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Estimating the parameters of exponentially damped/undamped sinusoids in noise: A non-iterative approach

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Abstract

A non-iterative method is proposed for estimating the parameters of damped/undamped exponential signals in noise. In this paper we present a new method based on the decomposition of the noise space of an extended order model. It is observed that the proposed method provides better estimates even at lower signal-to-noise ratio than currently existing non-iterative techniques in terms of lower mean squared errors.

Zusammenfassung

Für die Parameterschätzung von durch Rauschen gestörten gedämpften/ungedämpften exponentiellen Signalen wird eine nichtiterative Methode vorgeschlagen. In dieser Arbeit stellen wir eine neue Methode vor, die sich auf eine Zerlegung des gestörten Signalraums eines Modells mit erweiterter Ordnung stützt. Es wird beobachtet, daß die vorgeschlagene Methode bessere Schätzwerte mit geringerem mittlerem quadratischen Fehler liefert, selbst bei geringerem Signal-zu-Rausch-Verhältnis, als augenblicklich existierende nichtiterative Verfahren.

Résumé

Une méthode non-iterative est proposée pour estimer les paramètres de signaux exponentiels amortis/non-amortis dans du bruit. Dans cet article nous proposons une nouvelle méthode basée sur la décomposition de l'espace du bruit sur un modèle d'ordre élevé. On observe que la méthode proposée donne de meilleures estimées, même pour un rapport signal-sur-bruit faible, que les techniques non-iteratives existantes, ce en termes d'erreurs quadratiques moyennes.

Keywords: Centro-symmetric; Cramer–Rao bound, ESPRIT; Exponential signals; Noise space decomposition; Singular value decomposition

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1. Introduction

Estimating the parameters of superimposed exponential signals in noise is a very important problem in signal processing. We consider the following model of multiple superimposed damped exponential signals in complex valued white Gaussian noise $w(n)$, i.e.

$$y(n) = \sum_{k=1}^{M} a_k \exp(s_k n) + w(n), \quad n = 1, \ldots, N, \quad (1.1)$$

where $s_k = -\alpha_k + \mathrm{j} \cdot 2\pi f_k$, $k = 1, \ldots, M$, are complex numbers (α_ks are non-negative) and a_ks, $k = 1, \ldots, M$, are the complex amplitudes. α_ks are the damping factors and f_ks are the corresponding frequencies, $\mathrm{j} = \sqrt{-1}$. Given a sample of size N, the problem is to estimate α_ks and f_ks assuming M to be known.

This is an old problem [6], and the readers are referred to [13] for an extensive list of references. It is observed [5, 14] that the general purpose algorithms such as Gauss–Newton or its variants encounter lots of difficulties to obtain the maximum likelihood estimates. Considerable amount of research has been done in the past 20 years [1,4,5,7,13–17] to obtain the maximum likelihood estimates efficiently. However, all these methods have the drawback of high computational complexity and dependence on the initial value chosen.

Although maximum likelihood extimates are optimal in the sense that it reaches the Cramer–Rao lower bound, it always involves an iterative search procedure. Due to that lots of non-iterative procedures have been attempted by many researchers. It is observed that among the different non-iterative methods the ESPRIT of [12] with centro-symmetric sample covariance matrix (we call it ESPRIT only) gives the best performance [2].

In this paper, we propose a new method named as Noise Space Decomposition (NSD) method, which uses the extended order modelling and decomposition of the noise space by singular value decomposition technique similar to that of [16]. The new method involves solving some linear equations and rooting of an M degree polynomial. It is observed that the new method has a clear advantage over ESPRIT in terms of lower mean squared errors even at low SNR.

2. Estimation procedure

Consider the following $N - L \times L + 1$ data matrix

$$A = \begin{bmatrix} y(1) & \cdots & y(L+1) \\ \vdots & \ddots & \vdots \\ y(N-L) & \cdots & y(N) \end{bmatrix} \quad (2.1)$$

for any $N - M \geqslant L \geqslant M$. Let us denote the matrix $\boldsymbol{T}$ by $\boldsymbol{T} = (1/N)\boldsymbol{A}^*\boldsymbol{A}$ where * denotes the conjugate transpose of a matrix or of a vector. Observe that when the data are noiseless, the matrix $\boldsymbol{T}$ has rank $\boldsymbol{M}$. Let the singular value decomposition of $\boldsymbol{T}$ be as follows:

