Generalized Logistic Distributions

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Abstract

In this paper we discuss different properties of the two generalizations of the logistic distributions, which can be used to model the data exhibiting a unimodal density having some skewness present. The first generalization is carried out using the basic idea of Azzalini [2] and we call it as the skew logistic distribution. It is observed that the density function of the skew logistic distribution is always unimodal and log-concave in nature. But the distribution function, failure rate function and different moments can not be obtained in explicit forms and therefore it becomes quite difficult to use it in practice. The second generalization we propose as a proportional reversed hazard family with the base line distribution as the logistic distribution. It is also known in the literature as the Type-I generalized logistic distribution. The density function of the proportional reversed hazard logistic distribution may be asymmetric but it is always unimodal and log-concave. The distribution function, hazard function are in compact forms and the different moments can be obtained in terms of the ψ function and its derivatives. We have proposed different estimators and performed one data analysis for illustrative purposes.

Key Words and Phrases: Heavy tail distribution; L-Moment estimator; Log-concave density function; Maximum likelihood estimator; Proportional reversed hazard; Skew distribution.

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1 Introduction

The random variable X has the logistic distribution if it has the following cumulative distribution function (CDF);

$$F(x;\mu,\sigma) = \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}}; \quad -\infty < x < \infty, \tag{1}$$

for any arbitrary location parameter μ and for the scale parameter $\sigma > 0$. The probability density function (PDF) corresponding to the CDF (1) is

$$f(x;\mu,\sigma) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma\left(1+e^{-\frac{x-\mu}{\sigma}}\right)^2}; \quad -\infty < x < \infty. \tag{2}$$

Clearly, the PDF given in (2) is symmetric about the location parameter μ. From now on the logistic distribution with the PDF given in (2) will be denoted as $L(\mu,\sigma)$.

In this paper, we mainly consider two different generalizations of the logistic distribution by introducing skewness parameters. It may be mentioned that although several skewed distribution functions exist on the positive real axis, but not many skewed distributions are available on the whole real line, which are easy to use for data analysis purpose. The main idea is to introduce the skewness parameter, so that the generalized logistic distribution can be used to model data exhibiting a unimodal density function having some skewness present in the data, a feature which is very common in practice.

Recently, skewed distributions have played an important role in the statistical literature since the pioneering work of Azzalini [2]. He has provided a methodology to introduce skewness in a normal distribution. Since then a number of papers appeared in this area, see for example the monograph by Genton [7] for some recent references.

The first generalization is carried out using the idea of Azzalini [2] and we name it as the skew logistic distribution. It is observed that using the same basic principle of Azzalini

[2], the skewness can be easily introduced to the logistic distribution. It has location, scale and skewness parameters. It is observed that the PDF of the skew logistic distribution can have different shapes with both positive and negative skewness depending on the skewness parameter. It has heavier tails than the skew normal distribution proposed by Azzalini [2]. Although the PDF of the skew logistic distribution is unimodal and log-concave, but the distribution function, failure rate function and the different moments can not be obtained in explicit forms. Moreover, it is observed that even when the location and scale parameters are known, the maximum likelihood estimator of the skewness parameter may not always exist. Due to this problem, it becomes difficult to use this distribution for data analysis purposes.

The second generalization can be viewed as a family of proportional reverse hazard distribution functions, when the base line distribution is the two-parameter logistic distribution. It was originally proposed as a generalization of the logistic distribution by Ahuja and Nash [1], see also Johnson, Kotz and Balakrishnan [9], Balakrishnan and Leung [6], Olapade [10] and the references cited therein. It is also known in the literature as the Type-I generalized logistic distribution, but we prefer to call it as the proportional reversed hazard logistic (PRHL) distribution. The PRHL distribution has also location, scale and skewness parameters. It can be both positively and negatively skewed depending on the skewness parameter, but the PDF is always unimodal and log-concave. In this case, the distribution function, hazard function have explicit forms and moments can be expressed in terms of digamma and polygamma functions. Zelterman [13] showed that if all the three parameters are unknown, then the maximum likelihood estimators do not exist. However, it is observed that if the location parameter is known, then the maximum likelihood estimators of the scale and skewness parameters exist. We propose some alternate estimators and use them for analyzing one real data set.

The rest of the paper is organized as follows. In section 2, we briefly describe some basic properties of the logistic distribution and provide some new interpretations. In section 3, we introduce the skew logistic distribution and in section 4, we describe the PRHL distribution. Estimation of the different parameters are presented in Section 5. For illustrative purposes, one data set is analyzed in section 6. Finally conclusions appear in Section 7.

2 Some Basic Properties of the Logistic Distribution

In this section we briefly describe some of the basic properties of the logistic distribution, which will be used in its generalizations. The logistic distribution has been used in many different fields, for detailed description of the various properties and applications, the readers are referred to the monograph of Balakrishnan [5] or Chapter 23 of Johnson, Kotz and Balakrishnan [9]. In this section, we mainly assume $\mu = 0$ and $\sigma = 1$, *i.e.* the standard logistic distribution and we will denote $F(x; 0, 1) = F(x)$.

