\right]^{\frac{1}{\alpha}} \quad \text{and} \quad T_{p,2} = \left[-\frac{1}{\lambda_2}\ln(1-p)\right]^{\frac{1}{\alpha}} \tag{14}$$

for Cause 1 and Cause 2, respectively. Our criteria are based on estimating $\ln T_{p,1}$ and $\ln T_{p,2}$, as was used by Zhang and Meeker (2005). For fixed $0 \le w \le 1$, our Criterion 1 is;

$$C_1(P) = w\text{Var}(\ln \widehat{T}_{p,1}) + (1-w)\text{Var}(\ln \widehat{T}_{p,2}), \tag{15}$$

where $P = \{R_1, \ldots, R_m\}$ denotes the censoring plan and $\text{Var}(\ln \widehat{T}_{p,1})$ and $\text{Var}(\ln \widehat{T}_{p,2})$ denote the asymptotic variances of the MLEs of $\ln T_{p,1}$ and $\ln T_{p,2}$ respectively. Note that the weight function w depends on the practitioner. Depending on the relative importance of Cause 1 and Cause 2, w should be chosen. If both the Causes are equally important, then naturally $w = \frac{1}{2}$. Now, according to Criterion 1, for a given p and w, the scheme P_1 is better than P_2 if $C_1(P_1) < C_1(P_2)$.

One drawback of Criterion 1 is that it is a function of the quantile point p. Using the idea of Gupta and Kundu (2006) and Kundu (2008), we propose the following criterion, which is independent of p. We propose the following Criterion 2;

$$C_2(P) = w\int_0^1 \text{Var}(\ln \widehat{T}_{p,1})\mathrm{d}W_1(p) + (1-w)\int_0^1 \text{Var}(\ln \widehat{T}_{p,2})\mathrm{d}W_2(p), \tag{16}$$

where the censoring plan P, the weight w, $\text{Var}(\ln \widehat{T}_{p,1})$ and $\text{Var}(\ln \widehat{T}_{p,2})$ are the same as before. In this case $W_1(p) \ge 0$ and $W_2(p) \ge 0$ are two non-negative weight functions defined on $[0, 1]$ and they must satisfy

$$\int_0^1 \mathrm{d}W_1(p) = \int_0^1 \mathrm{d}W_2(p) = 1. \tag{17}$$

The two weight functions $W_1(\cdot)$ and $W_2(\cdot)$ have to be decided beforehand, depending on the problem. For example, if someone is interested in giving more stress at the center, then more weight should be given at $p = 0.5$. On the other hand, if tail probabilities are more important, then proper weights can be given for large p. Moreover, w also has to be chosen beforehand, similarly as in Criterion 1. In this case, also similarly as before, P_1 is better than P_2, if $C_2(P_1) < C_2(P_2)$. Observe that Criterion 1 is a special case of Criterion 2.

Note that, both for Criterion 1 and Criterion 2, we need to compute $\text{Var}(\ln \widehat{T}_{p,1})$ and $\text{Var}(\ln \widehat{T}_{p,2})$ and they can be obtained as follows;

$$\text{Var}(\ln \widehat{T}_{p,1}) = \begin{bmatrix} \frac{1}{\alpha^2}[\ln\lambda_1 - \psi] & -\frac{1}{\alpha\lambda_1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2}[\ln\lambda_1 - \psi] \\ -\frac{1}{\alpha\lambda_1} \end{bmatrix} \tag{18}$$

and

$$\text{Var}(\ln \widehat{T}_{p,2}) = \begin{bmatrix} \frac{1}{\alpha^2}[\ln\lambda_2 - \psi] & -\frac{1}{\alpha\lambda_2} \end{bmatrix} \begin{bmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2}[\ln\lambda_2 - \psi] \\ -\frac{1}{\alpha\lambda_2} \end{bmatrix} \tag{19}$$

respectively. Here $\psi = \ln(-\ln(1-p))$ and V_{ij} is the (i, j)th element of the matrix V and $V = E^{-1}$.

