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Modified Sarhan–Balakrishnan singular bivariate distribution

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ABSTRACT

Recently Sarhan and Balakrishnan [2007. A new class of bivariate distribution and its mixture. Journal of Multivariate Analysis 98, 1508–1527] introduced a new bivariate distribution using generalized exponential and exponential distributions. They discussed several interesting properties of this new distribution. Unfortunately, they did not discuss any estimation procedure of the unknown parameters. In this paper using the similar idea as of Sarhan and Balakrishnan [2007. A new class of bivariate distribution and its mixture. Journal of Multivariate Analysis 98, 1508–1527], we have proposed a singular bivariate distribution, which has an extra shape parameter. It is observed that the marginal distributions of the proposed bivariate distribution are more flexible than the corresponding marginal distributions of the Marshall–Olkin bivariate exponential distribution, Sarhan–Balakrishnan's bivariate distribution or the bivariate generalized exponential distribution. Different properties of this new distribution have been discussed. We provide the maximum likelihood estimators of the unknown parameters using EM algorithm. We reported some simulation results and performed two data analysis for illustrative purposes. Finally we propose some generalizations of this bivariate model.

1. Introduction

Recently Sarhan and Balakrishnan (2007) introduced a new bivariate distribution based on the generalized exponential (GE) and exponential distributions. From now on we call this distribution as the SB distribution. They derived several interesting properties of this new distribution. They obtained the marginal and conditional probability density functions (PDFs) and different moments from the moment generating function (MGF). Unfortunately, they did not discuss any estimation procedure of the unknown parameters of their proposed model. The SB model has four unknown parameters (with the presence of the scale parameter) and it is not immediate how to obtain any estimators of the unknown parameters. Moreover, without a proper estimation procedure it may be difficult to use this distribution in practice.

The main aim of this paper is to introduce a new singular bivariate (SBV) model using the similar idea as of Sarhan and Balakrishnan (2007). This new model has an extra shape parameter than the SB model. It may be mentioned that the four-parameter SB model has two shape and one scale parameters, whereas the proposed SBV model has three shape and one scale parameters. That makes SBV model more flexible than the SB bivariate model. SBV model has an absolute continuous part and a singular part, like the Marshall–Olkin bivariate exponential distribution, SB bivariate model or the bivariate generalized exponential model of Kundu and Gupta (2009). Both the absolute continuous part and the singular part can take various shapes, depending on the parameter values. We discuss various properties of this new distribution and provide different application areas. The SBV model can be observed as a competing risk model or as a shock model also.

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The marginal and conditional distributions of the SBV model have been obtained. It is observed that the marginal distributions can be obtained as weighted generalized exponential distributions. Shapes, hazard functions and different moments of the marginal distributions, are discussed. It is observed that the PDFs of the marginals can be decreasing or unimodal and the hazard functions can be increasing, decreasing or bathtub shaped. The shape of the hazard function makes it more flexible than the Marshall–Olkin bivariate exponential distribution, SB bivariate model or the bivariate generalized exponential model. Moment generating functions and different product moments of the SBV model cannot be obtained in closed forms. They can be expressed in terms of infinite series, but for certain restrictions on the shape parameters, the infinite expressions become finite only.

The proposed SBV model has four unknown parameters. To compute the maximum likelihood estimators (MLEs) directly one needs to solve a four dimensional optimization problem. It is not immediate how to solve these four non-linear equations simultaneously. To avoid that we treat this problem as a missing value problem and use the EM algorithm to compute the MLEs of the unknown parameters. For implementing the EM algorithm, at each M-step one needs to solve four one dimensional optimization problems. They are much easier to solve than the direct four dimensional optimization problem. The bootstrap technique can be used very easily to construct the confidence intervals of the unknown parameters. We have performed some simulation studies and for illustrative purposes we have analyzed two data sets using SBV model. It is observed that the proposed model and the EM algorithm work quite well in practice. Finally we discuss some generalizations of the SBV model.

The rest of the paper is organized as follows. In Section 2, we introduce the SBV model and provide two interpretations. Different properties of the SBV model namely, moment generating function, the product moments, marginal distributions, conditional distributions and hazard functions are discussed in Section 3. The implementation of the EM algorithm is provided in Section 4. Simulation results and data analysis have been presented in Section 5. Some generalizations and the conclusions appear in Section 6.

2. A singular bivariate model

Let U_1, U_2, U_3, be three mutually independent random variables and

$$U_i \sim \mathrm{GE}(\alpha_i, \lambda), \quad i = 1, 2, 3$$

Here '$\sim$' means follows or has the distribution, and $\mathrm{GE}(\alpha, \lambda)$ denotes the generalized exponential distribution with the shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ with the cumulative distribution function (CDF)

$$F(x; \alpha, \lambda) = (1 - e^{-\lambda x})^{\alpha}; \quad x > 0 \tag{1}$$

and the probability density function (PDF)

$$f(x; \alpha, \lambda) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}; \quad x > 0 \tag{2}$$

Define the random variables

$$X_1 = \min\{U_1, U_3\} \quad \text{and} \quad X_2 = \min\{U_2, U_3\}$$

Then we say the bivariate vector (X_1, X_2) has the SBV distribution with the shape parameters $\alpha_1, \alpha_2, \alpha_3$ and the scale parameter λ. It is assumed that $\alpha_i > 0$ for $i = 1, 2, 3$ and $\lambda > 0$. We will denote this distribution as $\mathrm{SBV}(\alpha_1, \alpha_2, \alpha_3, \lambda)$. Now for the rest of the discussions in Sections 2 and 3, we assume that $\lambda = 1$ for brevity, although the results are true for general λ only. The SBV distribution with $\lambda = 1$, will be denoted by $\mathrm{SBV}(\alpha_1, \alpha_2, \alpha_3)$. The PDF, CDF and the survival function of a $\mathrm{GE}(\alpha, 1)$ will be denoted by $f(\cdot; \alpha)$, $F(\cdot; \alpha)$ and $S(\cdot; \alpha)$, respectively. The following interpretations can be provided for SBV model.

Competing risks model: Suppose a system has two components, say 1 and 2 and the survival times of the two components are denoted by (X_1, X_2). It is assumed that there are three different causes of failures, which may affect the system. Due to cause 1, only component 1 can fail and similarly, due to cause 2 only component 2 can fail. But due to cause 3, both the components fail at the same time. If the lifetime distributions of the different causes are U_1, U_2, U_3, respectively, and they are independent, then (X_1, X_2) follows SBV model.

Shock model: Suppose the shocks are generated at random from three different sources, say 1, 2, and 3. The inter-arrival times between the shocks in source i follow the distribution U_i. Suppose these shocks are affecting a system with two components, say 1 and 2. If the shock generated from source 1 reaches the system, component 1 fails immediately. Similarly, if the shock generated from source 2 reaches the system, component 2 fails. But if the shock generated from source 3 hits the system both the components fail immediately. If (X_1, X_2) denote the survival time of the components, then (X_1, X_2) follows SBV model. The following results will provide the joint survival function (JSF) and joint PDF of SBV distribution.