Note that the standard logistic distribution can be obtained as

$$f(x; 0, 1) = \int_0^\infty e^{-x} e^{-\alpha e^{-x}} e^{-\alpha} d\alpha, \tag{3}$$

i.e., it is the exponential mixture of extreme value distributions. It is a symmetric distribution around 0 with the points of inflection at $\pm$ 0.53. The shape of the logistic distribution is very similar to that of the normal distribution, but it is more peaked in the center and has heavier tails than the normal distribution. This particular heavy tailed property of the logistic distribution can be used to produce heavy tailed skewed distribution, which is often required for data analysis purposes.

If $X \sim L(0, 1)$ (here $\sim$ means 'has the distribution'), the moment generating function of

X is

$$M_X(t) = \Gamma(1-t)\Gamma(1+t); \qquad -1 < t < 1, \tag{4}$$

the cumulant generating function;

$$Q_X(t) = \ln M_X(t) = \ln \Gamma(1-t) + \ln \Gamma(1+t) \qquad -1 < t < 1. \tag{5}$$

The first four central moments of $L(0,1)$ are $\mu_1 = 0$, $\mu_2 = \frac{\pi^2}{3}$, $\mu_3 = 0$ and $\mu_4 = \frac{7\pi^4}{15}$. The $k-th$ probability weighted moment is

$$\beta_k = E(X(F(X))^k) = \int_{-\infty}^{\infty} x e^{-x}(1+e^{-x})^{-(k+1)-1}, \tag{6}$$

therefore, the first four L-moments, see Hosking [8], are $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 0$ and $\lambda_4 = \frac{1}{6}$.

Interestingly, for $L(0,1)$, the hazard function $h(x)$ and the distribution function $F(x)$ are identical. Therefore, $L(0,1)$ has an increasing failure rate and it increases from 0 to 1. Since it is symmetric about 0, the reversed hazard rate at x is the hazard rate at $-x$. Therefore, the reversed hazard rate $r(x)$ and the survival rate $S(x) = 1 - F(x)$ are identical. Moreover, for all $-\infty < x < \infty$,

$$h(x)r(x) = f(x), \qquad h(x) + r(x) = 1, \quad \text{and} \quad \frac{h(x)}{r(x)} = e^x. \tag{7}$$

The mean residual life function or remaining life expectation at age x is

$$e(x) = E(X - x | X > x) = \frac{\ln F(x)}{F(x)}, \tag{8}$$

and it characterizes the logistic distribution. Another interesting property of the $L(0,1)$ distribution and which is usually used to generate logistic distribution from the uniform generator is the following. If U is a uniform random variable on $(0,1)$, then $\ln\left(\frac{U}{1-U}\right)$ has the $L(0,1)$ distribution. Moreover, it can be easily shown by simple transformation technique that if $G(\cdot)$ is any absolutely continuous distribution function of the random variable Y, then $\ln\left(\frac{G(Y)}{1-G(Y)}\right)$ and $\ln\left(\frac{h_Y(Y)}{r_Y(Y)}\right)$ follow $L(0,1)$, here $h_Y(\cdot)$ and $r_Y(\cdot)$ denote the hazard function and reversed hazard function of Y respectively. Suppose Y_1 and Y_2 are two independent exponential random variables with mean λ_1 and λ_2 respectively, then $\ln\left(\frac{Y_1}{Y_2}\right) \sim L\left(\ln\left(\frac{\lambda_1}{\lambda_2}\right), 1\right)$. When $\lambda_1 = \lambda_2$, then $\ln\left(\frac{Y_1}{Y_2}\right) \sim L(0,1)$. It provides new interpretations of the logistic distribution. We will see later that this particular idea can be used to generalize logistic distribution in various ways.

3 Skew Logistic Distribution

Azzalini [2] proposed the skew normal distribution, which has the following density function;

$$f_{SN}(x;\alpha) = 2\Phi(\alpha x)\phi(x); \qquad -\infty < x < \infty, \tag{9}$$

here α is the skewness parameter, $\phi(.)$ and $\Phi(.)$ are the density function and distribution function of the standard normal random variable. A motivation of the above model has been elegantly exhibited by Arnold *et al.* [3]. Although, Azzalini has been extended the standard normal distribution function to the form (9), but it has been observed that similar method can be applied to any symmetric density function. For example, if $f(\cdot)$ is any symmetric density function defined on $(-\infty,\infty)$ and $F(\cdot)$ is its distribution function, then for any $\alpha \in (-\infty,\infty)$,

$$2F(\alpha x)f(x); \qquad -\infty < x < \infty, \tag{10}$$

is a proper density function and it is skewed if $\alpha \neq 0$. This property has been studied extensively in the literature to study skew-t and skew-Cauchy distributions. Along the same line we define the skew logistic distribution with the skewness parameter α as follows. If a random variable X has the following density function

