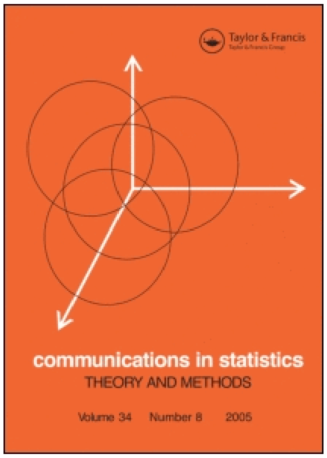

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CONSISTENCY OF THE UNDAMPED EXPONENTIAL SIGNAL MODEL ON A RESTRICTED PARAMETER SPACE

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Key Words and Phrases: least norm squares estimates; signal processing; non linear regression.

ABSTRACT

The main aim of this note is to prove the consistency of the least norm squares estimates of an undamped exponential signal model on a restricted parameter space and to compare different methods available in the literature to compute the least norm squares estimates.

1. INTRODUCTION

We consider the following model

$$y_t = \sum_{K=1}^{M} \alpha_K e^{i2\pi\omega_K t} + \epsilon_t; \qquad t = 1, \ldots n \tag{1.1}$$

Here $\alpha_1, \ldots \alpha_M$ are unknown complex numbers belonging to some compact subset of a complex Euclidean space, $i = \sqrt{-1}$, ω_K's are real numbers strictly between 0 and 1 and they are distinct. ϵ_t's are i.i.d. random variables with mean zero and equal variances for both the real and imaginary parts. The real and imaginary part are assumed to be independent. M is assumed to be known. This model is a well discussed model in signal processing literature (See Rao 1988).

We will see in the next section that the sufficient conditions stated by Jennrich (1969) do not hold true for the whole parameter space and that of Wu (1981) is difficult to verify in this situation. Strong consistency of the least norm squares estimates (LNSE) has been established on a restricted parameter space. Some simulations have been performed to compare the different methods avaliable in the literature to compute the LNSE.

2. SUFFICIENT CONDITIONS

Jennrich (1969) considered the general non-linear regression model when the parameters are real valued, the result has been extended to the complex parameters case by Kundu (1991). Suppose

$$S_n(\underset{\sim}{\theta}) = \sum_{t=1}^{n} |y_t - f(\underset{\sim}{\alpha}, \underset{\sim}{\omega}, t)|^2 \tag{2.1}$$

where $f(\underset{\sim}{\alpha}, \underset{\sim}{\omega}, t) = \sum_{K=1}^{M} \alpha_K e^{i2\pi\omega_K t}$, $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_M)$, $\omega = (\omega_1, \ldots, \omega_M)$ and $\underset{\sim}{\theta} = (\underset{\sim}{\alpha}, \underset{\sim}{\omega})$.

Any vector $\underset{\sim}{\theta}_n$ which minimizes (2.1) will be called the least norm squares estimators of $\underset{\sim}{\theta}$. The strong consistency of $\underset{\sim}{\theta}_n$ was proved in Jennrich (1969) under the following assumptions :

$\frac{1}{n} D_n(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o)$ converges uniformly to a continuous function $D(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o)$ and $D(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o) = 0$ iff $\underset{\sim}{\theta}$

$= \underset{\sim}{\theta}_o$, here $\underset{\sim}{\theta}_o$ is the true parameter value and

$$D_n(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o) = \sum_{t=1}^{n} |f(\underset{\sim}{\alpha}, \underset{\sim}{\omega}, t) - f(\underset{\sim}{\alpha}^o, \underset{\sim}{\omega}, t)|^2$$

Unfortunately even for the first order model

$$f(\underset{\sim}{\alpha}, \underset{\sim}{\omega}, t) = e^{i2\pi\omega t}$$

$D(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o) = 0$ for all $\underset{\sim}{\theta}$. Which does not satisfy Jennrich's sufficient condition. Therefore the result of Jennrich cannot be applied directly, on the other hand the assumption A or A' of Wu (1981, p 506) is difficult to verify in this situation.

Assumption 1. Let $\underset{\sim}{\alpha} \in H_1$, where H_1 is a compact subset of a complex Euclidean space $\mathbb{C}^M$ and $\underset{\sim}{\alpha}_o$, the true parameter value, is an interior point of H_1. $\underset{\sim}{\omega} \in H_2$, H_2 is a subset of [0,1]X...X[0,1] which has finite number of points.

Assumption 2. ϵ_t's are i.i.d. with mean zero and finite variances for both the real and imaginary part.