Theorem 2.1. *If* $(X_1, X_2) \sim \mathrm{SBV}(\alpha_1, \alpha_2, \alpha_3)$, *then the JSF is*

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\ &= S(x_1; \alpha_1)S(x_2; \alpha_2)S(z; \alpha_3) \\ &= (1 - (1 - e^{-x_1})^{\alpha_1})(1 - (1 - e^{-x_2})^{\alpha_2})(1 - (1 - e^{-z})^{\alpha_3}) \end{aligned}$$

where $z = \max\{x_1, x_2\}$.

Proof. Trivial. □

Theorem 2.2. *If* $(X_1,X_2) \sim \text{SBV}(\alpha_1,\alpha_2,\alpha_3)$, *then the joint PDF of* (X_1,X_2) *can be written as follows*:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \textit{if } 0 < x_1 < x_2 \\ f_2(x_1,x_2) & \textit{if } 0 < x_2 < x_1 \\ f_0(x) & \textit{if } x_1 = x_2 = x \end{cases}$$

where

$$f_1(x_1,x_2) = f(x_1;\alpha_1)[f(x_2;\alpha_2)+f(x_2;\alpha_3)-f(x_2;\alpha_2+\alpha_3)]$$

$$f_2(x_1,x_2) = f(x_2;\alpha_2)[f(x_1;\alpha_1)+f(x_1;\alpha_3)-f(x_1;\alpha_1+\alpha_3)]$$

$$f_0(x) = \frac{\alpha_1}{\alpha_1+\alpha_3}f(x;\alpha_1+\alpha_3)+\frac{\alpha_2}{\alpha_2+\alpha_3}f(x;\alpha_2+\alpha_3)-\frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3}f(x;\alpha_1+\alpha_2+\alpha_3)$$

Proof. The expressions for $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ can be obtained from $(\partial^2/\partial x_1 \partial x_2)S_{X_1,X_2}(x_1,x_2)$ for $x_1 < x_2$ and for $x_1 > x_2$, respectively. Now using the facts

$$\int_0^\infty \int_0^{x_2} f_1(x_1,x_2)\,dx_1\,dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1,x_2)\,dx_2\,dx_1 + \int_0^\infty f_0(x)\,dx = 1$$

$$\int_0^\infty \int_0^{x_2} f_1(x_1,x_2)\,dx_1\,dx_2 = \frac{\alpha_2}{\alpha_1+\alpha_2}\int_0^\infty f(x;\alpha_1+\alpha_2)\,dx + \frac{\alpha_3}{\alpha_1+\alpha_3}\int_0^\infty f(x;\alpha_1+\alpha_3)\,dx - \frac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3}\int_0^\infty f(x;\alpha_1+\alpha_2+\alpha_3)\,dx$$

and

$$\int_0^\infty \int_0^{x_1} f_2(x_1,x_2)\,dx_2\,dx_1 = \frac{\alpha_1}{\alpha_1+\alpha_2}\int_0^\infty f(x;\alpha_1+\alpha_2)\,dx + \frac{\alpha_3}{\alpha_2+\alpha_3}\int_0^\infty f(x;\alpha_2+\alpha_3)\,dx - \frac{\alpha_1+\alpha_3}{\alpha_1+\alpha_2+\alpha_3}\int_0^\infty f(x;\alpha_1+\alpha_2+\alpha_3)\,dx$$

the result immediately follows. □

The SBV distribution has both an absolute continuous part and a singular part similar to the bivariate exponential model of Marshall and Olkin (1967), the bivariate models proposed by Sarhan and Balakrishnan (2007) and the bivariate generalized exponential model of Kundu and Gupta (2009). The function $f_{X_1,X_2}(\cdot,\cdot)$ may be considered to be a density function if it is understood that its first two terms are density functions with respect to two dimensional Lebesgue measure and the third term is a density function with respect to one dimensional Lebesgue measure, see for example Bemis et al. (1972) for a nice discussions on it. Moreover there is a positive mass on the line $x_1 = x_2$. It is well known that although in one dimension the singular distribution may not be a very practical assumptions but in higher dimensions it occurs quite naturally, see Marshall and Olkin (1967). Note that

$$f_{X_1,X_2}(x_1,x_2) = pf_a(x_1,x_2)+(1-p)f_s(x_1,x_2) \tag{3}$$

where

$$\begin{aligned} p &= \int_0^\infty \int_0^{x_2} f_1(x_1,x_2)\,dx_1\,dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1,x_2)\,dx_2\,dx_1 \\ &= 1+\frac{\alpha_3}{\alpha_1+\alpha_3}+\frac{\alpha_3}{\alpha_2+\alpha_3}-\frac{\alpha_1+\alpha_2+2\alpha_3}{\alpha_1+\alpha_2+\alpha_3} \\ &= \alpha_3\left[\frac{1}{\alpha_1+\alpha_3}+\frac{\alpha_1}{(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}\right] \end{aligned} \tag{4}$$

$$f_a(x_1,x_2) = \frac{1}{p}\begin{cases} f_1(x_1,x_2) & \text{if } x_1 < x_2 \\ f_2(x_1,x_2) & \text{if } x_2 < x_1 \end{cases} \tag{5}$$