$$\boldsymbol{T} = \sum_{i=1}^{L+1} \hat{\sigma}_i^2 \hat{U}_i \hat{U}_i^*, \quad (2.2)$$

here $\hat{\sigma}_1 > \hat{\sigma}_2 > \cdots > \hat{\sigma}_{L+1}$ are the ordered eigenvalues of $\boldsymbol{T}$ and $\hat{U}_i$ is the normalized eigenvector corresponding to $\hat{\sigma}_i^2$. The subspace generated by $\{\hat{U}_1, \ldots, \hat{U}_M\}$ is denoted by $\mathscr{S}$ and that of $\{\hat{U}_{M+1}, \ldots, \hat{U}_{L+1}\}$ is denoted by $\mathscr{N}$. We call $\mathscr{S}$ the signal subspace and $\mathscr{N}$ the noise subspace. Let $\boldsymbol{B}_1$ be any basis of the noise subspace $\mathscr{N}$. We write

$$\boldsymbol{B}_1 = \begin{bmatrix} b_{1,1} & \cdots & b_{1,L+1-M} \\ \vdots & \ddots & \vdots \\ b_{L+1,1} & \cdots & b_{L+1,L+1-M} \end{bmatrix}. \quad (2.3)$$

Since it is well known [11] that in the noiseless case there exists a unique vector $\boldsymbol{g} = (g_1, \ldots, g_{M+1})$ such that

$$\sum_{k=1}^{M+1} g_k y(t+k) = 0 \quad \text{for } t = 0, \ldots, N-M-1,$$

$$\|\boldsymbol{g}\| = 1 \text{ and } g_1 > 0, \quad (2.4)$$

therefore we can say that there exists a unique basis of $\mathcal{N}$ which has the following form:

$$B_2 = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ g_2 & g_1 & \cdots & 0 \\ \cdot & g_2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ g_{M+1} & \cdot & \cdots & g_1 \\ 0 & g_{M+1} & \cdots & \cdot \\ 0 & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & g_{M+1} \end{bmatrix}. \tag{2.5}$$

The unknown $(s_1, \dots, s_M)$ can be obtained from $(g_1, \dots, g_{M+1})$ as the roots of the following polynomial equation:

$$g_1 Z^M + g_2 Z^{M-1} + \cdots + g_{M+1} = 0. \tag{2.6}$$

Therefore, our main aim is to estimate $\boldsymbol{g}$ from the observed data and once we obtain $\boldsymbol{g}$ we can easily obtain the estimate of the parameters using (2.6). Since $\hat{U}_{M+1}, \dots, \hat{U}_{L+1}$ forms a basis of $\mathcal{N}$, so let us take

$$\boldsymbol{B}_1 = [\hat{U}_{M+1} \ \dots \ \hat{U}_{L+1}]$$
$$= \begin{bmatrix} b_{1,1} & \cdots & b_{1,L+1-M} \\ \vdots & \ddots & \vdots \\ b_{L+1,1} & \cdots & b_{L+1,L+1-M} \end{bmatrix}. \tag{2.7}$$

Our main idea is to obtain a basis of $\mathcal{N}$ which is of the similar form as (2.5). Let us write

$$\boldsymbol{B}_1^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{B}_{11}^{\mathrm{T}} & \boldsymbol{B}_{12}^{\mathrm{T}} \\ L+1-M \times M+1 & L+1-M \times L-M \end{bmatrix}. \tag{2.8}$$

Now observe that $\boldsymbol{B}_{12}$ is a random matrix, i.e. each element of $\boldsymbol{B}_{12}$ is a random variable which are not perfectly correlated. Since it is well known that any random matrix is of full rank, $\boldsymbol{B}_{12}$ is of rank $L - M$. Therefore there exists an $L - M + 1$ vector $\boldsymbol{X}_1 \neq \boldsymbol{0}$ such that

$$\boldsymbol{B}_{12}\boldsymbol{X}_1 = \boldsymbol{0}. \tag{2.9}$$

Consider the $(M + 1)$ vector $\hat{\boldsymbol{g}}^1$, where

$$\hat{\boldsymbol{g}}^1 = (\hat{g}_{1,1}, \dots, \hat{g}_{1,M+1}) = \boldsymbol{B}_{11}\boldsymbol{X}_1, \tag{2.10}$$

by properly normalizing we can make $\hat{g}_{1,1} > 0$ and $\|\hat{\boldsymbol{g}}_1\| = 1$. The above argument can be easily generalized and therefore we can conclude that there exist vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_{L-M+1}$ such that