$$f_1(x;\alpha) = \frac{2e^{-x}}{(1+e^{-x})^2(1+e^{-\alpha x})}; \qquad -\infty < x < \infty, \tag{11}$$

then we say that X has a skew-logistic (SL) distribution with skewness parameter α. For brevity we will denote it by $SL(\alpha)$. Clearly (11) is a proper density function and $\alpha = 0$, corresponds to the standard logistic distribution. Note that the location parameter $\mu \in (-\infty, \infty)$ and the scale parameter $\lambda > 0$ can be easily introduced in (11) as follows;

$$f_1(x; \alpha, \mu, \lambda) = \frac{2\lambda e^{-\lambda(x-\mu)}}{(1 + e^{-\lambda(x-\mu)})^2 (1 + e^{-\alpha\lambda(x-\mu)})}; \quad -\infty < x < \infty, \tag{12}$$

and we will denote it $SL(\alpha, \mu, \lambda)$. Therefore $SL(\alpha, 0, 1) = SL(\alpha)$. Before discussing different properties of $SL(\alpha)$, first let us look at the shapes of the density function of $SL(\alpha)$ for different values of α in Figure 1.

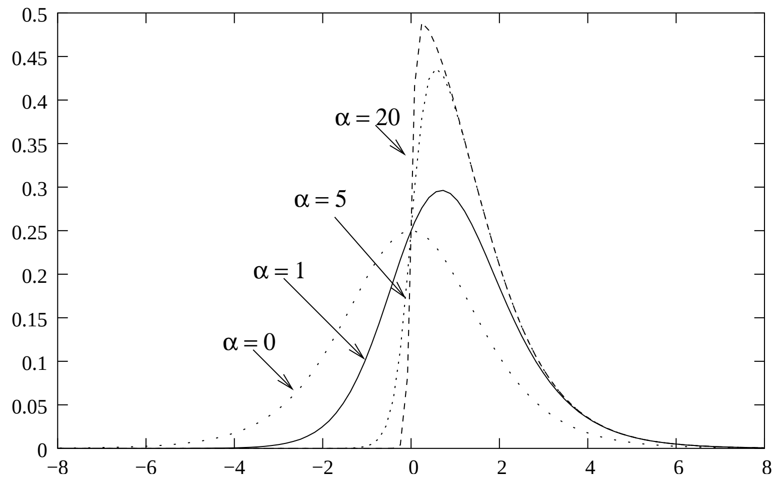

Figure 1: Different density functions of the skew logistic distribution.

From Figure 1 it is clear that $SL(\alpha)$ is positively skewed when α is positive. It takes similar shapes on the negative side for $\alpha < 0$. Therefore, $SL(\alpha)$ can take positive and negative skewness. As α goes to $\pm\infty$, it converges to the half logistic distribution. Comparing with the shapes of the skew normal density function of Azzalini [2], it is clear that $SL(\alpha)$ produces heavy tailed skewed distribution than the skew normal ones. For large values of α, the tail behaviors of the different members of the $SL(\alpha)$ family are very similar, which is apparent from (11) also. It is clear from Figure 1 that the tail behaviors of the different family members of $SL(\alpha)$ are same for large values of α. Some of the properties which are true for skew normal distribution are also true for skew logistic distribution. For example:

- PROPERTY 1: Consider X and Y to be independent and identically distributed (*i.i.d.*) standard logistic distributions with distribution and density functions as $F(\cdot)$ and $f(\cdot)$ respectively. The conditional distribution of X given $Y < \alpha X$ is given by $2F(\alpha x)f(x)$.

- PROPERTY 2: The skew logistic distribution is log-concave and therefore it is unimodal. For a given α, the mode can be obtained as the root of the following equation

$$\frac{\alpha e^{-\alpha}}{1 + e^{-\alpha x}} = \frac{1 - e^{-x}}{1 + e^{-x}}.$$

- PROPERTY 3: The skew logistic random variable has increasing failure rates for all values of α and hence it has decreasing mean residual life.

- PROPERTY 4: The skew logistic random variable has decreasing reversed hazard rate.

- PROPERTY 5: Let $X_{1:m} < \ldots < X_{m:m}$ be the order statistics from a sample of size m from $SL(\alpha)$, then $X_{i:m}$ for $i = i, \ldots, m$ has a log-concave density function.

COMMENTS: It should be mentioned that although we have mentioned these properties for skew logistics distribution, but there is nothing particular about the logistic distribution. The base distribution can be changed to any other symmetric distribution with log-concave density function, then the above properties will still be true.

GENERATION OF SKEW LOGISTIC DISTRIBUTION: Note that the generation of skew logistic random numbers is quite straight forward mainly using the property 1. The following algorithm can be used to generate Z, which has $SL(\alpha)$ distribution, for a given α, from uniform $(0, 1)$ random deviates.

- Step 1: Suppose U_1 and U_2 are two uniform random numbers. Consider $X = -\ln \frac{1 - U_1}{U_1}$ and $Y = -\ln \frac{1 - U_2}{U_2}$.

