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Consistent estimates of super imposed exponential signals when some observations are missing

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Abstract

Methods are proposed for estimating the parameters of undamped exponential signals when observations are missing. Some consistency results have been established. The finite sample behavior of the proposed methods have been studied by Monte Carlo simulation.

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Key words: Exponential signals; Consistent estimates; Missing observations; Modal analysis; Matrix pencil

1. Introduction

The problem of detecting signals in the midst of noise arises in communications, radio location of objects, seismic signal processing and computer-assisted medical diagnosis. For example, in electromagnetic pulse (EMP) situations (Ricketts et al., 1976; Sircar, 1987), the EMP pickup can be characterized by a sum of complex exponentials whose parameters are to be determined. The parameters are means of coding the various pulse wave forms, and the signal approximation thus obtained can be readily employed to analyze responses in various subsystems under an EMP environment.

In this paper we consider the following undamped exponential model

$$y_k = \sum_{j=1}^{M} \alpha_j \, e^{i\beta_j k} + \varepsilon_k, \quad k = 1, 2, \ldots, N. \tag{1.1}$$

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Here α_j's are unknown complex number and β_j's are real numbers lying between $(0, 2\pi)$ and they are distinct. The ε_k's are i.i.d. complex-valued random variables with mean zero and finite variance for both the real and imaginary parts. The real and imaginary parts are assumed to be independent. M is assumed to be known $i = \sqrt{-1}$.

This is an important poblem in what might be called modal analysis in the signal processing literature. The literature pertaining to the multiple sinusoid model goes back almost 200 years. Prony (1795) in his classic paper proposed a solution to the problem of fitting sums of exponentials to equispaced data. The wide applicability of the model has lead to numerous books and papers over the past 20 years. To name a few, see Kay and Marple (1981), Marple (1987), Kay (1987), Froberg (1969), Tufts and Kumaresan (1982), Bai et al. (1987), Kumaresan et al. (1986), Breslar and Macovski (1986), Rao (1988), Kundu (1993) and for recent references see the Ph.D. Thesis of Kannan (1992). The existing methods of estimation of β's are mainly based on the fact that the time points are equidistant.

The aim of this paper is to obtain an efficient method for estimating the non-linear parameters, namely β's, when one or more observations are missing. If some observations are missing none of the above methods can be applied directly because the data points are no more equidistant. Furthermore, for model identification, i.e. the estimation of M, it is observed (Kundu, 1992) that Information Theoretic Criteria work quite well for large sample sizes but for small sample sizes, Rao (1988) suggested a cross-validation technique for this problem. To obtain the cross-validation error for a particular M, it is required to estimate efficiently α's and β's in presence of the missing observations. Note that once we obtain an efficient estimator of the non-linear parameters, the estimation of the linear parameters can be obtained by simple linear regression.

In Section 2 we develop a method of estimation, which is a modified version of what is usually called Pisarenko's method or the EquiVariance Linear Prediction (EVLP) method (see Rao, 1988). The strong consistency of the method is established in Section 3. It is well known that Pisarenko's method is a poor performer among all the methods of modal analysis particularly for small samples. Some modifications like reduced rank linear prediction, matrix pencil, equations error that form the basis for high performance modal analysis at low to moderate signal-to-noise ratios (SNR), are suggested in Section 4. Some simulation results are presented in Section 5 and finally we draw conclusions in Section 6.

2. Modified Pisarenko's method

First we give a procedure when just one observation is missing. Then we generalize the method when more than one observation is missing.

Here $M<N$ and let Y be a $N-M\times M+1$ Hankel data matrix as follows:

$$Y=\begin{bmatrix} Y_1 & \cdots & Y_{M+1} \\ \vdots & & \vdots \\ Y_{m-M} & \cdots & Y_m \\ \vdots & & \vdots \\ Y_m & \cdots & Y_{m+M} \\ \vdots & & \vdots \\ Y_{N-M} & & Y_N \end{bmatrix}. \tag{2.1}$$

