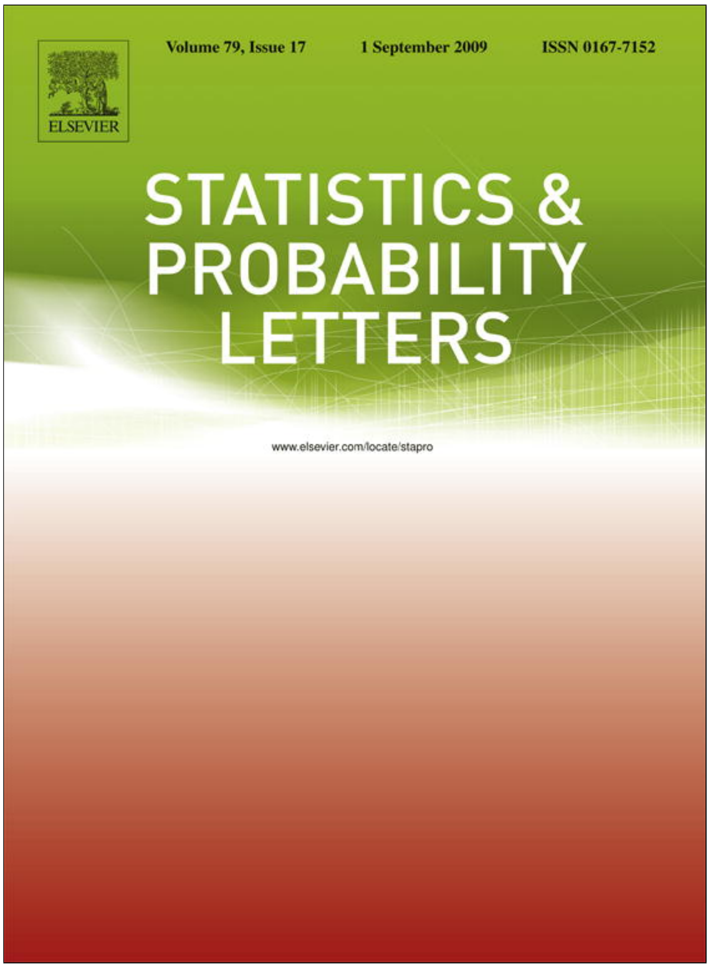
Volume 79, Issue 17
1 September 2009
ISSN 0167-7152
ELSEVIER
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Statistics and Probability Letters 79 (2009) 1839–1846

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution

Debasis Kundu [a,*], Mohammad Z. Raqab [b]

[a] *Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India*

[b] *Department of Mathematics, University of Jordan, Amman 11942, Jordan*

ARTICLE INFO

Article history:
Received 18 December 2008
Received in revised form 28 April 2009
Accepted 18 May 2009
Available online 18 June 2009

ABSTRACT

In this paper we consider the estimation of the stress–strength parameter $R = P(Y < X)$, when X and Y are independent and both are three-parameter Weibull distributions with the common shape and location parameters but different scale parameters. It is observed that the maximum likelihood estimators do not exist in this case, and we propose a modified maximum likelihood estimator, and also an approximate modified maximum likelihood estimator of R. We obtain the asymptotic distribution of the modified maximum likelihood estimators of the unknown parameters and it can be used to construct the confidence interval of R. Analyses of two data sets have also been presented for illustrative purposes.

1. Introduction

In this paper we consider the problem of estimating the stress–strength parameter $R = P(Y < X)$ when X and Y are independent and both are three-parameter Weibull-distributed random variables. It is further assumed that both have the same shape and location parameters but different scale parameters. The estimation of R is very common in the statistical literature. For example, if X is the strength of a system which is subjected to a stress Y, then R is a measure of system performance and arises quite naturally in the mechanical reliability of a system. The system fails if and only if at anytime, the applied stress is greater than its strength. This particular problem was first considered by McCool (1991) and later by Kundu and Gupta (2006). In both cases it is assumed that the common location parameter is known. In the literature the estimation of R in the case of Weibull or exponential distributions has been obtained under the assumption of common location parameters, and when it is known, see for example Kotz et al. (2003). The assumption of the common location parameters is not very unrealistic. For example a measuring equipment has a calibration problem, and the measurements have been taken in this particular equipment, then all the observations will have a location shift. In this paper also, it is assumed that the location parameters are same, but the common location parameter is unknown.

It should be mentioned that related problems have been widely used in the literature. Maximum likelihood estimation of R, when X and Y have bivariate exponential distribution, has been considered by Awad et al. (1981). Church and Harris (1970), Downtown (1973), Govidarajulu (1967), Woodward and Kelley (1977) and Owen et al. (1977) considered the estimation of R when both X and Y are normally distributed. A book length treatment of the different methods can be found in Kotz et al. (2003). Recently Kundu and Gupta (2005) and Raqab and Kundu (2005) considered the estimation of R for the two-parameter generalized exponential distribution and scaled Burr type-X distribution respectively. Although, extensive treatment for the stress–strength parameter for different models in the presence of shape and scale parameters, are available, but not much attempt has not been made when the location parameter is also present. Very recently Raqab et al. (2008) considered the statistical inference of the stress–strength parameter of the three-parameter generalized exponential distribution. It is observed that in the presence of location parameter, the analysis becomes quite non-trivial and that is the main purpose of this paper.

* Corresponding author.
E-mail addresses: kundu@iitk.ac.in (D. Kundu), mraqab@ju.edu.jo (M.Z. Raqab).