- Step 2: If $X < \alpha Y$ then $Z = X$, otherwise $Z = -X$.

Some Other Characteristics

It is observed that the mean of $SL(\alpha)$ increases from 0 to 1.3862 as α varies from 0 to ∞. The mode is 0 when α is 0, and then it increases as α increases and it reaches its maximum at $\alpha = 1.665$ and then gradually decreases to 0 as $\alpha \longrightarrow \infty$ The variance is a decreasing function of α. At $\alpha = 0$, the variance is 3.2899 and it decreases to 1.3681 as $\alpha \longrightarrow \infty$. The skewness is an increasing function of α and it increases from 0 to 1.540 as α varies from 0 to ∞. Note that the range of achievable skewness of the skew logistic distribution, namely $(-1.540, 1.540)$ is more than the range of achievable skewness of skew normal distribution, which is $\left(-\frac{\sqrt{(2/\pi)(4/\pi-1)}}{(1-2/\pi)^{3/2}}, \frac{\sqrt{(2/\pi)(4/\pi-1)}}{(1-2/\pi)^{3/2}}\right) \approx (-0.9953, 0.9953)$. Therefore, it is clear that the skew logistic distribution will be more flexible than the skew normal distribution for data analysis purpose.

4 Proportional Reversed Hazard Logistic Distribution

In the previous section we introduced the skew logistic distribution along the same line as Azzalini's skew normal distribution and discussed several properties. Although that is the very natural way of introducing skewness in the logistic distribution, but it is observed that different moments and the distribution function can not be expressed in compact form. Therefore, even if the skew logistic distribution has several desirable theoretical properties, it becomes very difficult to use it for data analysis purposes, particularly numerical problems become quite complicated if the data are censored.

In this section we discuss another generalization of the logistic distribution using the idea of the proportional reversed hazard family. This particular distribution is not completely new in the literature. It is already available and known as the Type-I generalized logistic distribution, see Johnson, Kotz and Balakrishnan [9]. The main aim of this section is to view this distribution as a proportional reversed hazard distribution and discuss its several properties. From now on, we name this distribution as the proportional reversed hazard logistic (PRHL) distribution. It is observed that the PRHL distribution has several advantages over skew logistic distribution. For example, PRHL distribution has explicit distribution function and hazard function. The moments can be obtained in terms of the known ψ functions and its derivatives. Therefore, it is expected that the PRHL distribution can be more useful than the skew logistic distribution for data analysis purposes.

4.1 Background

The following definitions are needed for further development.

Definition: Suppose X is a absolutely continuous random variable with density function and distribution function as $f(\cdot)$ and $F(\cdot)$ respectively, then the reversed hazard function, say $r(\cdot)$ of X can be defined as follows;

$$r(x) = \frac{f(x)}{F(x)}. \tag{13}$$

Definition: Suppose X is a absolutely continuous random variable with CDF $F(\cdot)$ and reversed hazard function $r(\cdot)$. The family of random variables with reversed hazard function of the form $\{\alpha r(\cdot);\ \alpha > 0\}$ is called the proportional reversed hazard family and the CDF $F(\cdot)$ is called the base line CDF of that family.

Therefore, if Y is a member of the reversed hazard family with the base line CDF and PDF as $F(\cdot)$ and $f(\cdot)$, respectively. If the reversed hazard function of Y is $\alpha r(\cdot)$ for some $\alpha > 0$, then the distribution function of Y, say $G(\cdot)$, becomes;

$$G(x) = [F(x)]^{\alpha}, \tag{14}$$

and the corresponding PDF becomes

$$g(x) = \alpha \left[F(x)\right]^{\alpha-1} f(x). \tag{15}$$

4.2 PRHL Distribution

Based on the discussions of the previous section, we can define the PRHL family as the proportional reversed hazard family with the base line distribution as the logistic distribution. Therefore, if a random variable X has the following PDF;

$$f_2(x;\alpha) = \frac{\alpha e^{-x}}{\left(1+e^{-x}\right)^{\alpha+1}}; \quad -\infty < x < \infty, \tag{16}$$

for $\alpha > 0$, then we say X follows proportional reversed hazard logistic (PRHL) distribution with proportionality parameter α. For brevity, we will denote it by PRHL(α). It is clear that (16) is a proper density function and it has the following CDF;

$$F_2(x;\alpha) = \left(1+e^{-x}\right)^{-\alpha}; \quad -\infty < x, < \infty. \tag{17}$$

Its hazard function and the reversed hazard function can be explicitly expressed as

$$h_2(x;\alpha) = \frac{\alpha e^{-x}}{\left(1+e^{-x}\right)^{\alpha+1} - \left(1+e^{-x}\right)}; \quad -\infty < x, < \infty, \tag{18}$$

and

$$r_2(x;\alpha) = \frac{\alpha e^{-x}}{1+e^{-x}}; \quad -\infty < x, < \infty, \tag{19}$$

respectively. In this case also the location parameter $\mu \in (-\infty, \infty)$ and scale parameter $\lambda > 0$ also can be easily introduced in (16) as follows;

$$f_2(x;\alpha,\mu,\lambda) = \frac{\alpha\lambda e^{-\lambda(x-\mu)}}{\left(1+e^{-\lambda(x-\mu)}\right)^{\alpha+1}}; \quad -\infty < x < \infty, \tag{20}$$

and we will denote it by PRHL(α, μ, λ). In this case PRHL(α) = PRHL(α,0,1). In the next subsection we discuss different properties of PRHL(α) which can be easily translated to PRHL(α, μ, λ) very easily.