We first assume that only the mth observation is missing. Let Y_m be the matrix obtained from Y by removing the rows which contain mth observation, i.e. if we denote

$$Y=[Y_1\ Y_2\ \cdots\ Y_{N-M}]^{\mathrm{T}}, \tag{2.2}$$

then

$$\begin{aligned} Y_m &= [Y_{M+1}\ Y_{M+2}\ \cdots\ Y_{N-M}]^{\mathrm{T}} \quad \text{if } 1\leqslant m\leqslant M+1 \\ &= [Y_1\ \cdots\ Y_{m-M+1}\ Y_{M+1}\ Y_{N-M}]^{\mathrm{T}} \quad \text{if } M+2\leqslant m\leqslant N-M-1 \\ &= [Y_1\ \cdots\ Y_{N-M}]^{\mathrm{T}} \quad \text{if } N-M\leqslant m\leqslant N. \end{aligned} \tag{2.3}$$

Consider the $M+1\times M+1$ matrix $R_m=Y_m^*Y_m$. Here * denotes the complex conjugate transpose of a matrix. Let $b=(b_1,\dots,b_{M+1})$ such that $b^*b=1$; $b_1>0$, be the normalized eigenvector corresponding to the smallest eigenvalue of R_m. The following prediction error polynomial equation

$$B(Z)=b_1+b_2Z+\cdots+b_{M+1}Z^M=0 \tag{2.4}$$

has M roots, namely $\hat{\rho}_k\,e^{i\hat{\beta}_k}$, where $\hat{\rho}_k>0$ and $\hat{\beta}_k\in(0,2\pi)$ for $k=1,2,\dots,M$. We can take $(\hat{\beta}_1,\dots,\hat{\beta}_M)$ with proper rearrangement as the estimates of $(\beta_1,\dots,\beta_M)$.

If more than one observation is missing the same approach can be used, i.e. suppose $m_1,\dots,m_r$th observations are missing, then construct the matrix $Y_{m_1m_2\cdots m_r}$ from Y by deleting the rows of Y which contains at least one of these m_i, $i=1,\dots,r$. Proceeding exactly as before we can have the estimates of β's.

Observe that if no observation is missing, this method coincides with the degenerate version of linear prediction method, usually called Pisarenko's (see Pisarenko, 1973) method or EVLP method (see Bai et al., 1987 or Rao, 1988). This method actually finds its basis in a theorem of Caratheodory. It is observed that every singular non-negative definite symmetric Toeplitz matrix has a spectral representation (the Herglotz representation) in which the spectrum is a line spectrum. It then follows that the roots of the prediction error polynomial $B(z)$, as defined in (2.4), are the line frequencies (see Tufts and Kumaresan, 1982).

3. Consistency of the modified Pisarenko's methods

We first prove the consistency of the modified Pisarenko's method when only one observation is missing and the missing observation follows a particular pattern. We finally prove the result when the observation misses arbitrarily. The result can be easily generalized for more than one observation case.

Lemma 3.1. *Let $Y_1, \ldots, Y_N$ be a sample from the model* (1.1). *Let the m_Nth observation be missing from the sample of size N. If $\hat{\beta}_1, \ldots, \hat{\beta}_M$ are the estimates of $\beta_1, \ldots, \beta_M$, obtained by the method described in Section* 2 *and* $\lim_{N\to\alpha}(m_N/N)=c$, $0\leqslant c\leqslant 1$, *then $\hat{\beta}_1, \ldots, \hat{\beta}_M$ are consistent estimates of $\beta_1, \ldots, \beta_M$.*

Proof. First let us assume $M+2\leqslant m_N\leqslant N-M-1$ for all N. Then

$$Y_{m_N}=\begin{bmatrix} Y_1 & \cdots & Y_{M+1} \\ \vdots & & \vdots \\ Y_{m_N-1-M} & \cdots & Y_{m_N-1} \\ \cdots & \cdots & \cdots \\ Y_{m_N+1} & \cdots & Y_{m_N+1+M} \\ \vdots & & \vdots \\ Y_{N-M} & & Y_N \end{bmatrix}=\begin{bmatrix} B_{1N} \\ \cdots \\ B_{2N}\end{bmatrix}, \tag{3.1}$$

$$\frac{1}{N}Y^*_{m_N}Y_{m_N}=\frac{1}{N}B^*_{1N}B_{1N}+\frac{1}{N}B^*_{2N}B_{2N}$$

$$=\frac{m_N-1-M}{N}\left(\frac{1}{m_N-1-M}B^*_{1N}B_{1N}\right)+\frac{N-m_N}{N}\left(\frac{1}{N-m_N}B^*_{2N}B_{2N}\right).$$