0167-7152/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2009.05.026

Note that when the location parameter is unknown for a three-parameter Weibull distribution, it is no longer a regular family. In fact for a three-parameter Weibull distribution, the maximum likelihood estimators (MLEs) of the unknown parameters do not exist. The non-existence of the MLEs are quite common in many embedded models, see for example Cheng et al. (1992), and the references cited therein. The condition on the non-existence of the MLEs for the three-parameter Weibull distribution can be found in Table 1 of Shao et al. (2004). The likelihood function monotonically increases along a particular parameter values. In this case using the idea of Smith (1985), we consider the modified MLEs (MMLEs) and they have to be obtained by solving a non-linear equation. We propose a simple iterative scheme to solve this non-linear equation. Although, we could not prove its convergence, but it is observed that in practice the proposed iterative scheme works very well. We derive the asymptotic distribution of the MMLE of the unknown parameters and based on that we obtain the asymptotic distribution of the MMLE of R. Using the asymptotic distribution, we obtain the asymptotic confidence interval of R.

Since the MMLEs cannot be obtained explicitly, using the idea of Kundu and Gupta (2006), we propose approximate MMLEs (AMMLEs) and they can be obtained explicitly. It is observed that the performance of the MMLEs and AMMLEs are very similar in nature. Therefore, for all practical purposes, the computation of the MMLEs can be avoided. Moreover, AMMLEs may be used as initial guesses to compute the MMLEs.

We use the following notation. A Weibull distribution with the shape, scale and location parameters, α, θ and μ respectively, will be denoted by $WE(\alpha, \theta, \mu)$ and the corresponding probability density function $f(x; \alpha, \theta, \mu)$ for $\alpha > 0$, $\theta > 0$ is as follows;

$$f(x; \alpha, \theta, \mu) = \frac{\alpha}{\theta}(x-\mu)^{\alpha-1}e^{-\frac{1}{\theta}(x-\mu)^{\alpha}}; \quad x > \mu. \tag{1}$$

The rest of the paper is organized as follows. In Section 2 we provide the modified maximum likelihood estimators. The asymptotic distribution of the modified maximum likelihood estimators are provided in Section 3. The approximate modified maximum likelihood estimators are provided in Section 4. Analyses of two data sets are presented in Section 5. Finally the conclusions and discussion appear in Section 6.

2. Modified maximum likelihood estimator

Suppose X and Y follow $WE(\alpha, \theta_1, \mu)$ and $WE(\alpha, \theta_2, \mu)$ respectively, and they are independent. Then it can be easily shown that

$$R = P(Y < X) = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{2}$$

Now to compute the MLE of R, we need to compute the MLEs of θ_1 and θ_2. Moreover, to compute the MLEs of θ_1 and θ_2, we need to compute the MLEs of α and μ also. Suppose $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ are independent random samples from $WE(\alpha, \theta_1, \mu)$ and $WE(\alpha, \theta_2, \mu)$, respectively. Our problem is to estimate R from the given samples. Let us denote the ordered X_i's and ordered Y_j's as $\{X_{(1)} < \cdots < X_{(n)}\}$ and $\{Y_{(1)} < \cdots < Y_{(m)}\}$ respectively. Based on the observations X_i's and Y_j's the likelihood function of $\alpha, \theta_1, \theta_2$ and μ is

$$l(\alpha, \theta_1, \theta_2, \mu) \propto \alpha^{m+n}\theta_1^{-n}\theta_2^{-m}\prod_{i=1}^{n}\left(x_{(i)}-\mu\right)^{\alpha-1} \times \prod_{j=1}^{m}\left(y_{(j)}-\mu\right)^{\alpha-1} e^{-\frac{1}{\theta_1}\sum_{i=1}^{n}\left(x_{(i)}-\mu\right)^{\alpha-1}} \times e^{-\frac{1}{\theta_2}\sum_{j=1}^{m}\left(y_{(j)}-\mu\right)^{\alpha-1}} \times 1_{\{z>\mu\}}, \tag{3}$$

where $z = \min\{x_{(1)}, y_{(1)}\}$, and $1_{\{z>\mu\}}$ is an indicator function takes the value 1 or 0, if $z > \mu$ or $z \leq \mu$ accordingly.

It is clear that if $\alpha < 1$, for $\theta_1 > 0, \theta_2 > 0$ as μ approaches z, the likelihood function $l(\alpha, \theta_1, \theta_2, \mu)$ gradually increases to infinity. Therefore, MLEs of $\alpha, \theta_1, \theta_2, \mu$ do not exist. To estimate the parameters, one can proceed by first estimating the location parameter μ by its natural consistent estimator $\widetilde{\mu} = z$, see Smith (1985). The modified log-likelihood function based on the $(m+n-1)$ observations after ignoring the smallest observation and replacing μ by its estimator $\widetilde{\mu} = z$, is given by for $x_{(1)} < y_{(1)}$

$$\begin{aligned} l(\alpha, \theta_1, \theta_2, \widetilde{\mu}) = {} & (m+n-1)\ln\alpha - (n-1)\ln\theta_1 - m\ln\theta_2 + (\alpha-1)\sum_{i=2}^{n}\ln(x_{(i)}-\widetilde{\mu}) \\ & + (\alpha-1)\sum_{j=1}^{m}\ln(y_{(j)}-\widetilde{\mu}) - \frac{1}{\theta_1}\sum_{i=2}^{n}(x_{(i)}-\widetilde{\mu})^{\alpha} - \frac{1}{\theta_2}\sum_{j=1}^{m}(y_{(j)}-\widetilde{\mu})^{\alpha} \end{aligned} \tag{4}$$