4.3 Different Properties of PRHL Distribution

First let us look at the shapes of the density functions of PRHL(α) for different values of α in Figure 2. It is quite clear from the figures that the shapes of the density functions of PRHL

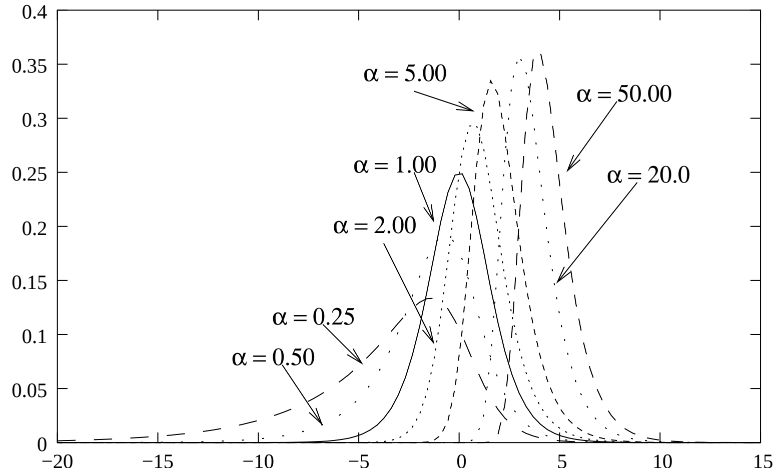

Figure 2: Density functions of PRHL distributions for different values of α.

distributions are quite different than the skew logistic distributions. It is positively skewed for $\alpha > 1$ and negatively skewed for $0 < \alpha < 1$. Because of this reason, the proportionality constant α can also be termed as the skewness parameter. For $\alpha = 1$ the PRHL distribution coincides with the standard logistic distribution and it is symmetric. The density function of PRHL(α) is log-concave for all values α. Therefore, it also satisfies the properties 3, 4 and 5 as described in section 3 for skew logistic distribution.

If the random variable X follows PRHL(α), then the moment generating function (MGF) of X is

$$M_X(t) = E(e^{tX}) = \alpha \int_{\infty}^{\infty} e^{-(1-t)x}(1+e^{-x})^{-(\alpha+1)}dx = \frac{\Gamma(1-t)\Gamma(\alpha+t)}{\Gamma(\alpha)}. \quad (21)$$

Therefore, the mean, variance and different moments can be easily obtained. The mean and variance of X can be written as

$$E(X) = \psi(\alpha) - \psi(1), \qquad Var(X) = \psi'(\alpha) + \psi'(1), \quad (22)$$

here $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ and $\psi'(x) = \frac{d}{dx}\psi(x)$, known as digamma and polygamma functions respectively. The mean is an increasing function of α and the variance is a decreasing function of α. The mean is increasing to ∞ and the variance is decreasing to $\frac{\pi^2}{6}$ as $\alpha \longrightarrow \infty$. The coefficient of variation is

$$\frac{\sqrt{\psi'(\alpha) - \psi'(1)}}{\psi(\alpha) - \psi(1)}. \tag{23}$$

The skewness and kurtosis are

$$\frac{\psi''(\alpha) - \psi''(1)}{(\psi'(\alpha) - \psi'(1))^{\frac{3}{2}}} \quad \text{and} \quad \frac{\psi'''(\alpha) - \psi'''(1)}{(\psi'(\alpha) - \psi'(1))^2} \tag{24}$$

respectively. The skewness varies between $(-2.0, 1.1396)$. Interestingly, the kurtosis has a bathtub shape, initially it decreases from 6 to 1.1479 and then it increases to 2.4. The other moments of X also can be obtained in terms of higher derivatives of $\psi(\cdot)$. Similarly, different L-moments also can be expressed in terms of the ψ function and its derivatives. The first three L-moments are as follows;

$$\lambda_1 = \mu + \sigma[\psi(\alpha) - \psi(1)], \quad \lambda_2 = \sigma[\psi(2\alpha) - \psi(\alpha)], \quad \lambda_3 = \sigma[2\psi(3\alpha) - 3\psi(2\alpha) + \psi(\alpha)]. \tag{25}$$

The PRHL distribution is unimodal with mode at $\ln(\alpha + 1)$. It has the median at $-\ln(2^{\frac{1}{\alpha}} - 1)$. The mean, median and mode are all non-linear functions of α and as α tends to ∞ all of them tend to ∞. In this respect the PRHL distribution is quite different than the SL distribution. It is observed that as $\alpha \longrightarrow \infty$, (mean - mode) $\longrightarrow$ Euler's constant $\approx$ 0.5772 and (median - mode) $\longrightarrow$ - $\ln(\ln 2)$. The survival function at the median is always 0.5. The survival function at the mean is $1 - \left(1 + e^{\psi(1)-\psi(\alpha)}\right)^{-\alpha}$, which is a decreasing function of α and the survival function at the mode is $1 - \left(\frac{\alpha+1}{\alpha+2}\right)^{\alpha}$, which is an increasing function of α and it increases to $(1 - e^{-1})$.