Now for illustrative purposes, we present the optimal sampling scheme for different objective functions (OF) for selected combinations of m, n, α, λ_1 and λ_2. We have used the minimum trace criterion (OF-1), minimum determinant criterion (OF-2) and the minimum sum of the variances of the pth percentile estimators for different choices of p, namely $p = 0.1$ (OF-3), $p = 0.99$ (OF-4). Finally, we have also computed the optimal sampling scheme based on the minimum sum of the weighted variances (OF-5) under the assumption that $W_1(p) = W_2(p) = 1$ for all $0 \le p \le 1$. In calculating OF-3 to OF-5, it is also assumed that $w = \frac{1}{2}$.

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Table 5
The optimal censoring scheme for different objective functions when $\alpha = 2$, $\lambda_1 = 0.6$, $\lambda_2 = 0.4$, $m = 5$ and $n = 10, 15, 20, 25$ and 30. The relative efficiency (RE) and the relative expected time (RT) of Type-II censoring scheme with respect to the optimum censoring scheme are reported.

OF	n	R_1	R_2	R_3	R_4	R_5	RE (%)	RT (%)
1	10	0	5	0	0	0	97.0	56.9
	15	0	10	0	0	0	97.4	45.1
	20	0	15	0	0	0	98.2	38.6
	25	0	20	0	0	0	98.8	34.4
	30	0	0	25	0	0	99.2	34.5
2	10	5	0	0	0	0	93.7	54.9
	15	0	10	0	0	0	95.1	45.1
	20	0	15	0	0	0	96.8	38.6
	25	0	0	20	0	0	98.1	37.8
	30	0	0	25	0	0	98.8	34.5
3	10	0	0	0	0	5	100.0	100.0
	15	0	0	0	0	10	100.0	100.0
	20	0	0	0	0	15	100.0	100.0
	25	0	0	0	0	20	100.0	100.0
	30	0	0	0	1	24	99.9	55.4
4	10	5	0	0	0	0	89.4	54.9
	15	10	0	0	0	0	90.0	43.0
	20	15	0	0	0	0	91.3	36.7
	25	20	0	0	0	0	92.5	32.5
	30	25	0	0	0	0	93.4	29.5
5	10	5	0	0	0	0	98.0	54.9
	15	10	0	0	0	0	98.0	43.0
	20	15	0	0	0	0	98.4	36.7
	25	20	0	0	0	0	98.7	32.5
	30	25	0	0	0	0	98.9	29.5

Table 6
The optimal censoring scheme for different objective functions when $\alpha = 2$, $\lambda_1 = 0.6$, $\lambda_2 = 0.4$, $n = 15$, and $m = 6$, 8 and 10. The relative efficiency (RE) and the relative expected time (RT) of Type-II censoring scheme with respect to the optimum censoring scheme are reported.

$m = 6$

OF	R_1	R_2	R_3	R_4	R_5	R_6	RE (%)	RT (%)
1	0	9	0	0	0	0	97.5	47.6
2	0	9	0	0	0	0	94.7	47.6
3	0	0	0	0	0	9	100.0	100.0
4	9	0	0	0	0	0	90.0	46.1
5	9	0	0	0	0	0	98.2	46.1

$m = 8$

OF	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	RE (%)	RT (%)
1	0	7	0	0	0	0	0	0	98.1	53.5
2	0	7	0	0	0	0	0	0	94.9	53.5
3	0	0	0	0	0	0	0	7	100.0	100.0
4	7	0	0	0	0	0	0	0	91.9	52.7
5	0	7	0	0	0	0	0	0	98.8	53.5

$m = 10$

OF	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	RE (%)	RT (%)
1	0	0	5	0	0	0	0	0	0	0	98.8	61.2
2	0	5	0	0	0	0	0	0	0	0	95.9	60.7
3	0	0	0	0	0	0	0	0	0	5	100.0	100.0
4	5	0	0	0	0	0	0	0	0	0	94.1	60.2
5	0	2	3	0	0	0	0	0	0	0	99.4	61.0

Note that, in all cases, the optimization has to be performed numerically. They are discrete optimization problems. For given n, m, α, λ_1 and λ_2, the optimum censoring scheme with respect to a given criterion, can be found by exhaustive search for all possible R_i values satisfying (13). The results are reported in Tables 5 and 6.

