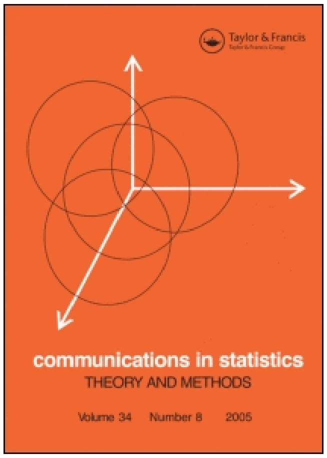

Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597238

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Online Publication Date: 01 January 1998
To cite this Article: Gupta, Rameshwar D. and Kundu, Debasis (1998) 'Hybrid censoring schemes with exponential failure distribution', Communications in Statistics - Theory and Methods, 27:12, 3065 - 3083
To link to this article: DOI: 10.1080/03610929808832273
URL: http://dx.doi.org/10.1080/03610929808832273

COMMUN. STATIST.—THEORY METH., 27(12), 3065–3083 (1998)

HYBRID CENSORING SCHEMES WITH EXPONENTIAL FAILURE DISTRIBUTION

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Key Words and phrases: Exponential distribution; hybrid censoring; two-sided confidence interval.

ABSTRACT

The mixture of Type I and Type II censoring schemes, called the hybrid censoring, is quite important in life-testing experiments. Epstein (1954, 1960) introduced this testing scheme and proposed a two-sided confidence interval to estimate the mean lifetime, θ, when the underlying lifetime distribution is assumed to be exponential. There are some two-sided confidence intervals and credible intervals proposed by Fairbanks *et al.* (1982) and Draper and Guttman (1987) respectively. In this paper we obtain the exact two-sided confidence interval of θ following the approach of Chen and Bhattacharya (1988). We also obtain the asymptotic confidence intervals in the Hybrid censoring case. It is important to observe that the results for Type I and Type II censoring schemes can be obtained as particular cases of the Hybrid censoring scheme. We analyze one data set and compare different methods by Monte Carlo simulations.

1. INTRODUCTION

Consider the following experiment: A total of n units is placed on test. The lifetime of each unit is independent and identically distributed (i.i.d.) exponential random variable with mean life θ. Therefore, the underlying probability density function of the lifetime is given by

$$f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}; \qquad x > 0, \theta > 0. \tag{1}$$

The test is terminated when a pre-chosen number, R, out of n items have failed or when a predetermined time, T, on the test has been reached. It is also assumed that the failed items are not replaced. This mixture of Type I and Type II censoring scheme is known as Hybrid censoring and it is quite important in reliability acceptance test in MIL-STD-781C (1977).

Epstein (1954) introduced this testing scheme and he proposed (Epstein; 1960) a two-sided confidence interval for θ without any formal proof. Fairbanks *et al.* (1982) modified slightly the proposition of Epstein (1960) and suggested a simple new set of confidence intervals. Chen and Bhattacharya (1988) obtained the exact distribution of $\hat{\theta}$, the maximum likelihood estimator of θ, and proposed a one-sided confidence interval. Draper and Guttman (1987) considered this problem from the Bayesian point of view and obtained the two-sided credible interval of the mean lifetime using the inverted gamma prior. But no where, at least not known to the authors, the comparison of the different methods is available in the literature. In this paper we propose a two-sided confidence interval based on the exact distribution of $\hat{\theta}$ following the approach of Chen and Bhattacharya (1988). It is observed that the set of confidence intervals as proposed by Fairbanks *et al.* (1982) is not exactly correct and we give some theoretical justification in favor of our comments. We derive the exact distribution as well as the asymptotic distribution of $\hat{\theta}$, under the Hybrid censoring scheme. We obtain the asymptotic confidence interval of θ and the exact expressions for $E(\hat{\theta})$ and $E(\hat{\theta}^2)$. Finally we compare different methods through Monte Carlo simulations and apply all the different methods on one data set.

It is important to observe that all the results regarding $\hat{\theta}$ in case of Type I and Type II censoring can be obtained as special cases of Hybrid censoring.