$$\boldsymbol{B}_1[\boldsymbol{X}_1 \ \dots \ \boldsymbol{X}_{L+1-M}]$$
$$= \begin{bmatrix} \hat{g}_{1,1} & 0 & \cdots & 0 \\ \hat{g}_{1,2} & \hat{g}_{2,1} & \cdots & 0 \\ \cdot & \hat{g}_{2,2} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hat{g}_{1,M+1} & \cdot & \cdots & \hat{g}_{L+1-M,1} \\ 0 & \hat{g}_{2,M+1} & \cdots & \cdot \\ 0 & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \hat{g}_{L+1-M,M+1} \end{bmatrix}, \tag{2.11}$$

where $\hat{g}_{k,1} > 0$ and $\|\hat{g}^k\| = \|(\hat{g}_{k,1}, \dots, \hat{g}_{k,M+1})\| = 1$ for $k = 1, \dots, L - M + 1$. Observe that in the noiseless situation

$$\hat{g}^1 = \hat{g}^2 = \cdots = \hat{g}^{L+1-M} = \boldsymbol{g}. \tag{2.12}$$

So it is quite reasonable that any one of the $\hat{g}^k$ for $k = 1, \dots, L - M + 1$ or all of them can be used to estimate $\boldsymbol{s} = (s_1, \dots, s_M)$. We use all the $\hat{g}^k$ for $k = 1, \dots, L - M + 1$ to estimate $\boldsymbol{s}$. We find $\hat{g}$ the average of $\hat{g}^k$s, $k = 1, \dots, L - M + 1$. We now use (2.6) to obtain the final estimate $\hat{\boldsymbol{s}}$ of $\boldsymbol{s}$.

3. Experimental results

In this section we present some simulations performed to examine the behavior of the proposed method for small samples. We have considered two simulation models to compare the performance of the different methods. We have kept the sample size fixed at 25.

Model 1 (*Undamped model*)

$$y(n) = a_1 \exp(s_1 n) + a_2 \exp(s_2 n) + w(n),$$
$$n = 1, \dots, 25,$$

where

$s_1 = j2\pi f_1 = j2\pi(0.6366)$, $s_2 = j2\pi f_2 = j2\pi(0.8754)$,

$a_1 = 2.5$ and $a_2 = 3.0$.

Model 2 (*Damped model*)

$$y(n) = a_1 \exp(s_1 n) + a_2 \exp(s_2 n) + w(n), \quad n = 1, \ldots, 25,$$

where

$s_1 = -\alpha_1 + j2\pi f_1 = -0.01 + j2\pi(0.52)$,

$s_2 = -\alpha_2 + j2\pi f_2 = -0.02 + j2\pi(0.42)$,

$a_1 = a_2 = 1.0$.

In all the cases the error random variable is white and complex Gaussian with variance $\sigma^2/2$. 500 independent trials using different $w(n)$ sequences are performed in all the cases. SNR is varied from 22 to 5 dB for model 1 and from 24 to 8 dB for model 2.

For both the models we obtained (s_1, s_2) by ESPRIT, NSD and also by one step modified NSD (see [3] for details) named as MNSD. For each s_i we computed MSE over 500 replications and also the corresponding Cramer–Rao lower bound (CRLB). It is observed that by changing L, the performance of the different estimates obtained by NSD method changes, similarly as ESPRIT. For $N = 25$, we observe that the best performance (min MSE) is obtained at $L = 15(\simeq \frac{3}{5}N)$ for NSD