The PRHL distribution can be seen as the exponential mixture of the extreme value distributions. It can be written as a difference of two log-gamma distributions. Moreover, it can also be written as the ratio of beta distributions of first kind, see Olapade [10] for details. It may be mentioned that the distribution of the sum of *i.i.d.* PRHL random variables can be obtained from the product of beta distributions.

4.4 Distribution of the Extreme values

In this subsection we provide some results regarding the distribution of the extreme values of the *i.i.d.* PRHL random variables. The following theorem shows that the distribution of the maximum of n independent PRHL random variables also follows PRHL distribution. It immediately shows that the model can be used to represent a parallel system.

Theorem 2: If $X_1 \ldots, X_n$ are independent random variables and $X_i \sim \text{PRHL}(\alpha_i)$, then $X_{(n)} = \max\{X_1, \ldots, X_n\} \sim \text{PRHL}(\sum_{i=1}^{n} \alpha_i)$.

The following theorem provides a characterization of the PRHL distribution in terms of the maximum. The proof is quite simple and therefore it is omitted.

Theorem 3: Suppose $X_1, \ldots, X_n$ are *i.i.d.* random variables. Then the X_i s are PRHL random variables iff $X_{(n)} = \max\{X_1, \ldots, X_n\}$ is also a PRHL random variable.

It is possible to provide the limiting distribution of the maximum and minimum order statistics. It is observed that after proper normalization, the distributions of the maximum and minimum of n *i.i.d.* PRHL random variables tend to the extreme value distributions of Type I (Gumbel type). The results can be stated as follows.

Theorem 4: Suppose $X_1, \ldots, X_n$ are *i.i.d.* random variables and $X_i \sim \text{PRHL}(\alpha)$, then for all $-\infty < x < \infty$, and $b_n = \ln n + \ln \alpha$,

$$\lim_{n \longrightarrow \infty} P(X_{(n)} - b_n \leq x) = e^{-e^{-x}}. \tag{26}$$

Theorem 5: Suppose $X_1, \ldots, X_n$ are *i.i.d.* random variables and $X_i \sim \text{PRHL}(\alpha)$, then for all $-\infty < x < \infty$, and $c_n = -\frac{1}{\alpha} \ln n$,

$$\lim_{n \longrightarrow \infty} P(X_{(1)} - c_n \le x) = 1 - e^{-e^{\alpha x}}. \tag{27}$$

5 Estimation

In this section we consider the estimation of the unknown parameters for both the distribution functions. First let us consider the skew logistic distribution.

5.1 Skew Logistic Distribution:

Suppose it is assumed that we have a sample of size n, say $\{x_1, \ldots, x_n\}$, from a skew logistic distribution. Let us assume at the beginning that the location and scale parameters are known and we want to estimate only the skewness parameter α. Therefore, with out loss of generality we can assume that $\{x_1, \ldots, x_n\}$ are from SL(α). The maximum likelihood estimate (MLE) of α can be obtained by solving the following normal equation;

$$\sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{1 + e^{-\alpha x_i}} = 0. \tag{28}$$

Therefore, the normal equation (28) clearly does not have a solution if all the $x_i \ge 0$ or if all the $x_i \le 0$. If all the $x_i > 0$ or all the $x_i < 0$, then it can be shown that the likelihood function is an increasing of $|\alpha|$ and therefore the MLE of α does not exist. When the MLE exists, it can be obtained by solving the non-linear equation (28). For each α and n, there is a positive probability that all $x_i > 0$ or all $x_i < 0$. In Table 1 the probabilities of the non-existence of the MLE for different values of α and n are provided. It is observed that for $\alpha = 10$, even when the sample size is 20, there is a more than 50% chance that the MLE of α will not exist. The probability increases as the shape parameter increases.

When all the three parameters are unknown, then the MLEs can be obtained by solving

$\alpha \longrightarrow$ $n \downarrow$	0.0	2.0	4.0	6.0	8.0	10.0
5	0.031	0.444	0.659	0.734	0.815	0.859
10	0.001	0.197	0.434	0.539	0.665	0.737
15	0.000	0.087	0.286	0.395	0.542	0.633
20	0.000	0.039	0.189	0.290	0.442	0.544

Table 1: The probability that all the $x_i > 0$ for different α and n are presented

the non-linear equations with three unknowns. Of course the problem becomes more complex in this situation and the non-existence of the MLEs naturally persists. Very recently, Sartori [12] suggested a bias prevention of the MLEs for skew normal and skew-t distributions. The method proposed by Sartori can be used here to provide bias corrected MLEs of the unknown parameters, it is not pursued here.