and for $x_{(1)} > y_{(1)}$

$$\begin{aligned} l(\alpha, \theta_1, \theta_2, \widetilde{\mu}) = {} & (m+n-1)\ln\alpha - n\ln\theta_1 - (m-1)\ln\theta_2 + (\alpha-1)\sum_{i=1}^{n}\ln(x_{(i)}-\widetilde{\mu}) \\ & + (\alpha-1)\sum_{j=2}^{m}\ln(y_{(j)}-\widetilde{\mu}) - \frac{1}{\theta_1}\sum_{i=1}^{n}(x_{(i)}-\widetilde{\mu})^{\alpha} - \frac{1}{\theta_2}\sum_{j=2}^{m}(y_{(j)}-\widetilde{\mu})^{\alpha}. \end{aligned} \tag{5}$$

Therefore, the modified MLEs of α, θ_1 and θ_2 can be obtained by maximizing $l(\alpha, \theta_1, \theta_2)$ with respect to α, θ_1 and θ_2. From the modified normal equation, it easily follows that for given α

$$\widetilde{\theta}_1(\alpha) = \frac{\sum_{i=2}^{n}(x_{(i)} - \widetilde{\mu})^\alpha}{n-1} \quad \text{and} \quad \widetilde{\theta}_2(\alpha) = \frac{\sum_{j=1}^{m}(y_{(j)} - \widetilde{\mu})^\alpha}{m} \quad \text{if } x_{(1)} < y_{(1)} \tag{6}$$

and

$$\widetilde{\theta}_1(\alpha) = \frac{\sum_{i=1}^{n}(x_{(i)} - \widetilde{\mu})^\alpha}{n} \quad \text{and} \quad \widetilde{\theta}_2(\alpha) = \frac{\sum_{j=2}^{m}(y_{(j)} - \widetilde{\mu})^\alpha}{m-1} \quad \text{if } x_{(1)} > y_{(1)}. \tag{7}$$

The profile modified log-likelihood function of α can be obtained as $l(\alpha, \widetilde{\theta}_1(\alpha), \widetilde{\theta}_2(\alpha), \widetilde{\mu})$. Therefore, the modified maximum likelihood estimator of α can be obtained by maximizing $l(\alpha, \widetilde{\theta}_1(\alpha), \widetilde{\theta}_2(\alpha), \widetilde{\mu})$ with respect to α. Now we will mention how to maximize (4), similar method can be adopted to maximize (5) also.

Without the additive constant, from (4), $l(\alpha, \widetilde{\theta}_1(\alpha), \widetilde{\theta}_2(\alpha), \widetilde{\mu})$ can be written as

$$l(\alpha, \widetilde{\theta}_1(\alpha), \widetilde{\theta}_2(\alpha), \widetilde{\mu}) = (m+n-1)\ln\alpha - (n-1)\ln\sum_{i=2}^{n}(x_{(i)} - \widetilde{\mu})^\alpha - m\ln\sum_{j=1}^{m}(y_{(j)} - \widetilde{\mu})^\alpha$$
$$+ \sum_{i=2}^{n}\ln(x_{(i)} - \widetilde{\mu})^\alpha + \sum_{j=1}^{m}(y_{(j)} - \widetilde{\mu})^\alpha. \tag{8}$$

Our problem is to maximize $l(\alpha, \widetilde{\theta}_1(\alpha), \widetilde{\theta}_2(\alpha), \widetilde{\mu})$ with respect to α. Using similar technique as in Kundu and Gupta (2006), the modified MLE of α can be obtained as a solution of the non-linear equation of the form

$$h(\alpha) = \alpha, \tag{9}$$

where

$$h(\alpha) = \frac{(m+n-1) + \sum_{i=2}^{n}\ln(x_{(i)} - \widetilde{\mu})^\alpha + \sum_{j=1}^{m}\ln(y_{(j)} - \widetilde{\mu})^\alpha}{\frac{\sum_{i=2}^{n}(x_{(i)}-\widetilde{\mu})^\alpha\ln(x_{(i)}-\widetilde{\mu})}{\frac{1}{n-1}\sum_{i=2}^{n}(x_k-\widetilde{\mu})^\alpha} + \frac{\sum_{j=1}^{n}(y_{(j)}-\widetilde{\mu})^\alpha\ln(y_{(j)}-\widetilde{\mu})}{\frac{1}{m}\sum_{i=2}^{m}(y_k-\widetilde{\mu})^\alpha}}. \tag{10}$$

Because α is a fixed point solution of the non-linear (9), therefore, it can be obtained by using a simple iterative procedure as

$$h(\alpha_{(j)}) = \alpha_{(j+1)}, \tag{11}$$

where $\alpha_{(j)}$ is the jth iterate of $\widetilde{\alpha}$. The iterative procedure should be stopped when the absolute difference between $\alpha_{(j)}$ and $\alpha_{(j+1)}$ is sufficiently small.

Once we obtain the MMLEs of $\alpha, \theta_1, \theta_2$ and μ, the MMLE of R is obtained as

$$\widetilde{R} = \frac{\widetilde{\theta}_1}{\widetilde{\theta}_1 + \widetilde{\theta}_2}. \tag{12}$$

3. Asymptotic distribution

In this section we would like to obtain the asymptotic distribution of $\widetilde{R}$, which can be used for constructing confidence interval. For constructing the asymptotic distribution of $\widetilde{R}$, we need the following results.