For example Type I and Type II cases can be obtained from the Hybrid case by putting R = n and T = ∞ respectively in the later. Although the exact distribution of $\hat{\theta}$ in case of Type II censoring is Chi-square but in case of Type I censoring the exact distribution of $\hat{\theta}$ is not very easy to obtain. Most of the text books on Reliability or Survival analysis (Bain; 1978, Barlow and Proschan; 1975 and Lawless; 1982) do not talk about the exact distribution of $\hat{\theta}$. It might be worth mentioning that the related problem of random sampling from a truncated exponential distribution has also found an application in the analysis of repairable systems. This is discussed in details in Bain and Engelherdt (1991, Chapter 9) and also the distribution of the sum of observations from a truncated exponential distribution is presented there. In the repairable systems framework, it turns out to be the same as the conditional density of the sum of the observed failure times given the number of failures for a time-censored Weibull process. This density is also given in its natural setting of sampling from a truncated exponential distribution in a paper by Bain *et al.* (1977), see also Bain and Weeks (1964).

The rest of the paper is organized as follows. In Section 2, we discuss the distribution of $\hat{\theta}$. Different confidence intervals and credible intervals are considered in Section 3. We perform the numerical experiments to compare different methods in Section 4. The data analysis are presented in Section 5. Finally we draw conclusions from our work in Section 6.

2. DISTRIBUTION OF $\hat{\theta}$

Let $X_1, \ldots, X_n$ be the lifetime of n items, where each X_i is i.i.d. exponential random variable with mean life θ and with the probability density function as defined in (1). We denote the ordered lifetime of these items by $X_{1,n} < \ldots <$ $X_{n,n}$. Let's also denote the $T^* = \min\{X_{R,n}, T\}$ and D^* = the number of units that fails up to and including T^*. Therefore, under Hybrid censoring scheme we observe either

$$\{X_{1,n} < \ldots < X_{R,n}\} \quad \text{if} \quad X_{R,n} < T \quad \text{or}$$
$$\{X_{1,n} < \ldots < X_{D^*,n}\} \quad \text{if} \quad D^* < R, \text{ and } X_{D^*,n} < T < X_{D^*+1,n}.$$

Based on the observations (2) the likelihood function of θ can be written (see Chen and Bhattacharya; 1988)

$$l(\theta) = \frac{n!}{(n-D^*)!}\theta^{-D^*}e^{-\frac{S}{\theta}}, \tag{3}$$

where

$$\begin{aligned} S &= \sum_{i=1}^{D^*} X_{i,n} + (n-D^*)T^* \qquad \text{if} \quad D^* \geq 1, \\ &= nT \qquad\qquad\qquad\qquad \text{if} \quad D^* = 0. \end{aligned}$$

It is interesting to note that (D^*, S) is a jointly sufficient statistic for θ if $D^* < R$, which is immediate from (3) but it is not complete for $n > 1$. The fact that (D^*, S) is not complete for $D^* < R$, when $n > 1$, follows from Bartoszewicz (1974), where he showed that (D^*, S) is not complete in the particular case when $R = n$ (Type I censoring scheme). Therefore, the result is true even for general R. On the other hand if $D^* = R$, then $S = \sum_{i=1}^{R} X_{i,n} + (n-R)X_{R,n}$ is a sufficient statistic and the distribution of S is Gamma with the shape parameter R and the scale parameter θ (Bain; 1978). Since Gamma is a complete family, S is a complete sufficient statistic in this case.

It can be easily observed that the maximum likelihood estimator (MLE) of θ is

$$\hat{\theta} = \frac{S}{D^*} \qquad \text{if} \quad D^* \geq 1, \tag{4}$$

and the MLE does not exist if $D^* = 0$. Now the distribution of $\hat{\theta}$ means the conditional distribution of $\hat{\theta}$ given $D^* \geq 1$. The basic idea is to obtain the distribution of S given $D^* = i$, for $i \geq 1$ and then obtain the unconditional distribution. It can be done directly using the convolution property of the uniform distribution (see Hoem; 1969) or it can be done through the conditional moment generating function approach as suggested by Chen and Bhattacharya (1988).