5.2 PRHL Distribution

Now suppose $\{x_1 \ldots, x_n\}$ is a random sample from PRHL distribution. First let us assume that the location and scale parameters are known and we are interested only estimating α. We are assuming $\mu = 0$ and $\lambda = 1$, without loss of generality. In this case we make the simple transformation

$$Y_i = -\ln\left(1 + e^{-X_i}\right), \quad \text{for} \quad i = 1, \ldots, n, \tag{29}$$

and then Y_i has the following exponential density function

$$f_Y(y) = \alpha e^{-\alpha y}, \quad \text{for} \quad 0 < y < \infty. \tag{30}$$

Therefore, the MLE of α always exists in this case and all the standard results on statistical inferences for exponential distribution function can be used here.

Now let us consider the case when all the three parameters are unknown. In this case Zelterman [13] showed that the MLEs of α, λ, μ do not exist. The method of moment estimators (MMEs) exist if and only if the sample skewness is between -2 and 1.1396 (the theoretical range of the population skewness). When the MMEs exist, they can be obtained by solving the three non-linear equations. The linear combinations of order statistics estimators namely the L-moment estimators (LMEs) as suggested by Hosking [8] can be applied here. Unlike the MMEs, the LMEs always exist. Note that, the computations of the MMEs or LMEs are quite involved when all the three parameters are unknown. We are suggesting the following alternative mainly for data analysis purposes. Although the MLEs of α, λ, μ do not exist if all of them are unknown, but the MLEs of α and λ exist, if μ is known. Therefore, for data analysis purposes, we suggest replacing the location parameter μ by the sample median and then obtain the MLEs of α and λ based on the transformed data.

6 Data Analysis

In this section we present the analysis of one real data set for illustrative purposes. It is a strength data originally considered by Badar and Priest [4]. The data represent the strength measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purposes we are considering the single fibers data set of 10 mm in gauge lengths with sample size 63. The data are presented below.

Data Set: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

The sample mean, variance and skewness are 3.0593, 0.3855 and 0.6278 respectively. First we try to fit the skew logistic distribution (12). We need to estimate three parameters. Since the skewness is independent of the location and scale parameters, we obtain the MME of α by equating the sample skewness with the population skewness. The MME of α is 0.5799 and then we obtain the MMEs of λ and μ as 2.6846 and 2.7818 respectively. Using 0.5799, 2.6846 and 2.7818 as the initial guesses of α, λ, μ, and using optimization routine of Press *et al.* [11], we obtain the MLEs of α, λ and μ as 0.5400, 2.6800 and 2.7740, respectively.

Next we try to fit the three-parameter PRHL distribution (20) to the data. It is well known, see Zelterman [13], that the MLEs do not exist in this case. Although, the MLEs do not exist, but the MMEs exist as the sample skewness is 0.6278 $\in$ (-2.0, 1.1396). The MMEs can be obtained by solving three non-linear equations and they are 1.5398, 2.5913 and 2.8084 for α, λ and μ respectively. The LMEs also can be obtained by solving three non-linear equations, and they are 3.2761, 2.2192 and 2.3339 for α, λ and μ respectively.

Now we would like to see how well the skew logistic and PRHL distributions with different parameter sets fit the data. We have plotted the density functions of
(i) SL(0.5799,2.6846,2.7818), (ii) SL(0.5400, 2.6800, 2.7740), (iii) PRHL(1.5398.2.5913,2.8084) and (iv) PRHL(3.2761,2.2192,2.3369) along with the histogram of the data in Figure 3. Note

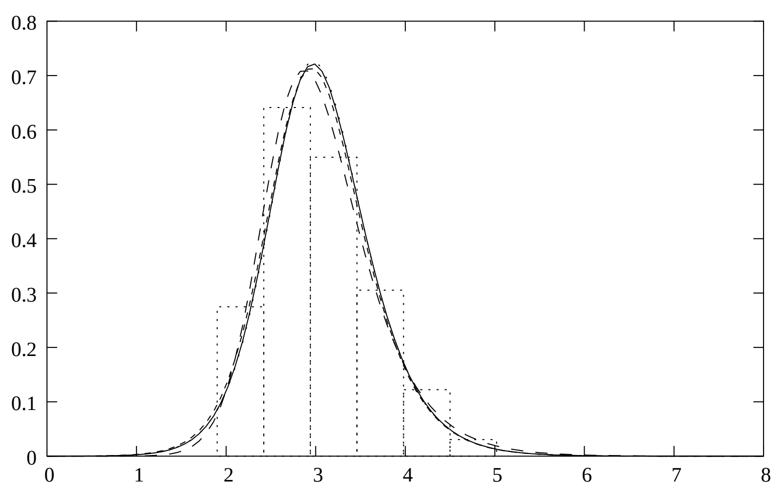

Figure 3: Fitted density functions and the histogram of the data.

that (i) and (ii) represent the skew logistic distributions when the parameters are obtained using MMEs and MLEs respectively. Similarly, (iii) and (iv) represent the PRHL distributions when the parameters are obtained using MMEs and LMEs respectively.