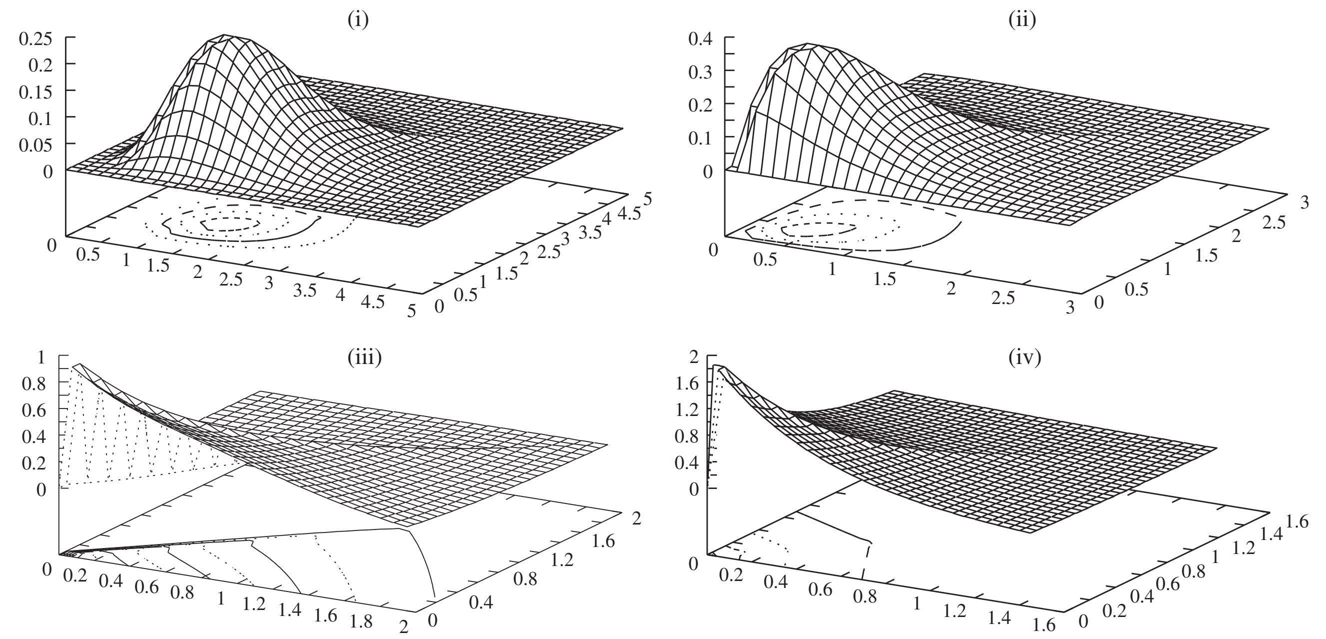

Fig. 1. The shape of the absolute continuous part of the joint probability density function of SBV for different values of α_1, α_2 and α_3, when $\lambda = 1$. Here (i) $\alpha_1 = \alpha_2 = \alpha_3 = 5$, (ii) $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$, (iii) $\alpha_1 = 5$, $\alpha_2 = \alpha_3 = 1$, (iv) $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

and

$$f_s(x_1, x_2) = \begin{cases} \dfrac{1}{1-p} f_0(z) & \text{if } x_1 = x_2 = z \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Here $f_a(x_1, x_2)$ is the absolute continuous part and $f_s(x_1, x_2)$ is the singular part. Shapes of the absolute continuous part are provided in Fig. 1.

3. Different properties

In this section we discuss different properties of the SBV model, namely the moment generating functions, product moments, marginal distributions, conditional distributions and bivariate hazard rate.

3.1. Moment generating function and product moments

In this section we present the moment generating function of (X_1, X_2) and its different product moments. We have the following results for $|t_1| < 1$ and $|t_2| < 1$.

Theorem 3.1. *If* $(X_1, X_2) \sim \text{SBV}(\alpha_1, \alpha_2, \alpha_3)$, *then*

$$M_{X_1,X_2}(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2}) = h_0(\alpha_1) h_0(\alpha_2) + h_1(\alpha_1, \alpha_3) - h_1(\alpha_1, \alpha_1 + \alpha_3) + h_2(\alpha_2, \alpha_3)$$
$$- h_2(\alpha_2, \alpha_2 + \alpha_3) + \frac{\alpha_1}{\alpha_1 + \alpha_3} h_0(\alpha_1 + \alpha_3) + \frac{\alpha_2}{\alpha_2 + \alpha_3} h_0(\alpha_1 + \alpha_3) - \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} h_0(\alpha_1 + \alpha_2 + \alpha_3)$$

where

$$h_1(\alpha, \beta) = \alpha\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{\alpha - 1}{i} \binom{\beta - 1}{j} \frac{1}{(j + 1 - t_2)} \times \frac{1}{(i + j + 2 - t_1 - t_2)}$$

$$h_2(\alpha, \beta) = \alpha\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{\alpha - 1}{j} \binom{\beta - 1}{i} \frac{1}{(j + 1 - t_1)} \times \frac{1}{(i + j + 2 - t_1 - t_2)}$$

$$h_0(\alpha) = \alpha \sum_{i=0}^{\infty} (-1)^i \binom{\alpha - 1}{i} \frac{1}{(i + 1 - t_1 - t_2)}$$

Proof. The result will follow mainly by expanding $(1-e^{-x})^{\alpha}$ in infinite series and exchanging the summation and integration. □

Similarly, the product moments can be written as follows.

Theorem 3.2. *For m and n non-negative integers*;

$$E(X_1^m X_2^n) = g_0(\alpha_1)g_0(\alpha_2) + g_1(\alpha_1,\alpha_3) - g_1(\alpha_1,\alpha_1+\alpha_3) + g_2(\alpha_2,\alpha_3) - g_2(\alpha_2,\alpha_2+\alpha_3) + \frac{\alpha_1}{\alpha_1+\alpha_3}g_0(\alpha_1+\alpha_3) + \frac{\alpha_2}{\alpha_2+\alpha_3}g_0(\alpha_2+\alpha_3) - \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3}g_0(\alpha_1+\alpha_2+\alpha_3)$$

where

$$g_1(\alpha,\beta) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\sum_{k=0}^{n}\binom{\alpha-1}{i}\binom{\beta-1}{j}\frac{n!(m+k)!}{k!(1+j)^{n+1-k}(2+i+j)^{m+k+1}}$$

$$g_2(\alpha,\beta) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\sum_{k=0}^{m}\binom{\alpha-1}{i}\binom{\beta-1}{j}\frac{m!(n+k)!}{k!(1+i)^{m+1-k}(2+i+j)^{n+k+1}}$$

$$g_0(\alpha) = \alpha\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha-1}{i}\frac{(m+n)!}{(1+i)^{m+n+1}}$$

Comment: Note that if α_1, α_2 and α_2 are integers, then the infinite sums become finite.

3.2. Marginal distributions

In this subsection we provide the marginal distribution functions and discuss some of their properties which will be useful for data analysis purposes.

Theorem 3.3. *The marginal PDFs of X_1 and X_2 for $x>0$ are given by*

$$f_{X_j}(x) = f(x;\alpha_j) + f(x;\alpha_3) - f(x;\alpha_j+\alpha_3); \quad j=1,2 \tag{7}$$

Proof. From the survival function of X_1 and X_2, the result immediately follows. □

Note that the marginal distribution is same as the distribution of the lifetime of a series system with two independent components each having GE distribution with different shape parameters and both having the scale parameter 1. Now we will discuss different lifetime properties of such a series system. Let us define the random variable $X=\min\{V_1,V_2\}$, where $V_1\sim \mathrm{GE}(\alpha,1)$ and $V_2\sim \mathrm{GE}(\beta,1)$. The survival function of X, say $S_X(\cdot;\alpha,\beta)$, can be written as

$$S_X(x;\alpha,\beta) = S(x;\alpha)S(x;\beta)$$

Therefore, the PDF of X for $x>0$ becomes

$$\begin{aligned} f_X(x;\alpha,\beta) &= f(x;\alpha) + f(x;\beta) - f(x;\alpha+\beta) \\ &= f(x;\alpha)S(x;\beta) + f(x;\beta)S(x;\alpha) \\ &= S(x;\alpha)S(x;\beta)(h(x;\alpha)+h(x;\beta)) \end{aligned} \tag{8}$$

Here $h(\cdot;\alpha)$ and $h(\cdot;\beta)$ are the hazard functions of V_1 and V_2, respectively. It is interesting to note that the PDF of X can be written as the mixture of two PDFs as follows:

$$f_X(x) = \frac{\beta}{\alpha+\beta}g(x;\alpha,\beta) + \frac{\alpha}{\alpha+\beta}g(x;\beta,\alpha) \tag{9}$$

where

$$g(x;\alpha,\beta) = \frac{\alpha(\alpha+\beta)}{\beta}e^{-x}(1-e^{-x})^{\alpha-1}(1-(1-e^{-x})^{\beta})$$