Now observe that the conditional moment generating function of $\hat{\theta}$ given $D^* > 0$ is given by

$$\begin{aligned} \Phi(\omega;\hat{\theta}) &= E(e^{\omega\hat{\theta}}|D^* > 0) \\ &= (1-q^n)^{-1}\left[\sum_{d=1}^{R-1}\binom{n}{d}\left(1-\frac{\omega\theta}{d}\right)^{-d}\left(1-q^{1-\frac{\omega\theta}{d}}\right)^{d}(q^{1-\frac{\omega\theta}{d}})^{n-d}\right] \end{aligned}$$

$$
\begin{aligned}
&+ (1-q^n)^{-1}\sum_{d=R}^{n}\binom{n}{d}C(R,D)\sum_{k=0}^{R-1}\sum_{j=0}^{d-R}(-1)^{d+k-R-j}(k+j+1)^{-1} \\
&\times \binom{R-1}{k}\binom{d-R}{j}q^{n-R-j}\left(1-q^{k+j+1-\frac{(n+k-R+1)\omega\theta}{R}}\right) \\
&\times \left(1-\frac{\omega\theta}{R}\frac{n+k-R+1}{k+j+1}\right)^{-1}\left(1-\frac{\omega\theta}{R}\right)^{-(R-1)}, \qquad (5)
\end{aligned}
$$

here $q = e^{-\frac{T}{\theta}}$ and $C(R,D) = \frac{d!}{(R-1)!(d-R)!}$. Now we would like to show that Type I and Type II censoring scheme can be obtained as the special cases of the Hybrid censoring scheme. Observe that as $T \to \infty$, $q \to 0$, therefore

$$
\begin{aligned}
\Phi(\omega;\hat{\theta}) &= E(e^{\omega\hat{\theta}}|D^* > 0) \to \left(1-\frac{\omega\theta}{R}\right)^{-R} \\
&\times \frac{n!}{(R-1)!(n-R)!}\sum_{k=0}^{R-1}(-1)^k(k+n-R+1)^{-1}\binom{R-1}{k} \\
&= \left(1-\frac{\omega\theta}{R}\right)^{-R}, \qquad (6)
\end{aligned}
$$

because of the identity

$$
\frac{n!}{(R-1)!(n-R)!}\sum_{k=0}^{R-1}(-1)^k(k+n-R+1)^{-1}\binom{R-1}{k} = 1. \qquad (7)
$$

(see Appendix for the proof of (7)). From (6), it is clear that the moment generating function of $\frac{2R\hat{\theta}}{\theta}$ is $(1-2\omega)^{-R}$, which is the moment generating function of a Chi-square random variable with $2R$ degrees of freedom in case of Type II censoring.

Now consider the case when T fixed and $R = n$. Observe that in this situation

$$
\begin{aligned}
\Phi(\omega;\hat{\theta}) &= E(e^{\omega\hat{\theta}}|D^* > 0) \\
&= (1-q^n)^{-1}\left[\sum_{d=1}^{n-1}\binom{n}{d}\left(1-\frac{\omega\theta}{d}\right)^{-d}\left(1-q^{1-\frac{\omega\theta}{d}}\right)^d(q^{1-\frac{\omega\theta}{d}})^{n-d}\right] \\
&+ (1-q^n)^{-1}\sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^k\left(1-q^{k+1-\frac{(k+1)\omega\theta}{n}}\right)\left(1-\frac{\omega\theta}{n}\right)^{-n} \\
&= (1-q^n)^{-1}\sum_{d=1}^{n}\binom{n}{d}\left(1-\frac{\omega\theta}{d}\right)^{-d}\left(1-q^{1-\frac{\omega\theta}{d}}\right)^d(q^{1-\frac{\omega\theta}{d}})^{n-d}. \qquad (8)
\end{aligned}
$$

Observe that (8) is the moment generating function of $\hat{\theta}$ in presence of Type I censoring (see Bartholmew; 1963).

Note that for Type I censoring from a one-parameter exponential distribution, the conditional distribution of the sum of observations given the number ($D^* = d$) of observations is the same as the distribution of the sum of observations in a random sample of size d, from a truncated exponential distribution, see Bain (1978) or Bain and Engelhardt (1991). The distribution function of the MLE is obtained as the expectation relative to D^* (truncated above 0) of the conditional distribution of $\frac{\sum_{i=1}^{d} X_{i,n}+(n-d)T}{d}$ given $D^* = d$.