Theorem 1. (a) *The marginal distribution of* $\widetilde{\mu} = \min\{X_{(1)}, Y_{(1)}\}$ *is given by*

$$P(\widetilde{\mu} \le t) = 1 - e^{-(t-\mu)^\alpha\left\{\frac{n}{\theta_1} + \frac{m}{\theta_2}\right\}}. \tag{13}$$

(b) *If* $m/n \to p > 0$, *as* $n, m \to \infty$ *then*

$$n^{\frac{1}{\alpha}}(\widetilde{\mu} - \mu) \stackrel{d}{=} \delta Z^{\frac{1}{\alpha}}, \tag{14}$$

here $\delta = \frac{1}{\theta_1} + \frac{p}{\theta_2}$, *Z is a standard exponential random variable and* $\stackrel{d}{=}$ *means equal in distribution.*

Proof of (a) Trivial.

Proof of (b) Note that

$$P(n^{1/\alpha}(\widetilde{\mu} - \mu) \le t) = P(\widetilde{\mu} \le n^{-1/\alpha}t + \mu) = 1 - e^{-\delta t^\alpha}.$$

Theorem 2. *The asymptotic distribution of* $(\widetilde{\alpha}, \widetilde{\theta}_1, \widetilde{\theta}_2)$ *as* $n \to \infty$, $m \to \infty$ *and* $m/n \to p$ *is as follows;*

$$\left(\sqrt{n}(\widetilde{\alpha} - \alpha), \sqrt{n}(\widetilde{\theta}_1 - \theta_1), \sqrt{n}(\widetilde{\theta}_2 - \theta_2)\right) \to N_3(0, A^{-1}(\alpha, \theta_1, \theta_2)),$$

where

$$A(\alpha, \theta_1, \theta_2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix},$$

and

$$a_{11} = \frac{1}{\alpha^2}\left[(1+p)(1+\Gamma''(2)) + 2\Gamma'(2)(\ln\theta_1 + p\ln\theta_2) + (\ln(\theta_1))^2 + p(\ln\theta_2)^2\right]$$

$$a_{22} = \frac{1}{\theta_1^2}, \qquad a_{12} = a_{21} = -\frac{1}{\alpha\theta_1}\left[\ln\theta_1 + \Gamma'(2)\right]$$

$$a_{33} = \frac{p}{\theta_2^2}, \qquad a_{13} = a_{31} = -\frac{p}{\alpha\theta_2}\left[\ln\theta_2 + \Gamma'(2)\right].$$

Proof. We use the following notations. If U_n and V_n are random variables, then $U_n = O_p(1)$ means as $n \to \infty$, U_n is bounded in probability. Similarly, if $V_n = o_p(1)$, means $n \to \infty$, V_n converges to zero in probability.

Case 1: $0 < \alpha < 2$

Let us denote $\bar{\alpha}$, $\bar{\theta}_1$ and $\bar{\theta}_2$ as the MLEs of α, θ_1 and θ_2, when μ is known. Then using Theorem 1 of Kundu and Gupta (2006), it follows that

$$\left(\sqrt{n}(\bar{\alpha} - \alpha), \sqrt{n}(\bar{\theta}_1 - \theta_1), \sqrt{n}(\bar{\theta}_2 - \theta_2)\right) \to N_3(0, A^{-1}(\alpha, \theta_1, \theta_2)). \tag{15}$$

From Theorem 4 of Smith (1985) it is observed that

$$\widetilde{\alpha} - \bar{\alpha} = \begin{cases} o_p(n^{-1/\alpha}) & \text{if } \alpha > 1 \\ o_p(\ln n/n) & \text{if } \alpha \le 1, \end{cases} \tag{16}$$

$$\widetilde{\theta}_1 - \bar{\theta}_1 = \begin{cases} o_p(n^{-1/\alpha}) & \text{if } \alpha > 1 \\ o_p(\ln n/n) & \text{if } \alpha \le 1 \end{cases} \quad \text{and} \quad \widetilde{\theta}_2 - \bar{\theta}_2 = \begin{cases} o_p(n^{-1/\alpha}) & \text{if } \alpha > 1 \\ o_p(\ln n/n) & \text{if } \alpha \le 1. \end{cases} \tag{17}$$

Since

$$\begin{aligned}(\sqrt{n}(\widetilde{\alpha} - \alpha), \sqrt{n}(\widetilde{\theta}_1 - \theta_1), \sqrt{n}(\widetilde{\theta}_2 - \theta_2)) &= (\sqrt{n}(\bar{\alpha} - \alpha), \sqrt{n}(\bar{\theta}_1 - \theta_1), \sqrt{n}(\bar{\theta}_2 - \theta_2)) \\ &\quad + (\sqrt{n}(\widetilde{\alpha} - \bar{\alpha}), \sqrt{n}(\widetilde{\theta}_1 - \bar{\theta}_1), \sqrt{n}(\widetilde{\theta}_2 - \bar{\theta}_2)),\end{aligned} \tag{18}$$

using (16) and (17), for $0 < \alpha \le 1$ and for $1 < \alpha < 2$, we have

$$\sqrt{n}(\widetilde{\alpha} - \bar{\alpha}) = o_p(1), \qquad \sqrt{n}(\widetilde{\theta}_1 - \bar{\theta}_1) = o_p(1), \qquad \sqrt{n}(\widetilde{\theta}_2 - \bar{\theta}_2) = o_p(1). \tag{19}$$

Therefore the result follows, using (18), (15) and (19).