Since Type-II censoring scheme is a particular choice of the general progressive censoring scheme and it is one of the most popular censoring schemes, the natural question is how Type-II censoring scheme behaves compared to the optimum censoring scheme. In Tables 5 and 6 we also report the relative efficiency of the Type-II censoring scheme with respect to the optimum censoring scheme for different objective functions, and we also report the relative expected experimental time of the Type-II progressive censoring scheme with respect to the corresponding expected experimental time of the optimum censoring scheme.

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It is interesting to observe that in most of the cases, OF-1 and OF-2 behave in a very similar fashion. Likewise, OF-4 and OF-5 are also very similar in nature. OF-3 and OF-4 behave just in the opposite manner, as expected. Comparing the optimal censoring scheme with the Type-II censoring scheme, it is observed that, for a Type-II censoring scheme, the relative efficiency is more than 90% in all the cases considered, where, as for most of the cases, the relative expected time taken by the Type-II censoring scheme is significantly smaller than the corresponding relative expected time taken by the optimum censoring scheme. That clearly justifies the overwhelming popularity of the Type-II censoring scheme.

One point should be noted; that the *optimum* censoring scheme depends on the parameter values. So one natural question is how to implement the *optimum* censoring scheme in practice. It is clear that one needs to have some idea about the range of the parameter values, from prior experience. If the prior information is not available, some pilot survey may be carried out to get a rough idea about the parameter values. Of course, parameter misspecification is an important issue, which has not been addressed here.

Moreover, although the total number of sampling schemes are finite, they can be quite large. For fixed m and n, a total $\binom{n-1}{m-1}$ of possible progressive censoring schemes are available. For example, when $n = 25$ and $m = 12$, then the possible number of censoring schemes is $\binom{24}{11} = 2496\,144$, which is quite large. So far, we do not have any efficient algorithm to find the optimal censoring scheme in this case. We propose the following sub-optimal censoring scheme, which can be obtained very efficiently. Note that for fixed n and m all the progressive censoring schemes $\{R_1, \ldots, R_m\}$, such that $R_1 + \cdots + R_m = n - m$, will belong to the convex hull generated by the points $(n-m, 0, \ldots, 0) \ldots (0, \ldots, 0, n-m)$. Therefore, a sub-optimal censoring scheme can be obtained by choosing the optimal censoring scheme among these extreme points on the convex hull. Interestingly, it is observed that, for most of the criteria, the optimal censoring scheme belongs to the extreme points. We could not provide any theoretical justification of this at present. More work is needed in this direction.

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Appendix

Proof of Lemma 1. Let us denote

$$v(\alpha) = \sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha}, \tag{20}$$

therefore,

$$v'(\alpha) = \sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha} \ln x_{i:m:n} \quad \text{and} \quad v''(\alpha) = \sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha} \left(\ln x_{i:m:n}\right)^2 .$$

Observe that

$$\left(\sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha} \left(\ln x_{i:m:n}\right)^2\right) \left(\sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha}\right) - \left(\sum_{i=1}^{m} (R_i + 1) x_{i:m:n}^{\alpha} \ln x_{i:m:n}\right)^2$$
$$= \sum_{1 \le i < j \le m} (R_i + 1) \left(R_j + 1\right) \left(\ln x_{i:m:n} - \ln x_{j:m:n}\right)^2 \ge 0.$$

Therefore,

$$v''(\alpha) v(\alpha) \ge \left(v'(\alpha)\right)^2 , \tag{21}$$

and it implies that the second derivative of $\ln p(\alpha)$ is negative. See, Balakrishnan and Kateri (2008) also, for a similar result on complete or Type-II censored data. ■

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