Now we obtain the mean and variance of $\hat{\theta}$ in the Hybrid censoring scheme as it was obtained for Type I censoring case by Bartholmew (1963) and Mandenhall and Lehman (1960). Observe that the usual technique of obtaining the the first two moments using (5) might prove complicated. However, we use the conditional density (Chen and Bhattacharya; 1988) of $\hat{\theta}$ given $D^* > 0$, to obtain the following expression for $E(\hat{\theta})$ and $E(\hat{\theta}^2)$.

$$
\begin{aligned}
E(\hat{\theta}) &= (1-q^n)^{-1}\{\sum_{d=1}^{R-1}\binom{n}{d}\sum_{k=0}^{d}\binom{d}{k}(-1)^k q^{n-d+k}\left(\theta+\frac{T}{d}(n-d+k)\right)\\
&+ \sum_{d=R}^{n}\binom{n}{d}C(R,D)\sum_{k=0}^{R-1}\sum_{j=0}^{d-R}(-1)^{d+k-R-j}(k+j+1)^{-1}\binom{R-1}{k}\binom{d-R}{j}\\
&\times\; q^{n-R-j}[(1-q^{k+j+1})\left(\frac{\theta(n+k-R+1)}{R(k+j+1)}+\frac{\theta(R-1)}{R}\right)\\
&\quad -q^{k+j+1}\frac{T}{R}(n+k-R+1)]\}, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
E(\hat{\theta}^2) &= (1-q^n)^{-1}\{\sum_{d=1}^{R-1}\binom{n}{d}\sum_{k=0}^{d}\binom{d}{k}(-1)^k q^{n-d+k}\\
&\times\left(\frac{d(d+1)\theta^2}{d^2}+\frac{2T(n-d+k)}{d}\theta+\frac{T^2}{d^2}(n-d+k)^2\right)+\\
&+ \sum_{d=R}^{n}\binom{n}{d}C(R,D)\sum_{k=0}^{R-1}\sum_{j=0}^{d-R}(-1)^{d+k-R-j}(k+j+1)^{-1}\\
&\quad\binom{R-1}{k}\binom{d-R}{j}q^{n-R-j}[(\frac{R(R-1)\theta^2}{R^2}+2\frac{\theta^2}{R^2}\left(\frac{n+k-R+1}{k+j+1}\right)^2\\
&+ 2\frac{\theta^2}{R^2}(R-1)\left(\frac{n+k-R+1}{k+j+1}\right))(1-q^{k+j+1})
\end{aligned}
$$

$$- \quad 2q^{k+j+1}\frac{T\theta}{R^2}(n+k-R+1)\left(R-1+\frac{n+k-R+1}{k+j+1}\right)$$
$$- \quad q^{k+j+1}\frac{T^2}{R^2}(n+k-R+1)^2]\}. \tag{10}$$

From (9) it can be seen that as $T \to \infty$, $E(\hat{\theta}) = \theta$, when $R = n$. Using the identity (7), when $R = n$, (9) becomes;

$$E(\hat{\theta}) = \frac{\theta}{(1-q^n)}(2-p^n) + \frac{Tn}{(1-q^n)}\sum_{d=1}^{n-1}\binom{n}{d}\frac{1}{d}q^{n-d}p^d. \tag{11}$$

(11) is the expected value of $\hat{\theta}$, in presence of Type I censoring. It is interesting to observe that the bias is always positive and as $T \to \infty$, (11) implies that $E(\hat{\theta}) \to \theta$, as it should be. The second term on the right hand side of (11) can be written by recursive relation (see Govindarajulu; 1963).

The asymptotic distribution of $\hat{\theta}$ is obtained by using the limiting properties of the maximum likelihood estimators. The asymptotic distribution of $\hat{\theta}$ is asymptotically normal with mean θ and variance

$$E\left(-\frac{\partial^2}{\partial\theta^2}log(l(\theta))\right) = E\left(\frac{2S}{\theta^3} - \frac{D^*}{\theta^2}\right). \tag{12}$$