Case 2: $\alpha \ge 2$

In this proof only we will denote the true values of α, θ_1, θ_2 and μ as α^0, θ_1^0, θ_2^0 and μ^0 respectively. Suppose $\gamma = (\alpha, \theta_1, \theta_2)$, $\gamma^0 = (\alpha^0, \theta_1^0, \theta_2^0)$ and $\widetilde{\gamma} = (\widetilde{\alpha}, \widetilde{\theta}_1, \widetilde{\theta}_2)$. Let $L = L(\alpha, \lambda_1, \lambda_2, \widetilde{\mu})$ be the log-likelihood function centered at $\widetilde{\mu}$. Moreover, let $G(\gamma, \widetilde{\mu}) = \frac{\partial}{\partial\gamma} L$ and let $H(\gamma, \widetilde{\mu}) = \frac{\partial}{\partial\gamma^2} L$ be the 3×1 derivative vector and 3×3 Hessian matrix of L respectively. By the definition of $\widetilde{\gamma}$, we have

$$G(\widetilde{\gamma}, \widetilde{\mu}) = 0. \tag{20}$$

Since conditioning of $\widetilde{\mu}$, $\widetilde{\gamma}$ is a $\sqrt{n}$ consistent estimator of γ^0, expanding $G(\widetilde{\gamma}, \widetilde{\mu})$ about γ^0, we get

$$G(\gamma^0, \widetilde{\mu}) + H(\gamma^0, \widetilde{\mu})(\widetilde{\gamma} - \gamma^0) + O_p(1) = 0. \tag{21}$$

Assuming that n is large enough and so that $\widetilde{\mu}$ can be replaced by μ^0 in $H(\gamma^0, \widetilde{\mu})$, we get

$$G(\gamma^0, \widetilde{\mu}) + H(\gamma^0, \mu^0)(\widetilde{\gamma} - \gamma^0) + O_p(1) = 0. \tag{22}$$

Therefore, rearranging the terms in (22) and multiplying by $\sqrt{n}$ we obtain

$$\begin{aligned}\sqrt{n}(\widetilde{\gamma} - \gamma^0) &= -\sqrt{n}H^{-1}(\gamma^0, \mu^0)G(\gamma^0, \widetilde{\mu}) + \left[\frac{1}{n}H(\gamma^0, \mu^0)\right]^{-1} \frac{O_p(1)}{\sqrt{n}} \\ &= -\sqrt{n}H^{-1}(\gamma^0, \mu^0)G(\gamma^0, \widetilde{\mu}) + o_p(1),\end{aligned} \tag{23}$$

since $\frac{1}{n}H(\gamma^0, \mu^0)$ converges to a positive definite matrix. Now expanding $G(\gamma^0, \widetilde{\mu})$ around μ^0, (see Theorem 4 of Mudholkar et al. (1996)) we have

$$\sqrt{n}(\widetilde{\gamma} - \gamma^0) = -\sqrt{n}H^{-1}(\gamma^0, \mu^0)G(\gamma^0, \mu^0) - \sqrt{n}(\widetilde{\mu} - \mu^0)H^{-1}(\gamma^0, \mu^0)\frac{\partial}{\partial \mu}G(\gamma^0, \mu^0) + o_p(1). \quad (24)$$

Since $\alpha \geq 2$, therefore $\frac{1}{\sqrt{n}}\left[\frac{\partial}{\partial \mu}G(\gamma^0, \mu^0)\right]$ converges to a multivariate normal distribution with mean zero and a finite variance–covariance matrix. It implies that the second term on the right-hand side of (24) is $o_p(1)$. Now the result follows immediately by observing the following facts

$$\frac{1}{\sqrt{n}}G(\gamma^0, \mu^0) \to N_3(0, A(\alpha, \theta_1, \theta_2)), \quad \text{and} \quad \frac{1}{n}H(\gamma^0, \mu^0) \to -A(\alpha, \theta_1, \theta_2).$$

4. Approximate modified maximum likelihood estimator

In the previous section we have observed that the maximum likelihood estimators do not exist and we have proposed to use the modified maximum likelihood estimator. It was also observed that the modified maximum likelihood estimators cannot be obtained in explicit forms. We need to solve a non-linear equation to compute the modified maximum likelihood estimators. In this section we will provide the approximate modified maximum likelihood estimators of the unknown parameters similarly as in Kundu and Gupta (2006). Using exactly the same procedure as in Kundu and Gupta (2006), the approximate modified maximum likelihood estimators of the unknown parameters can be easily obtained. We describe the estimators when $x_{(1)} < y_{(1)}$, but similarly, it can be defined for $x_{(1)} > y_{(1)}$ also. We need the following notations for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m$:

$$p_i = \frac{i}{n}, \qquad q_i = 1 - p_i, \qquad \bar{p}_j = \frac{j}{m+1}, \qquad \bar{q}_j = 1 - \bar{p}_j,$$
$$\delta_i = 1 + \ln q_i(1 - \ln(-\ln q_i)), \qquad \beta_i = -\ln q_i, \qquad \bar{\delta}_j = 1 + \ln \bar{q}_j(1 - \ln(-\ln \bar{q}_j)), \qquad \bar{\beta}_j = -\ln \bar{q}_j.$$