Let $m_N-1=v$ for brevity. Consider the matrix $(1/(v-M))B^*_{1N}B_{1N}=(b_{pq})$. Therefore,

$$b_{pq}=\frac{1}{v-M}\sum_{s=0}^{v-M-1}\left(\sum_{k=1}^{M}\bar{\alpha}_k\,\mathrm{e}^{-\mathrm{i}(p+s)\beta_k}+\bar{\varepsilon}_{p+s}\right)\left(\sum_{k=1}^{M}\alpha_k\,\mathrm{e}^{\mathrm{i}(q+s)\beta_k}+\varepsilon_{q+s}\right)$$

$$=\frac{1}{v-M}\sum_{s=0}^{v-M-1}\left[\left(\sum_{k=1}^{M}\bar{\alpha}_k\,\mathrm{e}^{-\mathrm{i}(p+s)\beta_k}\right)\left(\sum_{k=1}^{M}\alpha_k\,\mathrm{e}^{\mathrm{i}(q+s)\beta_k}\right)\right.$$

$$\left.+\left(\sum_{k=1}^{M}\alpha_k\,\mathrm{e}^{\mathrm{i}(q+s)\beta_k}\right)\bar{\varepsilon}_{p+s}+\left(\sum_{k=1}^{M}\bar{\alpha}_k\,\mathrm{e}^{-\mathrm{i}(p+s)\beta_k}\right)\varepsilon_{q+s}+\bar{\varepsilon}_{p+s}\varepsilon_{q+s}\right].$$

By the law of the iterated logarithm of M-dependent sequence

$$b_{pq} \xrightarrow{\text{a.s.}} (\Omega^* A\,\Omega + 2\sigma^2 I_{M+1})_{pq} + \mathrm{O}\left(\frac{\log\log v}{v}\right)^{1/2}, \tag{3.2}$$

where $A = \operatorname{diag}\{|\alpha_1|^2, \ldots, |\alpha_M|^2\}$ is a $M \times M$ matrix,

$$\Omega = \begin{bmatrix} e^{i\beta_1} & e^{2i\beta_1} & \cdots & e^{(M+1)i\beta_1} \\ \vdots & \vdots & & \vdots \\ e^{i\beta_M} & e^{2i\beta_M} & \cdots & e^{(M+1)i\beta_M} \end{bmatrix} \tag{3.3}$$

and $(\Omega^* A\,\Omega + 2\sigma^2 I_{M+1})_{pq}$ denotes the pqth element of the matrix $(\Omega^* A\,\Omega + 2\sigma^2 I_{M+1})$. Since similar result is true for $(1/(N-m_N))\, B_{2N}^* B_{2N}$ also, therefore, we can conclude that

$$\frac{1}{N}\, Y_{m_N}^* Y_{m_N} = \Omega^* A\,\Omega + 2\sigma^2 I_{M+1} + \mathrm{O}\left(\frac{\log\log N}{N}\right)^{1/2} \quad \text{a.s.} \tag{3.4}$$

Now observe that the eigenvalues of $\Omega^* A\,\Omega + 2\sigma^2 I_{M+1}$ are of the form $2\sigma^2 + \lambda_1 \geqslant 2\sigma^2 + \lambda_2 \geqslant \cdots \geqslant 2\sigma^2 + \lambda_M > 2\sigma^2$. Where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_M$ are M non-zero eigenvalues of $\Omega^* A\,\Omega$ and the $(M+1)$th eigenvalue is zero, since the rank of $\Omega^* A\,\Omega$ is M. From (3.4) it follows that

$$\frac{1}{N} Y_{m_N}^* Y_{m_N} \to \Omega^* A\,\Omega + 2\sigma^2 I_{M+1} \quad \text{a.s.} \tag{3.5}$$

Therefore, the eigenvector, with proper normalization as it is described in Section 2, corresponding to the smallest eigenvalue will converge to the unique eigenvector, with the same kind of normalization, corresponding to the zero eigenvalue of $\Omega^* A\,\Omega$. It is known (see Pisarenko, 1973) that if $(b_1, \ldots, b_M)$ is the eigenvector corresponding to the zero eigenvalue of $\Omega^* A\,\Omega$ and the polynomial equation is formed as (2.4) with these b_i's, then the M roots will be $e^{i\beta_1}, \ldots, e^{i\beta_M}$. Therefore, from (3.5) it follows that with proper rearrangements

$$\hat{\beta}_i \to \beta_i \quad \text{a.s. } i = 1, 2, \ldots, M. \tag{3.6}$$

For $m_N < M+2$ or $m_N > N-M-1$ it can be proved similarly. We can now prove the result for the general case.