Observe that,

$$\begin{aligned} E(S) &= \sum_{j=1}^{R-1} E\left(\sum_{i=1}^{j} X_{i,n} + (n-j)T|D=j\right)P(D=j) + nTP(D=0) \\ &+ \sum_{j=R}^{n} E\left(\sum_{i=1}^{R} X_{i,n} + (n-R)X_{R,n}|D=j\right)P(D=j), \end{aligned} \tag{13}$$

here D = the number of failure before time T. Observe that $D^* = D = j$, if $0 \le j \le R-1$; and $D^* = R \le j = D$ if $R \le j \le n$. Therefore,

$$\begin{aligned} E(S) &= \sum_{j=0}^{R-1}\binom{n}{j}p^j q^{n-j}\left(j\left[\theta - \frac{Te^{-\frac{T}{\theta}}}{1-e^{-\frac{T}{\theta}}}\right] + (n-j)T\right) + \sum_{j=R}^{n}\binom{n}{j}p^j q^{n-j} \\ &\times \left(\sum_{i=1}^{R-1} E(X_{i,n}|D=j) + (n-R+1)E(X_{R,n}|D=j)\right), \end{aligned} \tag{14}$$

where

$$E(X_{i,n}|D=j) =$$

$$\frac{j!}{(i-1)!(j-i)!}\frac{1}{p^j}\sum_{k=0}^{i-1}\sum_{m=0}^{j-i}(-1)^{k+j-i-m}\binom{i-1}{k}\binom{j-i}{m}\times$$
$$e^{-\frac{T(j-i-m)}{\theta}}\left[\frac{\theta}{(k+m+1)^2}\left(1-e^{-\frac{T(k+m+1)}{\theta}}\right)-\frac{T}{(k+m+1)}e^{-\frac{T(k+m+1)}{\theta}}\right],$$

and

$$E(D^*) = \sum_{i=0}^{R-1} i\binom{n}{i}p^i q^{n-i} + R\sum_{i=R}^{n}\binom{n}{i}p^i q^{n-i}. \tag{15}$$

Therefore, using (14) and (15), we can obtain (12).

3. CONFIDENCE INTERVALS AND CREDIBLE INTERVALS OF θ

In this section we discuss two-sided confidence intervals and credible intervals of θ. Fairbanks *et al.* (1982) proposed a set of two-sided confidence intervals for θ in the Hybrid censoring case, which can be given as follows;

$$\begin{aligned}
&\left[\frac{2S}{\chi^2_{2,\frac{\alpha}{2}}}, \infty\right] && \text{if } D^* = 0,\\
&\left[\frac{2S}{\chi^2_{2D^*+2,\frac{\alpha}{2}}}, \frac{2S}{\chi^2_{2D^*,1-\frac{\alpha}{2}}}\right] && \text{if } 1 \le D^* \le R-1,\\
&\left[\frac{2S}{\chi^2_{2R,\frac{\alpha}{2}}}, \frac{2S}{\chi^2_{2R,1-\frac{\alpha}{2}}}\right] && \text{if } D^* = R,
\end{aligned} \tag{16}$$

where S is as defined in Section 1. Observe that S depends on D^*. In fact they have given a proof in the 'with replacement' case and mention that the proof can be extended even for the 'without replacement' case. It is not very clear to us that how the proof will be in the 'without replacement' case. Because one of the major assumption in their proof ('with replacement' case) is that for $0 \le j \le R-1$,

$$P\{\text{j items fail at the decision time}\} = \frac{e^{-\frac{nT}{\theta}}\left(\frac{nT}{\theta}\right)^j}{j!}. \tag{17}$$

The result is no longer valid for the 'without replacement' case and their proof very much depends on this assumption.

Draper and Guttman (1987) also considered this same problem. They considered this problem from the Bayesian point of view. Under the assumption that θ has the following inverted Gamma prior

$$p(\theta) = \frac{a^r}{\Gamma(r)\theta^{r+1}} e^{-\frac{a}{\theta}} \qquad \text{when } a, r > 0, \tag{18}$$

they have shown that 100 $(1-\alpha)\%$ credible interval for θ is given by

$$\left[\frac{2(S+a)}{\chi^2_{2(D^*+r),\frac{\alpha}{2}}}, \frac{2(S+a)}{\chi^2_{2(D^*+r),1-\frac{\alpha}{2}}} \right]. \tag{19}$$