Moreover,

$$A_1 = \frac{\sum_{i=1}^{n-1} \beta_i \ln(x_{(i+1)} - x_{(1)})}{\sum_{i=1}^{n-1} \beta_i}, \qquad A_2 = \frac{\sum_{j=1}^{m} \bar{\beta}_j(\ln y_{(j)} - y_{(1)})}{\sum_{j=1}^{m} \bar{\beta}_j}, \qquad B_1 = \frac{\sum_{i=1}^{n-1} \delta_i}{\sum_{i=1}^{n-1} \beta_i}, \qquad B_2 = \frac{\sum_{j=1}^{m} \bar{\delta}_j}{\sum_{i=1}^{n} \bar{\beta}_j},$$

$$C = m + n,$$

$$D = \sum_{i=1}^{n-1} \delta_i(\ln(x_{(i)} - x_{(1)}) - A_1) - 2B_1 \sum_{i=1}^{n-1} \beta_i(\ln(x_{(i)} - x_{(1)}) - A_1)$$
$$+ \sum_{j=1}^{m} \bar{\delta}_j(\ln(y_{(j)} - y_{(1)}) - A_2) - 2B_2 \sum_{j=1}^{m} \bar{\beta}_j(\ln(y_{(j)} - y_{(1)}) - A_2)$$

$$E = \sum_{i=1}^{n-1} \beta_i(\ln(x_{(i)} - x_{(1)}) - A_1)^2 + \sum_{j=1}^{m} \bar{\beta}_j(\ln(y_{(j)} - y_{(1)}) - A_2)^2.$$

If

$$\widehat{\sigma} = \frac{-D + \sqrt{D^2 + 4E(m+n-1)}}{2(m+n-1)},$$

then the approximate maximum likelihood estimators of α, θ_1 and θ_2 are as follows;

$$\widehat{\alpha} = \frac{1}{\widehat{\sigma}}, \qquad \widehat{\theta}_1 = e^{\frac{1}{\widehat{\sigma}}(A_1 - B_1\widehat{\sigma})}, \qquad \widehat{\theta}_2 = e^{\frac{1}{\widehat{\sigma}}(A_2 - B_2\widehat{\sigma})}. \quad (25)$$

Once the approximate maximum likelihood estimators of α, θ_1 and θ_2 are obtained, the approximate maximum likelihood estimator of R can be obtained as

$$\widehat{R} = \frac{\widehat{\theta}_1}{\widehat{\theta}_1 + \widehat{\theta}_2}. \quad (26)$$

5. Data analysis

In this section we present two cases of data analysis results for illustrative purposes. The first case is illustrated through simulated data sets and the second case is illustrated through real life data sets.

Table 1
Data set 1 (simulated).

0.0116	0.1499	0.2400	0.2998	0.3560	0.3800	0.4044	0.4268	0.4376
0.4603	0.5532	0.6322	0.8410	0.8920	1.0550	1.1333	1.2511	1.3905
1.4364	1.6259	1.6581	2.1628	2.4506	3.5763	4.1416		

Table 2
Data set 2 (simulated).

0.0393	0.0394	0.0889	0.1354	0.1449	0.1694	0.2519	0.3362	0.3522
0.3647	0.4768	0.4953	0.5949	0.6091	0.6334	0.7579	0.7966	0.7979
0.8464	0.8766	1.0426	1.3485	1.7270	4.1001	5.2073		

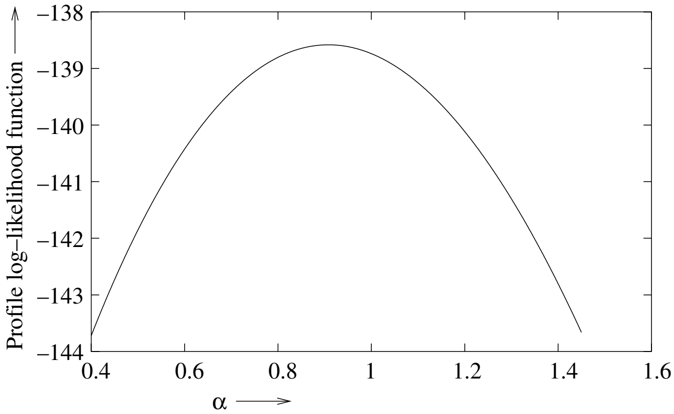

Fig. 1. Modified profile log-likelihood function for the data sets in Case-I.

Table 3
Data set 1 (gauge lengths of 20 mm).

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966
1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240
2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434
2.435	2.478	2.490	2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629
2.633	2.642	2.648	2.684	2.697	2.726	2.770	2.773	2.800	2.809	2.818
2.821	2.848	2.880	2.809	2.818	2.821	2.848	2.880	2.954	3.012	3.067
3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

Case 1: In this case we have generated two data sets using Weibull models, with the following parameter values; $\alpha = 1.0$, $\theta_1 = 1.5, \theta_2 = 1.0, \mu = 0$ and with $m = n = 25$. In this case $R = \frac{\theta_1}{\theta_1+\theta_2} = 0.6$. The data sets are presented in Tables 1 and 2.

The modified MLE of μ in this case is $\widetilde{\mu} = 0.0116$. After subtracting 0.0116 from all the data points, we plot the profile modified log-likelihood function (4) in Fig. 1. It is a unimodal function and it clearly indicates that the maximum should be close to 1. We have used the proposed iterative (9) as mentioned in Section 2, with the initial value of α as 1 and we obtain the modified MLE of α, $\widetilde{\alpha} = 0.9609$. Using $\widetilde{\alpha} = 0.9609$, we obtain $\widetilde{\theta}_1 = 1.1636$, $\widetilde{\theta}_2 = 0.9069$, $\widetilde{R} = 0.5620$. The corresponding 95% confidence interval based on asymptotic distribution is (0.4178, 0.7015). The approximate modified maximum likelihood estimator of α, θ_1 and θ_2 become $\widehat{\alpha} = 0.9475$, $\widehat{\theta}_1 = 1.2340$, $\widehat{\theta}_2 = 0.9215$. The approximate modified maximum likelihood estimator of R becomes $\widehat{R} = 0.5725$ and the corresponding 95% confidence interval becomes (0.4244, 0.7132).