Theorem 3.1. *Let $y_1, \ldots, y_N$ be a sample from the model* (1.1). *Let m_Nth observation be missing from the sample of size N. If $\hat{\beta}_1, \ldots, \hat{\beta}_M$* are the estimates of *$\beta_1, \ldots, \beta_M$, obtained by the method described in Section 2, and $1 \leqslant m_N \leqslant N$ is arbitrary then $\hat{\beta}_1, \ldots, \hat{\beta}_M$ are strongly consistent estimates of $\beta_1, \ldots, \beta_M$.*

Proof. From (3.2) and (3.4) it follows that

$$\frac{1}{N} Y^*_{m_N} Y_{m_N} = \Omega^* A \Omega + 2\sigma^2 I_{M+1} + \frac{m_N - 1 - M}{N} O\left(\frac{\log\log(m_N - 1 - M)}{m_N - 1 - M}\right)^{1/2}$$

$$+ \frac{N - m_N}{N} O\left(\frac{\log\log(N - m_N)}{N - m_N}\right)^{1/2}. \tag{3.7}$$

Now consider the different cases:

Case 1: m_N is bounded. Then,

$$\frac{m_N - 1 - M}{N} \to 0 \quad \text{and} \quad \frac{\log\log(N - m_N)}{N - m_N} \to 0. \tag{3.8}$$

Case 2: m_N is not bounded but $N - m_N$ is bounded. Then,

$$\frac{\log\log(m_N - 1 - M)}{m_N - 1 - M} \to 0 \quad \text{and} \quad \frac{N - m_N}{N} \to 0. \tag{3.9}$$

Case 3: m_N is not bounded and $N - m_N$ is also not bounded. Then,

$$\frac{\log\log(m_N - 1 - M)}{m_N - 1 - M} \to 0 \quad \text{and} \quad \frac{\log\log(N - m_N)}{N - m_N} \to 0. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) it follows that

$$\frac{1}{N} Y^*_{m_N} Y_{m_N} \to \Omega^* A \Omega + 2\sigma^2 I_{M+1} \quad \text{a.s.} \tag{3.11}$$

The rest of the proof follows exactly like that of Lemma 3.1. □

4. Some improved estimates

In this section we try to modify the existing methods like reduced rank linear prediction, matrix pencil or the equations error that form the basis of high-performance modal analysis at low-to-moderate SNR, in presence of missing observation.

4.1. *Modified FBLP method*

First let us assume that only one observation is missing, i.e. the mth observation. The method can be easily generalized when more than one observation is missing.

Observe that there exist $g_1, \ldots, g_L$, such that in the noiseless data when the mth. observation is missing the $g=(g_1, \ldots, g_L)$ satisfy the following equation:

$$\begin{bmatrix} Y_L & \cdots & Y_1 \\ \vdots & \cdots & \vdots \\ Y_{m-2} & \cdots & Y_{m-L} \\ Y_{m+L} & \cdots & Y_{m+1} \\ Y_{N-1} & \cdots & Y_{N-L} \\ Y_2^* & & Y_{L+1}^* \\ \vdots & \cdots & \vdots \\ Y_{m-L}^* & & Y_{m-1}^* \\ Y_{m+2}^* & & Y_{m+L+1}^* \\ \vdots & \cdots & \vdots \\ Y_{N-L+1}^* & & Y_N^* \end{bmatrix} \begin{bmatrix} g_L \\ \vdots \\ g_1 \end{bmatrix} = - \begin{bmatrix} Y_{L+1} \\ \vdots \\ Y_{m-1} \\ Y_{m+L+1} \\ Y_N \\ Y_1^* \\ \vdots \\ Y_{m-L-1}^* \\ Y_{m+1}^* \\ \vdots \\ Y_{N-L}^* \end{bmatrix}. \tag{4.1}$$