Chen and Bhattacharya (1988) used the exact distribution of $\hat{\theta}$ to obtain the one-sided confidence interval for θ. But their method can be easily used to obtain the two-sided confidence interval based on the exact distribution of $\hat{\theta}$ under the assumption that $P(\hat{\theta} \geq b)$ is an increasing function of θ as it was originally proposed by Chen and Bhattacharya (1988). Observe that

$$\begin{aligned} P(\hat{\theta} \leq b) &= (1-q^n)^{-1} \sum_{d=1}^{R-1} \binom{n}{d} \sum_{k=0}^{d} \binom{d}{k} (-1)^k q^{n-d+k} \int_0^{A_{kd}} f_{2d}(x)dx \\ &\times \sum_{d=R}^{n} \binom{n}{d} C(R,D) \sum_{k=0}^{R-1} \sum_{j=0}^{d-R} (-1)^{d+k-R+j} (k+j+1)^{-1} q^{n-R-j} \\ &\times \binom{R-1}{k} \binom{d-R}{j} \left[\int_0^b h_{kj}(x)dx - q^{k+j+1} \int_0^{b_k} h_{kj}(x)dx \right]. \end{aligned} \tag{20}$$

here $f_{2d}(x)$ denotes the chi-square probability density function with 2d degrees of freedom and

$$A_{kd} = \frac{2d}{\theta} < b - \frac{T(n-d+k)}{d} >, \quad b_k = < b - \frac{T(n+k-R+1)}{R} >, \tag{21}$$

and

$$\begin{aligned} h_{kj}(x) &= \left(\frac{R}{\theta}\right)^{R-1} \frac{R(k+j+1)}{\Gamma(R-1)\theta(n+k-R+1)} e^{-\frac{Rx(k+j+1)}{\theta(n+k-R+1)}} \\ &\times \int_0^x y^{R-2} e^{-\frac{Ry(n-R-j)}{\theta(n+k-R+1)}} dy, \end{aligned} \tag{22}$$

where $< a >$ means max $\{0,a\}$ and $\Gamma(.)$ is the Gamma function.

It can be easily seen from (20) that as $T \to \infty$, (20) becomes

$$P(\hat{\theta} \leq b) = \int_0^{\frac{2Rb}{\theta}} f_{2R}(x)dx, \tag{23}$$

and for $R = n$, (20) becomes

$$P(\hat{\theta} \;\leq\; b) = (1-q^n)^{-1} \{ \sum_{d=1}^{n-1} \binom{n}{d} \sum_{k=0}^{d} \binom{d}{k} (-1)^k q^{n-d+k} \int_0^{A_{kd}} f_{2d}(x)dx +$$

$$\int_0^b g(x)dx - n\sum_{k=0}^{n-1}(-1)^k(k+1)^{-1}\binom{n-1}{k}q^{k+1}\int_0^{c_k} g(x)dx\}, \quad (24)$$

where

$$g(x) = \left(\frac{n}{\theta}\right)^n \frac{1}{\Gamma(n)} e^{-\frac{nx}{\theta}} x^{n-1} \quad \text{and } c_k =< b - \frac{T(k+1)}{n} > . \quad (25)$$

Observe that under the assumption of Type I censoring (24) is the distribution function of $\hat{\theta}$. Now in both Hybrid censoring as well as in Type I censoring, we can obtain the symmetric $(1-\alpha)100\%$ confidence interval (b_1, b_2) by choosing b_1 and b_2 such that $P(\hat{\theta} \le b_1) = \frac{\alpha}{2}$ and $P(\hat{\theta} \le b_2) = 1 - \frac{\alpha}{2}$.

4. NUMERICAL EXPERIMENTS AND DISCUSSIONS

We perform some Monte Carlo simulations mainly to compare different methods and we present those results in this section. All the simulations are performed on PC-486 using the random deviate generator RAN2 of Press *et al.* (1986). We use RAN2 to generate the uniform random number generator and then using the transformation to obtain the exponential distribution. We consider the following cases.

Case 1a: $\theta = 1$, T = 1.0, R = 5, Case 1b: $\theta = 2$, T = 1.0, R = 5;

Case 2a: $\theta = 1$, T = 5.0, R = 5, Case 2b: $\theta = 2$, T = 5.0, R = 5;

Case 3a: $\theta = 1$, T = 1.0, R = 10, Case 3b: $\theta = 2$, T = 1.0, R = 10;

Case 4a: $\theta = 1$, T = 5.0, R = 10, Case 4b: $\theta = 2$, T = 5.0, R = 10.