Case 2: In this case we analyze the strength data, which was originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. We are thankful to Professor J.G. Surles for providing us the data. Single fibers were tested under tension at gauge lengths of 20 mm (Data Set 1) and 10 mm (Data Set 2), with sample sizes $n = 69$ and $m = 63$ respectively. The data are presented in Tables 3 and 4.

Several authors analyzed these data sets. Surles and Padgett (1998, 2001), Raqab and Kundu (2005) observed that the generalized Rayleigh distribution works quite well for these strength data. Kundu and Gupta (2006) analyzed these data sets using two-parameter Weibull distribution after subtracting 0.75 from both these data sets. After subtracting 0.75 from all the points of these data sets, Kundu and Gupta (2006) fitted Weibull distribution to both these data sets with equal shape parameter, and the Kolmogorov–Smirnov distance between the fitted and the empirical distribution functions for data set 1 and data set 2 are 0.0464 and 0.0767 and the corresponding p values are 0.9984 and 0.8525 respectively. Therefore, it indicates that three-parameter Weibull distribution with equal shape and location parameters can be fitted to data sets 1 and 2.

Table 4
Data set 2 (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445	2.454
2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659
2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937	2.977	2.996	3.030
3.125	3.139	3.145	3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332
3.346	3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628	3.852
3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020			

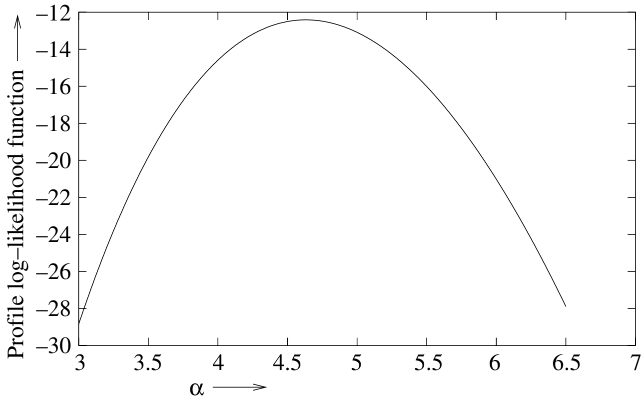

Fig. 2. Modified profile log-likelihood function for the data sets in Case-II.

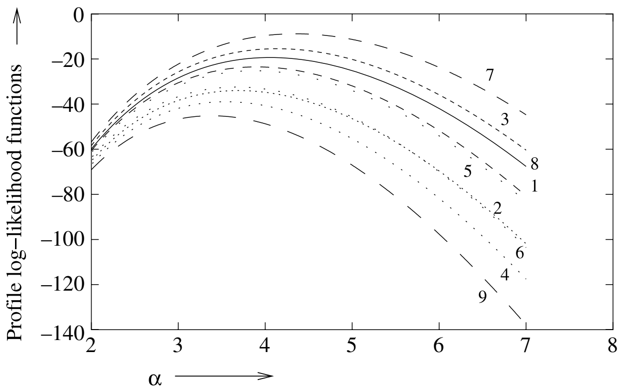

Fig. 3. Profile log-likelihood functions for different sets of (μ_1, μ_2) for the data sets in Case-II. Here 1: (0.5, 0.75), 2: (0.5, 1.0), 3: (0.75, 0.5), 4: (0.75, 1.0), 5: (1.0, 0.5), 6: (1.0, 0.75), 7: (0.5, 0.5), 8: (0.75, 0.75), 9: (1.0, 1.0).

In this case the modified MLEs of μ is $\widetilde{\mu} = 1.312$. After subtracting 1.312 from the rest of the data points we plot the profile modified log-likelihood function (4) in Fig. 2.

We obtain the modified MLEs of α, θ_1, θ_2, R as $\widetilde{\alpha} = 4.6344$, $\widetilde{\theta}_1 = 248.3652$, $\widetilde{\theta}_2 = 86.9579$ and $\widetilde{R} = 0.7406$. The 95% confidence interval of R based on asymptotic result is (0.6710, 0.8091). The approximate modified maximum likelihood estimator of α, becomes $\widehat{\alpha} = 4.3425$. The approximate modified maximum likelihood estimator of R becomes $\widehat{R} = 0.7265$ and the corresponding 95% confidence interval becomes (0.6553, 0.7976). In Kundu and Gupta (2006) when μ was assumed to be known, the MLEs of R and α were 0.7624 and 3.8770 respectively. Comparing the results with Kundu and Gupta (2006), it is clear that the location parameter plays an important role in estimation of α but the estimate of R is quite robust with respect to the location parameter estimate. Moreover, from the above experiments it is also clear that AMLEs can be used quite effectively in place of MLEs to avoid iteration method.