If $L+1<m \leqslant N-L-1$. For $m \leqslant L+1$ or $m>N-L-1$ it can be similarly defined. The system of equations can be written as

$$A_m g = -h_m, \tag{4.2}$$

where the matrix A_m and the vector h_m depend on the value of the missing observation. Now the minimum norm solution of g is given by

$$g = -(A_m^* A_m)^- A_m^* h_m, \tag{4.3}$$

where - is the pseudoinverse of A_m given by Rao (1971). Now if we use the usual linear prediction notation, $R=A_m^* A_m$ and $r=-A_m^* h$, then it can be easily shown as Tufts and Kumaresan (1982), that in the noiseless data

$$g = -\sum_{i=1}^{M} \frac{u_i^* r}{\gamma_i} u_i, \tag{4.4}$$

where $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_m > \gamma_{m+1} = \cdots = \gamma_L = 0$ are the eigenvalues of R and u_i, $i=1, \ldots, L$ are the corresponding orthonormal eigenvectors of R. We now form the prediction error polynomial equations with the vector g similar to (2.4), which is as follows:

$$H(z) = 1 + g_1 z + g_2 z^2 + \cdots + g_L z^L = 0. \tag{4.5}$$

Now Eq. (4.5) has L roots. It can be shown in the same way as Kumaresan (1982) that in the case of noiseless data, out of L roots of (4.5) M of them will be at $e^{i\beta_j}$, $j=1, \ldots, M$ and $L-M$ roots will be having magnitude strictly less than one and will be distributed uniformly over the unit circle.

In case of noisy data first estimate g from (4.4), form the polynomial equation (4.5) and obtain the L roots of the prediction polynomial equation. Once we obtain the

L roots, the estimate of the β_j's can be obtained from those M roots whose magnitudes are closest to one.

4.2. Modified matrix pencil method

Here also, first we assume that only mth. Observation is missing, and the concept can be generalized. Suppose

$$B_1 = \begin{bmatrix} Y_{m+1} & \cdots & Y_{m+1+L} \\ \vdots & \cdots & \vdots \\ Y_{N-1-L} & \cdots & Y_{N-1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} Y_{m+2} & \cdots & Y_{m+2+L} \\ \vdots & \cdots & \vdots \\ Y_{N-L} & \cdots & Y_N \end{bmatrix}$$

if $1 \leqslant m \leqslant L+2$,

$$B_1 = \begin{bmatrix} Y_1 & \cdots & Y_{L+1} \\ \vdots & \cdots & \vdots \\ Y_{m-L-2} & \cdots & Y_{m-2} \\ Y_{m+1} & \cdots & Y_{m+L+1} \\ \vdots & \cdots & \vdots \\ Y_{N-L-1} & \cdots & Y_{N-1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} Y_2 & \cdots & Y_{L+2} \\ \vdots & \cdots & \vdots \\ Y_{m-L-1} & \cdots & Y_{m-1} \\ Y_{m+2} & \cdots & Y_{m+L+2} \\ \vdots & \cdots & \vdots \\ Y_{N-L} & \cdots & Y_N \end{bmatrix}$$

if $L+2 < m < N-1-L$, or

$$B_1 = \begin{bmatrix} Y_1 & \cdots & Y_{L+1} \\ \vdots & \cdots & \vdots \\ Y_{m-2-L} & \cdots & Y_{m-2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} Y_2 & \cdots & Y_{L+2} \\ \vdots & \cdots & \vdots \\ Y_{m-L-1} & \cdots & Y_{m-1} \end{bmatrix}$$

if $n-1-L+2 \leqslant m \leqslant N$. Consider the generalized singular value for the matrix pencil $\{B_1, B_2\}$. It can be shown, similarly as Pillai (1989) that in the noiseless case the non-zero singular values of the above matrix pencil $\{B, B_2\}$ are $e^{-i\beta_j}$, $j=1, \ldots, M$.