Theorem : Under the assumption 1 and 2 the least norm squares estimator of the model (1.1) is strongly consistent.

Proof : It is enough to show from the sufficient conditions of Jennrich that $D_n(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o)$ converges uniformly on $H_1 \times H_2$ to a continuous function $D(\underset{\sim}{\theta}, \underset{\sim}{\theta}_o)$ which has a unique minimum at $\underset{\sim}{\theta} = \underset{\sim}{\theta}_o$.

Let $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_M)$, $\underset{\sim}{\alpha}_o = (\alpha_1^0, \ldots, \alpha_M^0) \in H_1$ and $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_M)$, $\underset{\sim}{\omega}^o = (\omega_1^0, \ldots, \omega_M^o) \in H_2$. Therefore

$$\frac{1}{n} D_n(\underset{\sim}{\alpha}, \underset{\sim}{\omega}, \underset{\sim}{\alpha}_o, \underset{\sim}{\omega}_o) = \frac{1}{n} \sum_{t=1}^{n} \left| \sum_{K=1}^{M} \alpha_K e^{i2\pi\omega_K t} - \sum_{K=1}^{M} \alpha_K^o e^{i2\pi\omega_K^o t} \right|^2$$

$$= \frac{1}{n} \sum_{t=1}^{n} \left\{ \sum_{K=1}^{M} \alpha_K e^{i2\pi\omega_K t} - \sum \alpha_K^o e^{i2\pi\omega_K^o} \right\} x$$

$$\left\{ \sum_{K=1}^{M} \bar{\alpha}_K e^{-i2\pi\omega_K t} - \sum \bar{\alpha}_K^o e^{-i2\pi\omega_K^o} \right\}$$

Let's assume $\underset{\sim}{\omega} \neq \underset{\sim}{\omega}^o$, we show that all the four components converges uniformly over $H_1 \times H_2$. First let's assume that none of the components of ω is equal to any one of the $\underset{\sim}{\omega}^o$, then the first term

$$\frac{1}{n} \sum_{t=1}^{n} \sum_{K=1}^{M} \sum_{\ell=1}^{M} \alpha_K \bar{\alpha}_\ell e^{i2\pi(\omega_K - \omega_\ell)t} =$$

$$\sum_{K=1}^{M} \sum_{\ell=1, \neq K}^{M} \alpha_K \bar{\alpha}_\ell \frac{1}{n} e^{i2\pi(\omega_K - \omega_\ell)} \frac{(1 - i2\pi n(\omega_K - \omega_\ell))}{(1 - e^{i2\pi}(\omega_K - \omega_\ell))} + \sum_{K=1}^{M} |\alpha_K|^2$$

converges to $\sum_{K=1}^{M} |\alpha_K|^2$ uniformly over $H_1 \times H_2$. Similarly it can be shown that $\frac{1}{n} D_n$ converges uniformly to $\sum_{K=1}^{M} \left\{ |\alpha_K|^2 + |\alpha_K^o|^2 \right\}$. In the same way, it can be shown that if r of the components of $\underset{\sim}{\omega}$ and $\underset{\sim}{\omega}^o$ are equal and if we assume without loss of generality that it is the first r of them, then $\frac{1}{n} D_n$ converges uniformly to $\sum_{K=1}^{r} |\alpha_K - \alpha_K^o|^2 + \sum_{K=r+1}^{M} (|\alpha_K|^2 + |\alpha_K^o|^2)$. It shows the uniqueness also.

Since in practice the parameter space is always finite in nature (say up to eight decimal places) therefore the Assumption 1 is not a restricted assumption at all.

3. COMPUTATION OF LNSE

In this section we will briefly mention the different methods available in the literature to compute the LNSE. Observe that we can write the model (1.1) in the following form:

$$Y = A(\omega)\alpha + \varepsilon \tag{4.1}$$

where $Y^T = (y_1,\dots,y_n)$, $\omega^T = (\omega_1,\dots,\omega_M)$, $\alpha^T = (\alpha_1,\dots,\alpha_M)$, $\varepsilon^T = (\varepsilon_1,\dots,\varepsilon_n)$ and

$$A(\omega) = \begin{bmatrix} e^{i2\pi\omega_1} & \dots & e^{i2\pi\omega_M} \\ \vdots & & \vdots \\ e^{i2\pi\omega_1 n} & \dots & e^{i2\pi\omega_M n} \end{bmatrix} \tag{4.2}$$