Therefore, several standard properties of the mixture distribution can also be immediately obtained. The CDF and the hazard function (HF) of X become

$$F_X(x;\alpha,\beta) = (1-e^{-x})^{\alpha} + (1-e^{-x})^{\beta} - (1-e^{-x})^{\alpha+\beta} \tag{10}$$

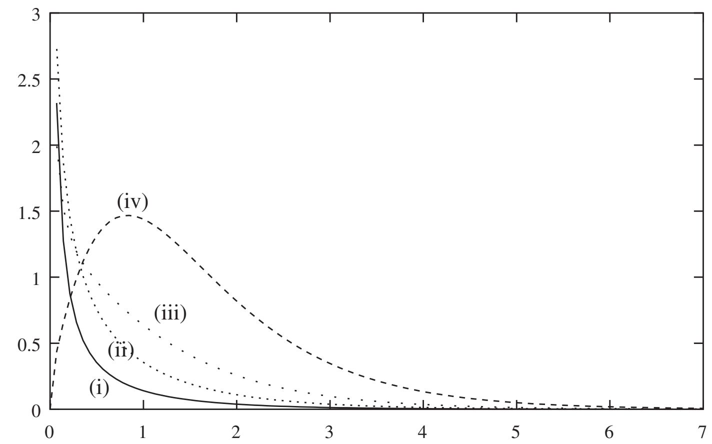

Fig. 2. Probability density function of the marginal distribution (8) for different values of α and β. Here (i) $\alpha=0.5$, $\beta=0.25$, (ii) $\alpha=0.5$, $\beta=0.75$, (iii) $\alpha=1.5$, $\beta=2.5$, (iv) $\alpha=2.0$, $\beta=0.5$.

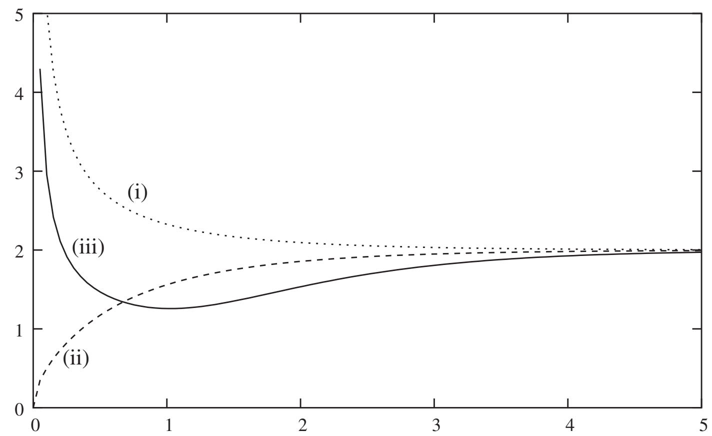

Fig. 3. Hazard function of the marginal distribution (10) for different values of α and β. Here (i) $\alpha=0.5$, $\beta=0.25$, (ii) $\alpha=1.5$, $\beta=2.5$, (iii) $\alpha=10.0$, $\beta=0.25$.

and

$$h_X(x;\alpha,\beta)=h(x;\alpha)+h(x;\beta) \tag{11}$$

respectively.

It is observed that the density function and hazard function of X can have different shapes. The density function can be decreasing or unimodal and the hazard functions can be decreasing, increasing or bathtub shaped, depending on α, β. If $\alpha>1$ and $\beta>1$ then the PDF becomes unimodal, otherwise it is a decreasing function. If both α and β are greater than one, then the HF becomes increasing and if both of them are less than one, the HF is a deceasing function. It can be bathtub shaped otherwise. Therefore, the HFs of the marginal distributions of SBV model can be either increasing, decreasing or bathtub shaped. Since the HFs of the marginals of SB model or the bivariate generalized exponential model can be either increasing or decreasing, and the hazard functions of the marginals of the Marshal–Olkin bivariate exponential model are constant, the SBV model is more flexible than the SB model, the bivariate generalized exponential or the Marshall–Olkin bivariate exponential models. Shapes of the density functions and hazard functions for different values of α and β are provided in Figs. 2 and 3.

The moment generating function (MGF) of X for $|t|<1$, can be written as

$$\begin{aligned} M_X(t)=Ee^{tX} &= \int_0^\infty e^{tx}f(x;\alpha)\,dx+\int_0^\infty e^{tx}f(x;\beta)\,dx-\int_0^\infty e^{tx}f(x;\alpha+\beta)\,dx \\ &= \frac{\Gamma(\alpha+1)\Gamma(1-t)}{\Gamma(\alpha-t+1)}+\frac{\Gamma(\beta+1)\Gamma(1-t)}{\Gamma(\beta-t+1)}-\frac{\Gamma(\alpha+\beta+1)\Gamma(1-t)}{\Gamma(\alpha+\beta-t+1)} \end{aligned} \tag{12}$$

The last equality of (12) follows from Gupta and Kundu (1999). Therefore, we can easily obtain different moments of X. For example

$$E(X) = \psi(\alpha+1) + \psi(\beta+1) - \psi(\alpha+\beta+1) - \psi(1)$$

$$E(X^2) = \psi'(\alpha+\beta+1) - \psi'(\alpha+1) - \psi'(\beta+1) + \psi'(1) + (\psi(\alpha+1) - \psi(1))^2 + (\psi(\beta+1) - \psi(1))^2 - (\psi(\alpha+\beta+1) - \psi(1))^2$$

Here $\psi(\cdot)$ and $\psi'(\cdot)$ are the digamma and polygamma functions, respectively.

3.3. Conditional distribution

The conditional PDF of X_1 given $X_2 = x_2$ has an absolute continuous part and a discrete part, similarly, the conditional PDF of X_2 given $X_1 = x_1$ also has the same form. We have the following result, the proof of which is trivial and therefore it is omitted.

Theorem 3.4. *The conditional PDF of X_1 given $X_2 = x_2$ has the following form*:

$$f_{X_1|X_2=x_2}(x_1) = p\frac{f_a(x_1,x_2)}{f_{X_2}(x_2)} + (1-p)\frac{f_s(x_1,x_2)}{f_{X_2}(x_2)}$$

where $p, f_a(\cdot,\cdot)$ and $f_s(\cdot,\cdot)$ are same as defined in (4), (5) *and* (6), *respectively*.