5. Numerical experiments

We performed some numerical experiments to observe how the proposed method works in finite samples. All these simulations were done on the HP-4000 computer at the Indian Institute Technology Kanpur using IMSL random deviate operator. We consider the same model used by Kundu (1993).

$$Y_t = e^{2.0it} + e^{3.5it} + e^{4.0it} + e^{5.5it} + \varepsilon_t, \quad t = 1, 2, \ldots, N. \tag{5.1}$$

Here ε_t are i.i.d. complex-valued random variables with mean zero and standard deviation σ for both the real and imaginary parts. The real and imaginary parts are assumed to be independent and normally distributed. We take $N=30$, and $\sigma^2=0.05$ (i.e. $\mathrm{SNR}=10\log(1/2\sigma^2)=10\,\mathrm{dB}$). One hundred different data sets of size 30 are generated using IMSL random deviate generator. For each data set one observation is missing at random. For each data set the prediction error polynomial (2.4) and its zeros are computed. The actual locations of sinusoids are shown by 'crossed mark'. The four zeros close to the signal frequency will be called signal zeros and the remaining as noise zeros or extraneous zeros. Figure 1 shows the location of the prediction error polynomial zeros in the Z plane. The results of the modified FBLP are reported in Figs. 2–5. We estimate the vector g by (4.4) for different L, namely $L=4$, $L=8$, $L=12$ and $L=14$. The zeros of the prediction error polynomial (4.5) are reported in Fig. 2 ($L=4$), Fig. 3 ($L=8$), Fig. 4 ($L=12$) and Fig. 5 ($L=14$). The results of the modified ESPRIT are reported in Figs. 6–8. We compute the generalized singular values of the matrix pencil (B_1, B_2) for different L as described in Section 4.2 and plot its conjugate in the Z plane. Fig. 6 corresponds to $L=4$, Fig. 7 corresponds to $L=8$ and Fig. 8 corresponds to $L=12$. The conjugate of the four eigenvalues close to the signal frequency location will be called as signal eigenvalues and the rest as noise eigenvalues.

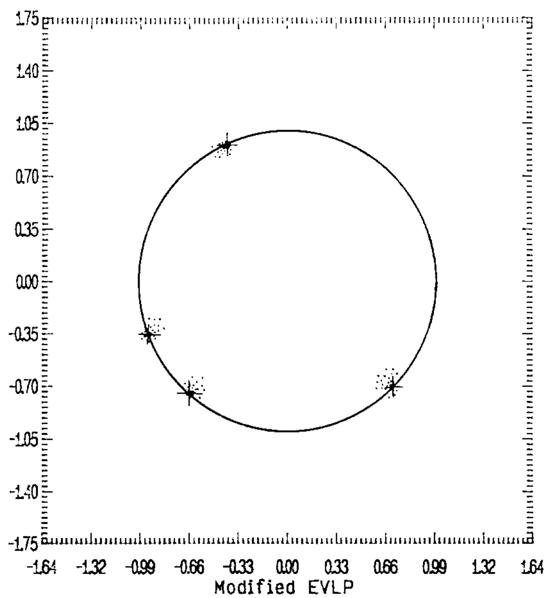
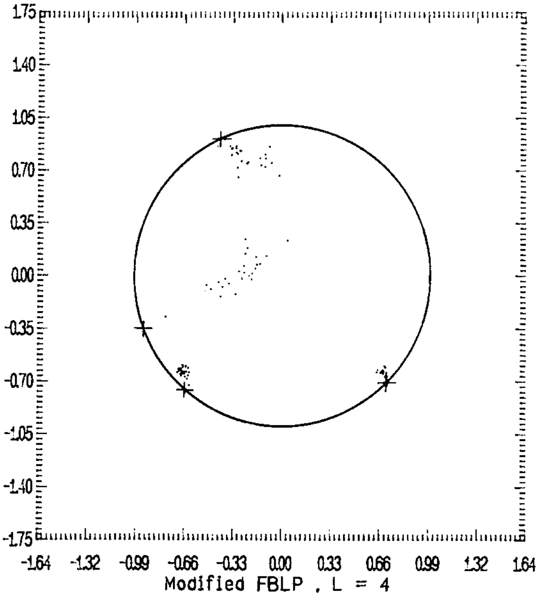


Fig. 1.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Modified FBLP , L = 4

Fig. 2.