It may be mentioned that $\widetilde{\mu}$ is an over estimate of μ. As has been suggested by one reviewer, we would like to see the effect of the different location parameters on estimation of the stress–strength parameter, where μ_1 and μ_2 are the location parameters of X and Y respectively. For illustrative purposes, we have taken different values of (μ_1, μ_2), namely, (0.5, 0.75), (0.5, 1.0), (0.75, 0.5), (0.75, 1.0), (1.0, 0.5), (1.0, 0.75), (0.5, 0.5), (0.75, 0.75), (1.0, 1.0) and potted their profile log-likelihood functions in Fig. 3. The respective estimates of the stress–strength parameter are 0.8276, 0.8709, 0.6843, 0.8216, 0.5889, 0.6803, 0.7663, 0.7623 and 0.7563 respectively. Therefore, it is clear that the estimation of the stress–strength parameter is quite robust when the location parameters are same, otherwise they affect the estimation of the stress–strength parameter quite significantly.

6. Conclusions and discussion

In this case we consider the estimation of the stress–strength parameter of three-parameter Weibull distribution when all the parameters are unknown. It is assumed that the two populations have the same shape and location parameters, but different scale parameters. We obtain the modified maximum likelihood estimators of the unknown parameters as the usual maximum likelihood estimators do not exist. We provide the asymptotic distributions of the modified maximum likelihood estimators, which have been used to construct the asymptotic confidence intervals based on pivotal quantities. We also provide the approximate modified maximum likelihood estimators by linearizing the non-linear normal equations. The approximate modified maximum likelihood estimators can be obtained simply by solving three linear equations, and hence they can be expressed explicitly. While analyzing the data sets it is observed that the modified maximum likelihood estimators and the approximate modified maximum likelihood estimators behave in a very similar manner, and therefore for all practical purposes the approximate modified maximum likelihood estimators can be used instead of the modified maximum likelihood estimator to avoid solving non-linear equation. Although we could not provide any theoretical justification, but it is observed that the estimation of the stress–strength parameter is quite robust with respect to the common location parameter. More work is needed in this direction.

Another natural question (also raised by a reviewer) is what would happen if all the parameters are arbitrary. Note that if all the parameters are arbitrary, then R does not have any compact form. It can be expressed only in terms of integration. Although, the estimation of the six different parameters and their properties can be established along the same line, but the estimation of R and its properties need further investigation.

Acknowledgements

The authors would like to thank the reviewers for their helpful comments. Part of the work of the first author is supported by a grant from the Department of Science and Technology, Government of India.

References

Awad, A.M., Azzam, M.M., Hamadan, M.A., 1981. Some inference results in $P(Y < X)$ in the bivariate exponential model. Communications in Statistics. Theory and Methods 10, 2515–2524.
Badar, M.G., Priest, A.M., 1982. Statistical aspects of fiber and bundle strength in hybrid composites. In: Hayashi, T., Kawata, K., Umekawa, S. (Eds.), Progress in Science and Engineering Composites. ICCM-IV, Tokyo, pp. 1129–1136.
Cheng, R.C.H., Evans, B.E., Iles, T.C., 1992. Embedded models in nonlinear regression. Journal of the Royal Statistical Society 54, 877–888.
Church, J.D., Harris, B., 1970. The estimation of reliability from stress strength relationships. Technometrics 12, 49–54.
Downtown, F., 1973. The estimation of $P(X > Y)$ in the normal case. Technometrics 15, 551–558.
Govidarajulu, Z., 1967. Two sided confidence limits for $P(X > Y)$ based on normal samples of X and Y. Sankhya B 29, 35–40.
Kotz, S., Lumelskii, Y., Pensky, M., 2003. The Stress-Strength Model and Its Generalizations. World Scientific Press, Singapore.
Kundu, D., Gupta, R.D., 2005. Estimation of $P(Y < X)$ for the generalized exponential distribution. Metrika 61 (3), 291–308.
Kundu, D., Gupta, R.D., 2006. Estimation of $P(Y < X)$ for Weibull distribution. IEEE Transactions on Reliability 55 (2), 270–280.
McCool, J.I., 1991. Inference on $P(Y < X)$ in the Weibull case. Communications in Statistics. Simulation and Computations 20, 129–148.
Mudholkar, G.S., Srivastava, D.K., Kollia, G.D., 1996. A generalization of the Weibull distribution with application to the analysis of survival data. Journal of the American Statistical Association 91, 1575–1583.
Owen, D.B., Craswell, K.J., Hanson, D.L., 1977. Non-parametric upper confidence bounds for $P(Y < X)$ and confidence limits for $P(Y < X)$ when X and Y are normal. Journal of the American Statistical Association 59, 906–924.
Raqab, M.Z., Kundu, D., 2005. Comparison of different estimators of $P[Y < X]$ for a scaled Burr type X distribution. Communications in Statistics. Simulation and Computation 34 (2), 465–483.
Raqab, M.Z., Madi, M.T., Kundu, D., 2008. Estimation of $P(Y < X)$ for the 3-parameter generalized exponential distribution. Communications in Statistics. Theory and Methods 37 (18), 2854–2864.
Shao, Q., Ip, W., Wong, H., 2004. Determination of embedded distributions. Computational Statistics & Data Analysis 46, 317–334.
Smith, R.L., 1985. Maximum likelihood estimation in a class of non-regular cases. Biometrika 72, 67–90.
Surles, J.G., Padgett, W.J., 1998. Inference for $P(Y < X)$ in the Burr type X model. Journal of Applied Statistical Sciences 7, 225–238.
Surles, J.G., Padgett, W.J., 2001. Inference for reliability and stress–strength for a scaled Burr-type X distribution. Lifetime Data Analysis 7, 187–200.
Woodward, W.A., Kelley, G.D., 1977. Minimum variance unbiased estimation of $P(Y < X)$ in the normal case. Technometrics 19, 95–98.