Theorem 3.5. *The conditional PDF of X_1 given $X_2 \geq x_2$ is*

$$f_{X_1|X_2\geq x_2}(x_1) = \begin{cases} f(x_1;\alpha_1) & \text{if } x_1 < x_2 \\ \dfrac{1}{S(x_2;\alpha_3)}\{f(x_1;\alpha_1) + f(x_1;\alpha_3) - f(x_1;\alpha_1+\alpha_3)\} & \text{if } x_1 > x_2 \end{cases}$$

Proof. The proof can be easily observed from

$$S_{X_1|X_2\geq x_2}(x_1) = P(X_1 > x_1|X_2 \geq x_2) = \begin{cases} S(x_1;\alpha_1) & \text{if } x_1 < x_2 \\ \dfrac{S(x_1;\alpha_1)S(x_1;\alpha_3)}{S(x_2;\alpha_3)} & \text{if } x_1 > x_2 \end{cases} \qquad \square$$

3.4. Bivariate failure rate

There are several ways to define bivariate failure rate. Basu (1971) first defined bivariate hazard function for an absolutely continuous bivariate distribution function. Since the proposed SBV model does not have an absolutely continuous bivariate distribution function, Basu's definition cannot be used here. It also may not uniquely define the joint distribution function. We adopt the notion proposed by Johnson and Kotz (1975) to define the bivariate hazard gradients, because it can be defined if X_1 and X_2 are two absolutely continuous distribution functions. Moreover, the bivariate hazard gradients uniquely define the bivariate joint distribution function.

The hazard gradients of the SBV are defined as $h_1(x_1,x_2) = -(\partial/\partial x_1)\ln S(x_1,x_2)$ and $h_2(x_1,x_2) = -(\partial/\partial x_2)\ln S(x_1,x_2)$, respectively. Note that

$$h_1(x_1,x_2) = -\frac{\partial}{\partial x_1}\ln S(x_1,x_2) = -\frac{\partial}{\partial x_1}\ln S(x_1|X_2 > x_2) = \begin{cases} h(x_1;\alpha_1) & \text{if } x_1 < x_2 \\ h(x_1;\alpha_1) + h(x_1;\alpha_3) & \text{if } x_1 > x_2 \end{cases}$$

Similarly, $h_2(x_1,x_2)$ also can be obtained from $h_1(x_1,x_2)$, by interchanging $x_1(\alpha_1)$ and $x_2(\alpha_2)$. Here $h(\cdot,\alpha)$ denotes the hazard function of GE$(\alpha,1)$ as before. Clearly, if $\alpha_i > 1$ for all $i = 1,2,3$, then SBV has bivariate increasing hazard rate (BIHR). Similarly, if $\alpha_i < 1$ for all $i = 1,2,3$, then SBV has bivariate decreasing hazard rate (BDHR). If (X_1,X_2) has BIHR (BDHR) and if $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are continuous increasing functions, then $(\psi_1(X_1),\psi_2(X_2))$ has also BIHR (BDHR). Moreover, for all α_i, for $i = 1,2,3$,

$$h_{X_1}(x_1;\alpha_1,\alpha_3) = h(x_1;\alpha_1) + h(x_1;\alpha_3) \geq h_1(x_1,x_2)$$

and

$$h_{X_2}(x_2;\alpha_2,\alpha_3) = h(x_2;\alpha_2) + h(x_2;\alpha_3) \geq h_2(x_1,x_2)$$

therefore, (X_1, X_2) is positive quadrant dependent. It implies, $(\psi_1(X_1), \psi_2(X_2))$ is also positive quadrant dependent, where $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are same as defined above.

4. Maximum likelihood estimators

In this section we discuss the problem of computing the maximum likelihood estimators (MLEs) of the unknown parameters of the SBV model. It is assumed that we have a sample of size n, of the form $\{(x_{11}, x_{12}), \ldots, (x_{n1}, x_{n2})\}$ from SBV$(\alpha_1, \alpha_2, \alpha_3, \lambda)$ and our problem is to estimate $\alpha_1, \alpha_2, \alpha_3, \lambda$ from the given sample. We use the following notations:

$$I_1 = \{i; x_{i1} < x_{i2}\}, \quad I_2 = \{i; x_{i1} > x_{i2}\}, \quad I_0 = \{i; x_{i1} = x_{i2} = y_i\}, \quad I = I_0 \cup I_1 \cup I_2$$

$$n_1 = |I_0|, \quad n_1 = |I_1|, \quad n_2 = |I_2|$$

Based on the above sample the log-likelihood function becomes

$$l(\alpha_1, \alpha_2, \alpha_3, \lambda) = \sum_{i \in I_1} \ln f_1(x_{i1}, x_{i2}) + \sum_{i \in I_2} \ln f_2(x_{i1}, x_{i2}) + \sum_{i \in I_0} f_0(y_i) \tag{13}$$

We need to maximize (13) with respect to the four unknown parameters. They cannot be obtained, as expected in closed form. It involves four dimensional optimization method to compute the MLEs. We propose to use the EM algorithm to compute the MLEs.

We treat this problem as missing value problem. Assume that for the bivariate random vector (X_1, X_2), there is an associated random vector (Λ_1, Λ_2) as follows:

$$\Lambda_1 = \begin{cases} 1 & \text{if } U_1 < U_3 \\ 3 & \text{if } U_1 > U_3 \end{cases} \quad \text{and} \quad \Lambda_2 = \begin{cases} 2 & \text{if } U_2 < U_3 \\ 3 & \text{if } U_2 > U_3 \end{cases}$$

Therefore, if $X_1 = X_2$, then $\Lambda_1 = \Lambda_2 = 3$, but if $X_1 < X_2$ or $X_1 > X_2$, then (Λ_1, Λ_2) is missing. If $(x_{1i}, x_{2i}) \in I_1$, then the possible values of (Λ_1, Λ_2) are $(1,2)$ or $(1,3)$, similarly, if $(x_{1i}, x_{2i}) \in I_2$, the possible values of (Λ_1, Λ_2) are $(1,2)$ or $(3,2)$ with non-zero probability.

Now we provide the 'E'-step and 'M'-step of the EM algorithm. In the 'E'-step we treat the observations belong to I_0 as complete observation and keep them intact. If the observation belong to I_1 or I_2, we treat it as a missing observation. If $(x_1, x_2) \in I_1$, we form the 'pseudo-observation' by fractioning (x_1, x_2) to two partially complete 'pseudo-observation' of the form $(x_1, x_2, u_1(\gamma))$ and $(x_1, x_2, u_2(\gamma))$, respectively. Here $\gamma = (\alpha_1, \alpha_2, \alpha_3, \lambda)$ and the fractional mass $u_1(\gamma)$, $u_2(\gamma)$ assigned to the 'pseudo-observation' is the conditional probability that the random vector (Λ_1, Λ_2) takes the values $(1,2)$ and $(1,3)$, respectively, given $X_1 < X_2$. Similarly, if $(x_1, x_2) \in I_2$, we form the 'pseudo-observation' of the form $(x_1, x_2, w_1(\gamma))$ and $(x_1, x_2, w_2(\gamma))$. Here the fractional mass $w_1(\gamma)$ or $w_2(\gamma)$ assigned to the 'pseudo-observation' is the conditional probability that the random vector (Λ_1, Λ_2) takes the values $(1,2)$ and $(3,2)$, respectively, given $X_2 < X_1$. Since

$$P(U_1 < U_3 < U_2) = \frac{\alpha_2 \alpha_3}{(\alpha_1 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)}$$

and

$$P(U_1 < U_2 < U_3) = \frac{\alpha_2 \alpha_3}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3)}$$

therefore,

$$u_1(\gamma) = \frac{\alpha_1 + \alpha_3}{2\alpha_1 + \alpha_2 + \alpha_3} \quad \text{and} \quad u_2(\gamma) = \frac{\alpha_1 + \alpha_2}{2\alpha_1 + \alpha_2 + \alpha_3}$$