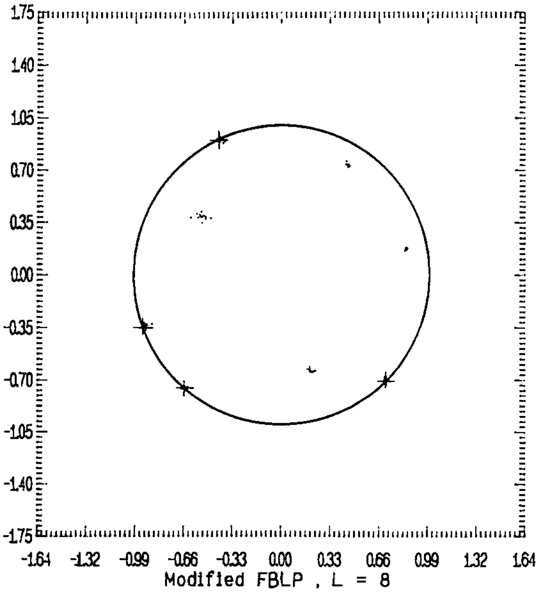
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Fig. 3.

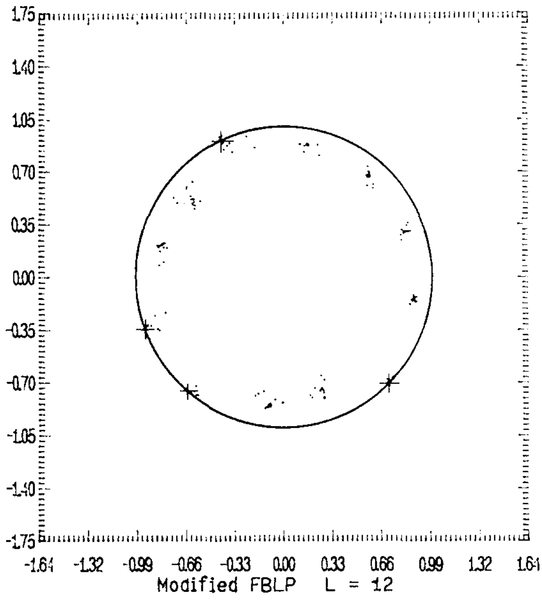
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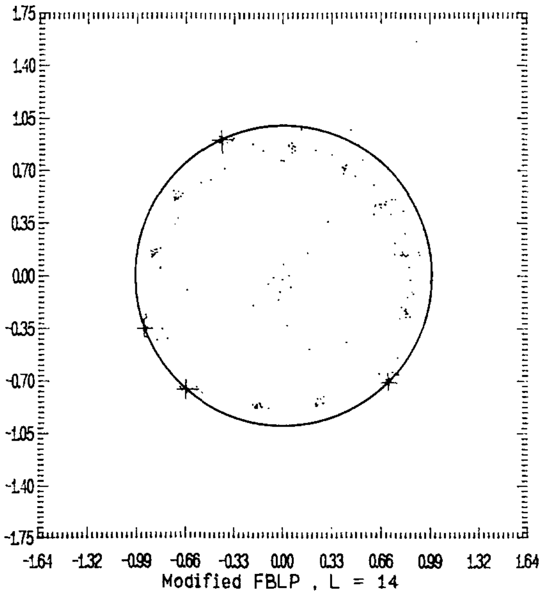
Fig. 4.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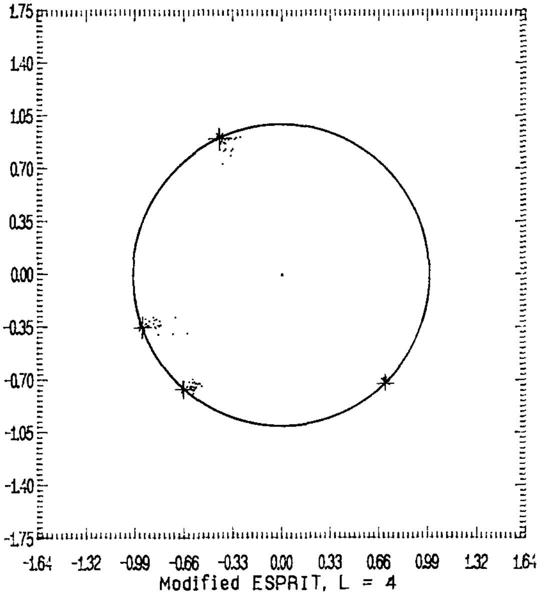
Modified FBLP , L = 14