Therefore the LNSE of (α,ω) can be obtained by minimizing

$$R(\alpha,\omega) = (Y - A(\omega)\alpha)^H(Y - A(\omega)\alpha) \tag{4.3}$$

with respect to (α,ω). Here 'H' denotes the complex conjugate transpose of a matrix or of a vector. It is interesting to note that α is separable from ω. For a fixed ω, the LNSE of α is

$$\hat{\alpha} = (A(\omega)^H A(\omega))^{-1} A^H(\omega) Y \tag{4.4}$$

Therefore the LNSE of ω can be obtained by minimizing

$$\begin{aligned} Q(\omega) &= (Y - A(\omega)\hat{\alpha})^H(Y - A(\omega)\hat{\alpha}) \\ &= Y^H [I - A(\omega)(A(\omega)^H A(\omega))^{-1} A^H(\omega)] Y \end{aligned} \tag{4.5}$$

with respect to ω. Once we obtain the LNSE of ω, the LNSE of α can be easily obtained by substituting $\hat{\omega}$, in place of ω in (4.4).

Different methods are avaliable in the literature to minimize (4.5). Parthasarathy and Tufts (1985) have suggested to minimize (4.5) by some standard non-linear minimization technique. Bresler and Macovski (1986) and Kumaresan, Scharf and Shaw (1986) independently suggested one iterative technique to minimize (4.5).

Recently Kundu (1993) has suggested a non-linear eigenvalue technique to minimize (4.5). A numerical comparison of all the different methods are presented in the next section.

4. NUMERICAL EXPERIMENTS

In this section we present some numerical results mainly to observe how the different methods behave for small samples. All these simulations are done using IMSL random deviate generator. We consider the following model:

$$y(t) = e^{2.0it} + e^{3.5it} + 2e^{4.0it} + 2e^{5.5it} + \epsilon(t)$$
$$t = 1,\ldots,N$$

Here $\epsilon(t)$ are i.i.d. complex valued random variables with mean zero and standard deviation σ for both the real and imaginary part. The real and imaginary parts are independent and normally distributed. Changing n and σ, 100 different sets of data are generated. Numerical results are reported for N = 20, 30, 40, and σ = .5, .1 and .01.

For each data set, we calculate LNSE of ω by using modified Prony algorithm (MP) as described in Kundu (1993), by Iterated Quadratic Maximum Likelihood (IQML) method as proposed by Bresler and Macovski (1986) and by using IMSL minimization package. We also calculate the Cramar- Rao lower bound (CRLB) in each case for comparison. We use the ordinary EVLP estimate (see Rao; 1988 or Kundu; 1993) as an initial estimate in all the cases. Observe that the stopping rule is different in all the methods. For MP we use the same stopping rule as it was used in Kundu (1993). For IQML and IMSL we use seven steps iterations. The mean estimate of the ω's and their mean squared errors over 100 replications are reported. We report all the

TABLE I

N = 20 σ = .01

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.387E-3)	3.50000 (.387E-3)	4.00000 (.194E-3)	5.50000 (.194E-3)
MP	2.00008 (.469E-3)	3.50014 (.458E-3)	4.00006 (.246E-3)	5.50008 (.247E-3)
IQML	2.00001 (.442E-3)	3.49998 (.458E-3)	4.00002 (.244E-3)	5.50003 (.248E-3)
IMSL	2.00079 (.991E-3)	3.50016 (.931E-3)	3.99912 (.537E-3)	5.49930 (.612E-3)

N = 20 σ = .1

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.387E-2)	3.50000 (.387E-2)	4.00000 (.194E-2)	5.50000 (.194E-2)
MP	2.00078 (.455E-2)	3.50147 (.458E-2)	4.00069 (.247E-2)	5.50088 (.248E-2)
IQML	1.99999 (.414E-2)	3.50056 (.446E-2)	4.00006 (.249E-2)	5.49946 (.247E-2)
IMSL	1.99758 (.917E-2)	3.49686 (.912E-2)	4.00097 (.537E-2)	5.50012 (.512E-2)

N = 20 σ = .50

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.194E-1)	3.50000 (.194E-1)	4.00000 (.970E-2)	5.50000 (.970E-2)
MP	2.00268 (.206E-1)	3.50691 (.207E-1)	4.00314 (.122E-1)	5.50296 (.122E-1)
IQML	2.00542 (.218E-1)	3.50271 (.221E-1)	4.00155 (.127E-1)	5.49751 (.125E-1)
IMSL	1.93032 (.891E-1)	3.40095 (.993E-1)	4.01943 (.581E-1)	5.50589 (.554E-1)