Similarly,

$$w_1(\gamma) = \frac{\alpha_2 + \alpha_3}{2\alpha_2 + \alpha_1 + \alpha_3} \quad \text{and} \quad w_2(\gamma) = \frac{\alpha_1 + \alpha_2}{2\alpha_2 + \alpha_2 + \alpha_3}$$

From now on, we write $u_1(\gamma)$, $u_2(\gamma)$, $w_1(\gamma)$ and $w_2(\gamma)$ as u_1, u_2, w_1 and w_2, respectively. If $\gamma=(\alpha_1,\alpha_2,\alpha_3,\lambda)$, then the log-likelihood function of the 'pseudo-data' can be written after some simplifications as

$$\begin{aligned}
l_{pseudo}(\gamma) &= n_0\ln\alpha_3 + n_0\ln\lambda - \lambda\sum_{i\in I_0} y_i + (\alpha_3-1)\sum_{i\in I_0}\ln(1-e^{-\lambda y_i}) \\
&\quad + \sum_{i\in I_0}\ln(1-(1-e^{-\lambda y_i})^{\alpha_1}) + \sum_{i\in I_0}\ln(1-(1-e^{-\lambda y_i})^{\alpha_2}) \\
&\quad + u_2\Bigg[n_1\ln\alpha_1 + n_1\ln\lambda - \lambda\sum_{i\in I_1}x_{1i} + (\alpha_1-1)\ln(1-e^{-\lambda x_{1i}}) + n_1\ln\alpha_3 + n_1\ln\lambda \\
&\quad - \lambda\sum_{i\in I_1}x_{2i} + (\alpha_3-1)\sum_{i\in I_1}\ln(1-e^{-\lambda x_{2i}}) + \sum_{i\in I_1}\ln(1-(1-e^{-\lambda x_{2i}})^{\alpha_2})\Bigg] \\
&\quad + u_1\Bigg[n_1\ln\alpha_1 + n_1\ln\lambda - \lambda\sum_{i\in I_1}x_{1i} + (\alpha_1-1)\ln(1-e^{-\lambda x_{1i}}) + n_1\ln\alpha_2 + n_1\ln\lambda \\
&\quad - \lambda\sum_{i\in I_1}x_{2i} + (\alpha_2-1)\sum_{i\in I_1}\ln(1-e^{-\lambda x_{2i}}) + \sum_{i\in I_1}\ln(1-(1-e^{-\lambda x_{2i}})^{\alpha_3})\Bigg] \\
&\quad + w_1\Bigg[n_2\ln\alpha_1 + n_2\ln\lambda - \lambda\sum_{i\in I_2}x_{1i} + (\alpha_1-1)\sum_{i\in I_2}\ln(1-e^{-\lambda x_{1i}}) + n_2\ln\alpha_2 + n_2\ln\lambda \\
&\quad - \lambda\sum_{i\in I_2}x_{2i} + (\alpha_2-1)\sum_{i\in I_2}\ln(1-e^{-\lambda x_{2i}}) + \sum_{i\in I_2}\ln(1-(1-e^{-\lambda x_{1i}})^{\alpha_3})\Bigg] \\
&\quad + w_2\Bigg[n_2\ln\alpha_2 + n_2\ln\lambda - \lambda\sum_{i\in I_2}x_{2i} + (\alpha_2-1)\sum_{i\in I_2}\ln(1-e^{-\lambda x_{2i}}) + n_2\ln\alpha_3 + n_2\ln\lambda \\
&\quad - \lambda\sum_{i\in I_2}x_{1i} + (\alpha_3-1)\sum_{i\in I_2}\ln(1-e^{-\lambda x_{1i}}) + \sum_{i\in I_2}\ln(1-(1-e^{-\lambda x_{1i}})^{\alpha_1})\Bigg] \\
&= l_0(\lambda) + l_1(\alpha_1,\lambda) + l_2(\alpha_2,\lambda) + l_3(\alpha_3,\lambda)
\end{aligned} \tag{14}$$

where

$$l_0(\lambda) = (n_0+2n_1+2n_2)\ln\lambda - \lambda\left[\sum_{i\in I_0}y_i + \sum_{i\in I_1\cup I_2}x_{1i} + \sum_{i\in I_1\cup I_2}x_{2i}\right]$$

$$\begin{aligned}
l_1(\alpha_1,\lambda) &= (n_1+w_1n_2)\ln\alpha_1 + (\alpha_1-1)\left[\sum_{i\in I_1}\ln(1-e^{-\lambda x_{1i}}) + w_1\sum_{i\in I_2}\ln(1-e^{-\lambda x_{1i}})\right] \\
&\quad + \sum_{i\in I_0}\ln(1-(1-e^{-\lambda y_i})^{\alpha_1}) + w_2\sum_{i\in I_2}\ln(1-(1-e^{-\lambda x_{1i}})^{\alpha_1})
\end{aligned}$$

$$\begin{aligned}
l_2(\alpha_2,\lambda) &= (n_2+n_1u_1)\ln\alpha_2 + (\alpha_2-1)\left[\sum_{i\in I_2}\ln(1-e^{-\lambda x_{2i}}) + u_1\sum_{i\in I_1}\ln(1-e^{-\lambda x_{2i}})\right] \\
&\quad + \sum_{i\in I_0}\ln(1-(1-e^{-\lambda y_i})^{\alpha_2}) + u_2\sum_{i\in I_1}\ln(1-(1-e^{-\lambda x_{2i}})^{\alpha_2})
\end{aligned}$$

$$\begin{aligned}
l_3(\alpha_3,\lambda) &= (n_0+u_2n_1+w_2n_2)\ln\alpha_3 + (\alpha_3-1)\left[\sum_{i\in I_0}\ln(1-e^{-\lambda y_i}) + u_2\sum_{i\in I_1}\ln(1-e^{-\lambda x_{1i}}) + w_2\sum_{i\in I_2}\ln(1-e^{-\lambda x_{1i}})\right] \\
&\quad + u_1\sum_{i\in I_1}\ln(1-(1-e^{-\lambda x_{2i}})^{\alpha_3}) + w_1\sum_{i\in I_2}\ln(1-(1-e^{-\lambda x_{1i}})^{\alpha_3}).
\end{aligned}$$