Fig. 5.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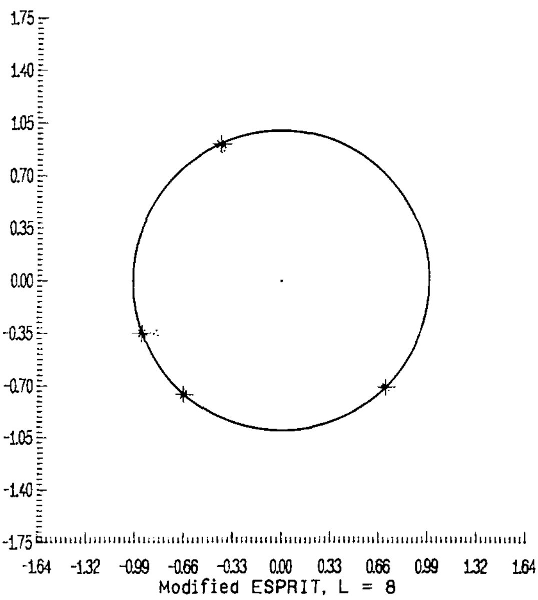
Modified ESPRIT, L = 4

Fig. 6.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Modified ESPRIT, L = 8

Fig. 7.

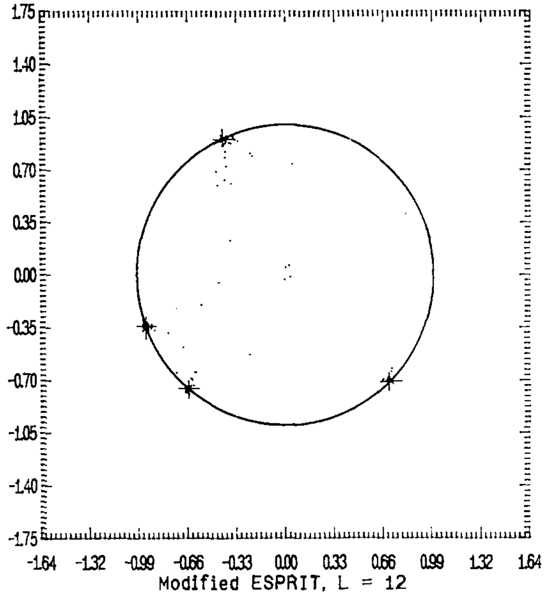

Fig. 8.

6. Conclusions

Some of the points are very clear from these experiments. First of all, the modified Pisarenko or modified EVLP does not perform very well, when some data are missing particularly for small sample and at low SNR. The variability of the signal zeros is quite high. The performance can be improved by using modified FBLP method with the proper choice of L. For $L=4$, the performance of the modified FBLP is also not very satisfactory. The spread of the signal zero is very high. However, as L is increased to 8, the performance is improved drastically. In this case, the four signal zeros are distinctly separated from the rest of the four extraneous zeros. The spread of the extraneous zero clusters are considerably less, thus reducing the chance of spurious frequency estimates. The spread of the signal zero clusters is also very small thus the variability of the frequency estimates is considerably reduced. But as L is increased to 12, the spread of the signal zeros as well as the noise zeros start increasing again. The eight noise zeros are distributed more or less uniformly over the unit circle but it is more scattered. The situation gets worsened as L is increased to 14. When one observation is missing it is not always possible to vary L, between M and $N-M/2$ as that of Tufts and Kumaresan (1982). For example, $N=30$, $L=15$ and the 15th observation is missing, then it is not possible to form the matrix equation (4.1). That is why, we have taken $L<N/2$.

The behavior of the modified ESPRIT is almost similar to that of the modified FBLP. When $L=4$, the spread of the signal eigenvalues are very high. The performance improves significantly when L is increased to 8. The spread of the signal eigenvalues is reduced to a great degree and the noise eigenvalues are mainly falling at zero. The situation starts getting worse as L is increased. As L is increased to 12, the spread of signal eigenvalues as well as the noise eigenvalues increase. It is becoming difficult to distinguish the two.

Considering all these, it is advisable to use either modified FBLP or modified ESPRIT for frequency estimation, when some, data are missing, with the proper choice of L (approximately equal to $N/3$).

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