The first three density functions are almost identical and it is not possible to distinguish them. But (iv) is slightly different than the other three. It is quite interesting that although we are fitting two different types of distributions, namely skew logistic and PRHL models, with different sets of parameter values, but the fitted density functions are quite similar. We have computed the Kolmogorov-Smirnov (K-S) distances of the fitted distribution functions and the empirical distribution function (EDF). It is observed that the for the first three distribution functions, the K-S distances match up to two decimal places. Because of that we just report the K-S distances of (iii) and (iv) and they are 0.1075, 0.0844 with the corresponding p-values 0.4607 and 0.7603 respectively. Therefore, based on the K-S distance, (iv) provides the best fit.

We also obtain the MLEs of α and λ assuming the location parameter to be known as the sample median, namely 2.996. In this case the MLEs of α and λ are 1.1240 and 2.7449 respectively. We denote this PRHL(1.1240,2.7449,2.9960) as (v). It is observed that the K-S distance between (v) and the EDF is 0.1135 and the corresponding p-value is 0.3653.

For comparison purposes, we also obtain the observed and the expected frequencies, the corresponding χ^2 values, based on (iii), (iv) and (v). They are presented in Table 2. From the χ^2 values and also from the K-S distances, it is observed that the PRHL distribution with the parameters estimated by the LMEs provides the best fit. It supports Hosking's observations that LMEs often provide better fit than the MMEs. Moreover, the PRHL distribution with the corresponding parameters estimated by the MLEs (replacing the location parameter by the sample median) may be used for practical purposes to avoid three dimensional optimization.

Class Intervals	Observed Frequency	Expected Frequency (iii)	Expected Frequency (iv)	Expected Frequency (v)
< 2.5	12	10.40	11.15	10.55
2.5 - 3.0	20	19.91	20.86	18.53
3.0 - 3.5	17	19.39	17.59	19.91
3.5 - 4.0	9	9.12	8.51	9.79
> 4.0	5	4.18	4.89	4.21
		$\chi^2 = 0.705$	$\chi^2 = 0.151$	$\chi^2 = 0.949$

Table 2: The observed and the expected frequencies of the three fitted PRHL distribution functions and the corresponding χ^2 values are presented.

7 Conclusions

In this paper we have considered two different generalizations of the symmetric logistic distribution, namely skew logistic distribution and PRHL distribution. The skew logistic distribution has been obtained using the idea of Azzalini and the base distribution as the logistic distribution instead of normal distribution. Since the standard logistic distribution has heavier tails than the standard normal distribution, therefore, it is expected, see Azzalini [2], that the skew logistic distribution has heavier tails than the skew normal distribution. It is observed that the range of the skewness of the skew logistic distribution is more than the skew normal distribution. Many properties of the skew logistic distribution and the skew normal distribution are quite similar. It can be used to model for both positively and negatively skewed data. But we feel, since the distribution function is not in a compact form it will be very difficult to use in practice if the data are censored.

The second generalization, namely the PRHL distribution, has been obtained as a proportional reversed hazard distribution using the logistic distribution as the base line distribution. It is observed that the PRHL distribution is unimodal and it can be used to model both left an right skewed data. Since the distribution function is in compact form, it can be used more conveniently than the skew logistic distribution to model censored data. In this paper we did not study the properties of the different estimators and their performances. Extensive simulations are needed for this purpose. The work is in progress and it will be reported later.

Note that, we had defined the proportional reversed hazard family and along the same line we can define the proportional hazard family also. It may be mentioned that if the random variable X belongs to the PRHL family, then $-X$ belongs to the proportional hazard logistic (PHL) family. Therefore, all the properties of the PHL family can be easily obtained from the PRHL family. Another generalization of the logistic distribution can be easily thought of from the various formulations of the PRHL distribution. For example if Y_1 and Y_2 are two independent gamma random variables both having scale parameter 1 and shape parameter as α and β respectively, then the PDF of $X = \ln Y_1 - \ln Y_2$ becomes;

$$f_X(x) = \frac{1}{B(\alpha,\beta)} \frac{e^{-\beta x}}{(1+e^{-x})^{\alpha+\beta}}. \tag{31}$$

Clearly (31) will have more flexibility than the PRHL(α) model. Similarly, if U has a Beta(α, β) distribution with first kind, then the density function of $X = \ln \frac{U}{1-U}$ is also same as (31). Moreover, if U follows $\text{Beta}_2(\alpha,\beta)$, then $-\ln U$ has also the same density function as (31). Finally we should mention that although we have studied the properties of skew logistic distribution and PRHL distribution only, but many properties are valid not only for logistic base distributions but for larger class of symmetric distributions, for example t, Laplace, Cauchy etc.

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