Now at the 'M'-step we need to maximize $l_{pseudo}(\gamma)$ with respect to the unknown parameters. For fixed λ, $l_1(\alpha_1,\lambda)$, $l_2(\alpha_2,\lambda)$ and $l_3(\alpha_3,\lambda)$ can be maximized with respect to α_1, α_2 and α_3, respectively, by one dimensional optimization process. If $\tilde{\alpha_1}(\lambda)$, $\tilde{\alpha_2}(\lambda)$, $\tilde{\alpha_3}(\lambda)$ maximize $l_1(\alpha_1,\lambda)$, $l_2(\alpha_2,\lambda)$ and $l_3(\alpha_3,\lambda)$, respectively, then the maximization with respect to λ can be obtained by maximizing $l_{pseudo}(\tilde{\alpha_1}(\lambda),\tilde{\alpha_2}(\lambda),\tilde{\alpha_3}(\lambda),\lambda)$, the pseudo-profile log-likelihood function of λ. If $\tilde{\lambda}$ maximizes the pseudo-profile log-likelihood function $l_{pseudo}(\tilde{\alpha_1}(\lambda),\tilde{\alpha_2}(\lambda),\tilde{\alpha_3}(\lambda),\lambda)$ then $\tilde{\alpha_1}(\tilde{\lambda})$, $\tilde{\alpha_2}(\tilde{\lambda})$, $\tilde{\alpha_2}(\tilde{\lambda})$, $\tilde{\lambda}$ become the next iterate of the EM algorithm. Therefore the following algorithm can be used to compute the MLEs of α_1, α_2, α_3 and λ.

Algorithm. *Step* 1: Take some initial guesses of γ, say $\gamma^{(0)}=(\alpha_1^{(0)},\alpha_2^{(0)},\alpha_3^{(0)},\lambda^{(0)})$.

Step 2: Compute $u_1(\gamma^{(0)})$, $u_2(\gamma^{(0)})$, $w_1(\gamma^{(0)})$ and $w_2(\gamma^{(0)})$.

Step 3: For given $u_1(\gamma^{(0)})$, $u_2(\gamma^{(0)})$, $w_1(\gamma^{(0)})$ and $w_2(\gamma^{(0)})$ maximize $l_1(\alpha_1,\lambda_0)$, $l_2(\alpha_2,\lambda_0)$ and $l_3(\alpha_3,\lambda_0)$ with respect to α_1, α_2 and α_3, respectively. Let they be $\alpha_1^{(1)}$, $\alpha_2^{(1)}$ and $\alpha_3^{(1)}$, respectively.

Step 4: Obtain $\lambda^{(1)}$ by maximizing $l_{pseudo}(\alpha_1^{(1)}(\lambda),\alpha_2^{(1)}(\lambda),\alpha_3^{(1)}(\lambda),\lambda)$, and therefore $\gamma^{(1)}=(\alpha_1^{(1)}(\lambda^{(1)}),\alpha_2^{(1)}(\lambda^{(1)}),\alpha_3^{(1)}(\lambda^{(1)}),\lambda^{(1)})$.

Step 5: Replace $\gamma^{(0)}$ by $\gamma^{(1)}$ and go back to Step 1. Continue the process until convergence takes place.

5. Simulation and data analysis

In this section we present some simulation results to show how the proposed EM algorithm works for different sample sizes. For illustrative purposes we analyze one simulated data and one real life data.

5.1. Simulation

We consider the following parameters $\alpha_1=\alpha_2=\alpha_3=\lambda=1.0$ and $n=25, 50, 75$ and 100. All the experiments are performed at the IIT Kanpur using Pentium IV machine and we have used RAN2 random number generator of Press et al. (1992). We have used $\varepsilon=10^{-5}$ and the initial guesses as 0.5 for all the parameters. We report the average estimates and the corresponding square root of the mean squared errors based on 1000 replications. The results are reported in Table 1.

Some of the points are quite clear from Table 1. It is observed that as sample size increases the average biases and the mean squared errors decrease as expected. The performance of the EM algorithm looks quite satisfactory and the convergence is also quite fast. It is also observed (not reported here) that the initial guesses do not play much role in the performance of the EM, except the number of iterations.

5.2. Simulated data

We have generated the data using $\alpha_1=1.5$, $\alpha_2=2.5$, $\alpha_3=2.0$, $\lambda=1$ and $n=30$. The data are presented below: (0.418 0.424), (0.106 0.851), (1.147 1.147), (0.529 1.795), (0.446 0.446), (0.326 0.326), (0.205 0.205), (1.106 1.106), (0.435 0.973), (0.935 0.850), (0.362 0.645), (0.257 1.464), (1.608 1.608), (0.628 0.628), (0.351 0.351), (0.885 0.791), (0.049 0.200), (1.088 1.128), (1.453 1.155), (0.878 0.878), (0.945 0.945), (0.850 0.850), (0.354 0.354), (0.345 2.308), (1.198 0.620), (0.525 0.504), (0.548 0.548), (2.837 1.057), (0.212 1.697), (2.356 1.348).

In this case $n_0=13$, $n_1=10$ and $n_2=7$. We have started the EM algorithm with the initial guesses of $\alpha_1=1.0$, $\alpha_2=1.0$, $\alpha_3=1.0$ and $\lambda=0.5$. We have taken the value of $\varepsilon=10^{-5}$. The EM iteration stops after six steps and provide the estimates of α_1, α_2, α_3 and λ as 1.5725, 3.1405, 1.9585 and 1.0932, respectively. The corresponding 95% confidence intervals based on bootstrapping are (1.0367, 2.8692), (1.7187, 4.7924), (1.3833, 3.8889) and (0.7640, 1.4650), respectively.

Now we would like to see how the EM algorithm converges in this case. We start with the above initial guesses and at each step the estimates are as provided in Table 2. It is observed that log-likelihood function is gradually increasing and it converges in six steps only. We have tried with other initial guesses also but the EM converges to the same point. It is observed that the profile pseudo-log-likelihood function is a unimodal function at each EM step, see Fig. 4, and that ensures the convergence of the EM also.

Table 1
Average estimates, the corresponding square root of the mean squared errors and the median number of iterations required for the EM to converge.

n	α_1	α_2	α_3	λ	Median no. iteration
25	1.1516 (0.1532)	1.1376 (0.1310)	1.1426 (0.1347)	1.1229 (0.0828)	4
50	1.0696 (0.0452)	1.0651 (0.0440)	1.0639 (0.0448)	1.0631 (0.0347)	3
75	1.0409 (0.0266)	1.0407 (0.0266)	1.0409 (0.0270)	1.0393 (0.0201)	3
100	1.0297 (0.0179)	1.0265 (0.0180)	1.0323 (0.0196)	1.0291 (0.0149)	3

Table 2
Estimates and log-likelihood values at different EM steps.