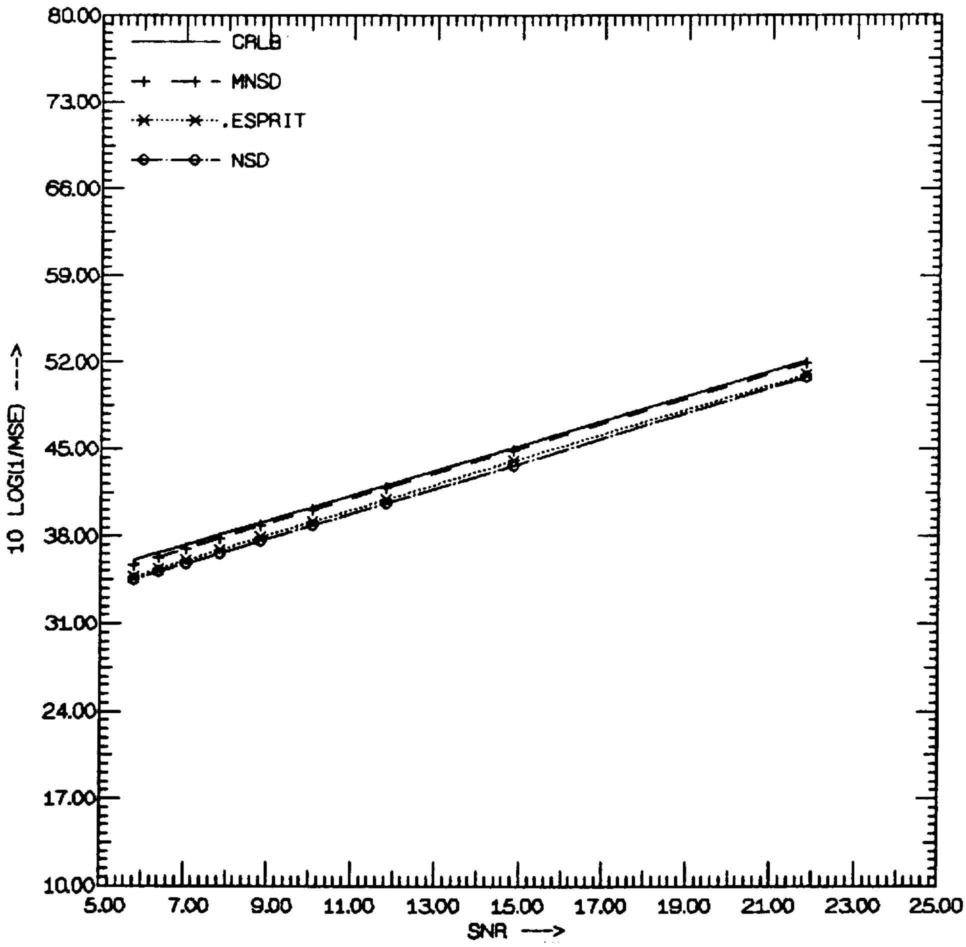

Fig. 1. MSE of f_2 for the model 1.

method and for ESPRIT the best performance is obtained at $L = 12(\simeq \frac{1}{2}N)$. We report the results of the best performance of the methods along with their CRLB. Since the performances are identical for the two models and for the different parameters, we have just reported the results corresponding to the frequency f_2 for model 1.

6. Conclusions

In this paper we consider the problem of estimating the parameters of damped/undamped exponential signals in Gaussian noise by some non-iterative procedures when the number of signals is known. If the number of signals is unknown, we can estimate the number of signals by the method of [8] and then use our method to estimate the parameters. The proposed method uses extended order modelling and decomposition of the noise space by singular value decomposition technique. The proof of consistency in the case of undamped exponential signals can be obtained similarly as [9]. A similar kind of method was mentioned in estimating direction of arrival of signals by Orfanidis [10] but no proof of consistency was provided. In our numerical experiments we observe that MNSD performs much better than ESPRIT or ordinary NSD at all SNR. The performances of ESPRIT and NSD are quite similar and NSD has a slight edge over ESPRIT in some cases. The MNSD estimates in fact attains the Cramer–Rao lower bound (CRLB) in almost all the cases considered.

The magnitude of L affects the performance of the NSD and ESPRIT substantially, but the performance of the MNSD estimate is not sensitive to the choice of L. Clearly L should be at least $M + 1$, but the natural question is why should it be larger than that? Although no theoretical justification has been provided in the literature, it is observed that extended order modelling always helps to improve the performance of the estimators. It seems more theoretical work is needed in this direction. Here we observe that as L increases the MSE starts decreasing for ESPRIT, NSD and MNSD. It reaches the minimum at $L = 12$ for ESPRIT and at $L = 15$ for NSD, when the sample size is 25.

Recently, one very elegant method, named as Markov-Based Eigen Analysis Method which makes optimal use of sample eigenvectors of the sample covariance matrix to obtain the signal parameter estimates has been proposed by Errikson et al. [2]. They have developed the method for sum of M undamped complex exponentials in presence of a random phase component. They showed by simulation study that the Markov method improves slightly over the Centro-Symmetric ESPRIT. It is however not very immediate how to apply Markov method in undamped exponential model without the random phase component or in the damped exponential model. It will be investigated later on.

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