TABLE II

N = 30 σ = .01

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.211E-3)	3.50000 (.211E-3)	4.00000 (.105E-3)	5.50000 (.105E-3)
MP	2.00007 (.236E-3)	3.49997 (.253E-3)	4.00062 (.110E-3)	5.50002 (.129E-3)
IQML	2.00003 (.237E-3)	3.50005 (.247E-3)	3.99996 (.117E-3)	5.49996 (.122E-3)
IMSL	1.99907 (.510E-3)	3.49643 (.643E-3)	3.99943 (.331E-3)	5.49948 (.212E-3)

N = 30 σ = .1

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.211E-2)	3.50000 (.211E-2)	4.00000 (.105E-2)	5.50000 (.105E-2)
MP	2.00074 (.220E-2)	3.49981 (.254E-2)	4.00030 (.111E-2)	5.50028 (.129E-2)
IQML	2.00035 (.238E-2)	3.50049 (.248E-2)	3.99963 (.117E-2)	5.49962 (.128E-2)
IMSL	2.00045 (.987E-2)	3.50194 (.931E-2)	4.00078 (.611E-2)	5.50028 (.539E-2)

N = 30 σ = .50

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000 (.105E-1)	3.50000 (.105E-1)	4.00000 (.527E-2)	5.50000 (.527E-2)
MP	2.00365 (.108E-1)	3.49938 (.131E-1)	4.00162 (.538E-2)	5.50124 (.549E-2)
IQML	2.00378 (.124E-1)	3.50184 (.138E-1)	3.99832 (.598E-2)	5.49762 (.599E-2)
IMSL	1.92465 (.632E-1)	3.49560 (.541E-1)	4.12463 (.245E-1)	5.51468 (.277E-1)

TABLE III

$N = 40 \quad \sigma = .01$

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000	3.50000	4.00000	5.50000
	(.137E-3)	(.137E-3)	(.685E-4)	(.685E-4)
MP	2.00003	3.49999	4.00001	5.50001
	(.180E-3)	(.182E-3)	(.686E-4)	(.725E-4)
IQML	2.00002	3.50002	3.99998	5.49998
	(.149E-3)	(.160E-3)	(.688E-4)	(.722E-4)
IMSL	1.99989	3.49968	3.99974	5.49986
	(.320E-3)	(.414E-3)	(.108E-3)	(.114E-3)

$N = 40 \quad \sigma = .1$

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000	3.50000	4.00000	5.50000
	(.137E-2)	(.137E-2)	(.685E-3)	(.685E-3)
MP	2.00004	3.49984	3.99985	5.50015
	(.181E-2)	(.181E-2)	(.692E-2)	(.835E-2)
IQML	2.00025	3.50017	3.99982	5.49983
	(.169E-2)	(.171E-2)	(.698E-2)	(.841E-2)
IMSL	1.99988	3.49131	4.00203	5.50023
	(.615E-2)	(.712E-2)	(.321E-2)	(.330E-2)

$N = 40 \quad \sigma = .50$

Method\	ω_1	ω_2	ω_3	ω_4
CRLB	2.00000	3.50000	4.00000	5.50000
	(.137E-2)	(.137E-2)	(.685E-3)	(.685E-3)
MP	2.00039	3.49994	4.00013	5.50018
	(.181E-2)	(.181E-2)	(.692E-3)	(.835E-3)
IQML	2.00025	3.50017	3.99982	5.49983
	(.169E-2)	(.171E-2)	(.698E-3)	(.841E-3)
IMSL	1.98669	3.52242	4.13035	5.50567
	(.735E-2)	(.690E-2)	(.351E-2)	(.376E-2)

5. CONCLUSIONS

The numerical results confirm the consistency of the LNSE on a restricted parameter space. In all the three cases, for fixed σ as N increases the MSE decreases and for fixed N as σ decreases MSE decreases. Comparing the behavior of the three methods it is observed that MP and IQML behave almost similarly in all the situations and both of them are much better than IMSL. Since it is known that (Kundu; 1994) if we start from a good starting value MP procedure converges almost surely, therefore it is advisable to use MP at least for large sample with EVLP as a starting value. It is observed that MP and IQML reach the CRLB particularly when N is large and σ is small. It seems that general purpose algorithm does not work very well and special purpose algorithm is needed to obtain the LNSE for this problem. Although consistency can't be achieved for the whole parameter space, for all practical purposes the restricted parameter space is sufficient to us as far as estimation is concerned.

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