Step no.	α_1	α_2	α_3	λ	Log-likelihood
1	1.5703	3.0260	1.9784	1.0887	−65.1856
2	1.5721	3.1280	1.9615	1.0928	−64.9271
3	1.5724	3.1392	1.9587	1.0931	−64.9013
4	1.5727	3.1405	1.9586	1.0932	−64.8982
5	1.5725	3.1405	1.9585	1.0932	−64.8979
6	1.5725	3.1405	1.9585	1.0932	−64.8979

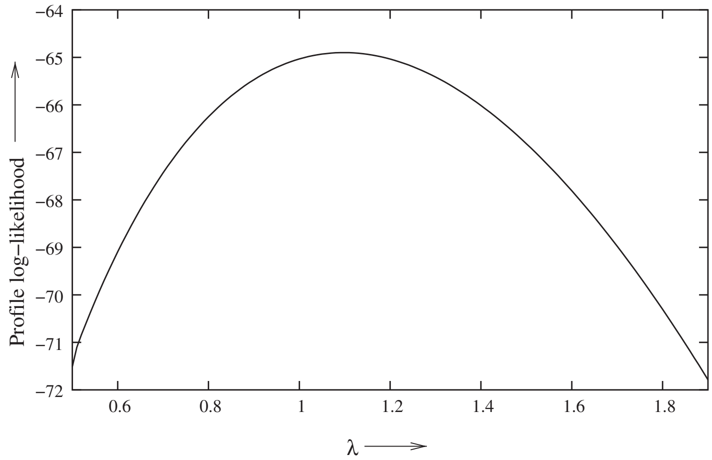

Fig. 4. The profile of the pseudo-log-likelihood function at the first EM step.

Table 3
American Football League (NFL) data.

X_1	X_2	X_1	X_2	X_1	X_2
2:03	3:59	5:47	25:59	10:24	14:15
9:03	9:03	13:48	49:45	2:59	2:59
0:51	0:51	7:15	7:15	3:53	6:26
3:26	3:26	4:15	4:15	0:45	0:45
7:47	7:47	1:39	1:39	11:38	17:22
10:34	14:17	6:25	15:05	1:23	1:23
7:03	7:03	4:13	9:29	10:21	10:21
2:35	2:35	15:32	15:32	12:08	12:08
7:14	9:41	2:54	2:54	14:35	14:35
6:51	34:35	7:01	7:01	11:49	11:49
32:27	42:21	6:25	6:25	5:31	11:16
8:32	14:34	8:59	8:59	19:39	10:42
31:08	49:53	10:09	10:09	17:50	17:50
14:35	20:34	8:52	8:52	10:51	38:04

5.3. *Real data*

These data are from the American Football (National Football League) matches played on three consecutive weekends in 1986. It has been originally published in 'Washington Post' and it is also available in Csorgo and Welsh (1989).

In this bivariate data set, the variables are the 'game time' to the first points scored by kicking the ball between goal posts (X_1) and the 'game time' to the first points scored by moving the ball into the end zone (X_2). These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game. The data (scoring times in minutes and seconds) are represented in Table 3. We have analyzed the data by converting the seconds to the decimal minutes, i.e. 2:03 has been converted to 2.05.

The variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal, (ii) $X_1 = X_2$ means the first score is a converted touchdown, (iii) $X_1 > X_2$ means the first score is an unconverted touchdown or safety. In this case the ties are exact because no 'game time' elapses between a touchdown and a point-after conversion attempt. It should be noted that the possible scoring times are restricted by the duration of the game but it has been ignored similarly as in Csorgo and Welsh (1989).

If we define the following random variables:

U_1 = time to first field goal

U_2 = time to first safety or unconverted touchdown

U_3 = time to first converted touchdown

then, $X_1 = \min\{U_1, U_3\}$ and $X_2 = \min\{U_2, U_3\}$. Therefore, (X_1, X_2) has a similar structure as the Marshall–Olkin bivariate exponential model or the proposed SBV model. Csorgo and Welsh (1989) analyzed the data using the Marshall–Olkin bivariate exponential model but concluded that it does not work well, because X_2 may be exponential but X_1 is not. It is observed that the empirical hazard function of X_1 is an increasing function.

We analyze the data using the proposed SBV model. We have taken the initial guesses of α_1, α_2, α_3 and λ are all equal to 1. The EM algorithm converges after 23 steps and the estimates of α_1, α_2, α_3 and λ become 1.7818, 4.5675, 1.1703 and 0.6858, respectively. The corresponding log-likelihood value is -153.2379. The corresponding 95% bootstrap confidence intervals are (1.2395, 3.0714), (2.6401, 7.8814), (0.8455, 1.8581) and (0.5208, 0.9420), respectively. Note that once we have the estimates of α_1, α_2, α_3 and λ, then $E(U_1)$ = expected time to the first field goal, $E(U_3)$ = expected time to converted touchdown or $P(U_1 < U_3)$ etc. which are of interest, can be easily estimated.

It is observed that the empirical hazard functions of both X_1 and X_2 are increasing functions. The estimated hazard function obtained from the proposed SBV model also indicates that. The chi-square test also supports this fact. Therefore, it may be more reasonable to fit SBV model than the Marshall–Olkin bivariate exponential model in this case.

One referee has suggested to use the four-parameter (with the addition of the scale parameter) SB-model also to analyze this data set. We have used the same notation as in Sarhan and Balakrishnan (2007) and we have used λ as the scale parameter. The estimates of λ_1, λ_2, λ_0 and λ become 5.5919, 6.6921, 0.8173 and 1.1112, respectively. The corresponding log-likelihood value is -165.8216. The 95% bootstrap confidence intervals of λ_1, λ_2, λ_0 and λ are (3.1395, 7.0714), (3.6401, 9.1141), (0.7211, 1.1581) and (0.8208, 1.3412), respectively. From the log-likelihood values, we suggest to use the proposed SBV model than the SB model.

6. Conclusions

In this paper we have proposed a new bivariate singular distribution and discussed several properties. This model has been obtained using similar technique as of Sarhan and Balakrishnan (2007) and using generalized exponential models. It is observed that the marginal distributions of the proposed model are more flexible than the corresponding marginal distributions of the Marshall–Olkin bivariate exponential model, Sarhan and Balakrishnan (2007) model or the bivariate generalized exponential model of Kundu and Gupta (2009). We have proposed the EM algorithm to compute the maximum likelihood and the performance is quite satisfactory.

It is known that the GE distribution is a proportional reversed hazard model. A distribution function $F(x;\alpha)$ is said to belong to proportional reversed hazard model if for $\alpha > 0$,

$$F(x;\alpha) = (F_0(x))^{\alpha} \tag{15}$$

where $F_0(\cdot)$ is the base distribution function. Since the introduction of the exponentiated Weibull distribution by Mudholkar and Srivastava (1993), see also Mudholkar et al. (1995), several proportional reversed hazard models have been introduced in the literature; exponentiated Rayleigh, exponentiated gamma, exponentiated Pareto, see for example the review article by Gupta and Kundu (2007) or the Gupta and Gupta (2007) and the references cited there.

Although we have introduced the SBV model using the generalized exponential distribution but similar technique can be used for other proportional reverse hazard models and most of the properties can be extended for the general case. Moreover, the EM algorithm is also possible to use for computing the MLEs of the unknown parameters. Work is in progress and it will be reported later.

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