For cases 1a, 1b, 2a and 2b we take sample sizes as $n = 5$, 10, (small), 50 (medium) and 100 (large). For cases 3a, 3b, 4a and 4b we take sample sizes as $n = 10$, 15 (small), 50 (medium) and 100 (large). For each data set, we compute the 95% confidence intervals by the methods proposed by Fairbanks *et al.* (1982) (FMD), the confidence intervals obtained from the true distribution of $\hat{\theta}$ (True) and also the confidence intervals obtained from the asymptotic distribution of $\hat{\theta}$ (Asymp) as in section 3. We finally compute 95% credible intervals proposed by Draper and Guttman (1987). We consider three different Bayes priors, namely the non-informative prior (Bayes 1), the prior mean is exactly equal to the true mean (Bayes 2) and the prior mean is more than the true mean. We have taken the prior mean to be three in our simulations (Bayes 3). For Bayes 2 and Bayes 3, we take the shape parameter of the prior to be two, i.e. r = 2. For the asymptotic confidence interval if the lower limit is negative, we replace it by zero. For each procedure we repeat the experiment one thousand times. We compute the average length and the coverage probability over one thousand replications. The results are reported in Tables 1 to 4. In each box the first quantity represents the average length and the second quantity represents the coverage probability.

From the tables some of the points are quite clear. It is observed that for fixed T and R as θ increases the length of the intervals also increases. For fixed θ and T as R increases the length of the intervals decreases and for fixed θ and R as T increases the length of the intervals decreases for all the methods. As n increases, for fixed R and T the length of the intervals decreases but it seems the intervals are not decreasing to zero as sample size increases. Therefore, we can't say that the MLE's are consistent as n tends to infinity for fixed R and T. All the methods more or less maintain the coverage probability at the nominal level (95% in our case) but there is a difference about their performances with respect to the average length of the confidence intervals or the credible intervals. It is observed that Bayes 2 has the minimum length, which is not very surprising. Because, in that case we have a good prior knowledge of the unknown parameter θ. Clearly, Bayes 2 has an unfair advantage over the other and it is quite difficult to use in practice because one usually may not have the information required for its use. One situation where Bayes 2 can be used as follows. Suppose two experiments are conducted under the almost identical conditions. One has the information of the first experiment and then that can be used to infer about the prior for the second.

The length of the confidence intervals obtained from the exact distribution of $\hat{\theta}$ are similar than the rest except Bayes 2. It is observed that the asymptotic approximation does not work well in this case, it may be due to the fact that although n is large but R is quite small, namely 5 and 10 in our simulations. From the tables it is clear that the Bayes estimation with non-informative prior does not work well although the credible intervals where the prior mean is higher than the true mean, work reasonably well. Most of the times it is even better than FMD. Comparing all of these, we recommend to use the

Table 1a

$\theta = 1$, T = 1.0, R = 5

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
5*	16.56, .96	4.23, .96	28.73, .71	2.49, .95,	3.63, .95,	8.32, .92
10	3.84, .95	2.36, .97	5.82, .83	1.82, .95,	2.41, .96,	8.21, .97
50	2.55, .95	1.72, .95	4.15, .83	1.68, .95,	2.23, .95,	4.08, .98
100	2.55, .95	1.72, .95	4.15, .85	1.68, .95,	2.23, .95,	3.94, .97

* MLE did not exist 7 times.

Table 1b

$\theta = 2$, T = 1.0, R = 5

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
5*	51.21, .96	14.32, .94	86.26, .96	4.81, .96,	5.60, .98,	18.02, .73
10@	28.70, .95	8.72, .94	38.13, .87	4.68, .95,	5.15, .96,	13.90, .92
50	5.21, .95	3.40, .95	6.76, .93	3.36, .95,	3.58, .96,	11.17, .97
100	5.21, .95	3.40, .95	6.76, .93	3.36, .95,	3.58, .96,	7.71, .96

* MLE did not exist 89 times. @ MLE did not exist 14 times.

Table 2a

$\theta = 1$, T = 5.0, R = 5

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
5	2.68, .95	1.87, .97	4.25, .84	1.70, .95,	2.26, .95,	7.59, .97
10	2.60, .94	1.74, .97	4.15, .83	1.68, .95,	2.26, .96,	7.39, .98
50	2.60, .95	1.70, .95	4.15, .83	1.68, .95,	2.23, .95,	4.08, .98
100	2.60, .95	1.70, .95	4.15, .85	1.68, .95,	2.23, .95,	2.95, .97

Table 2b

$\theta = 2$, T = 5.0, R = 5

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
5	7.86, .95	4.67, .95	9.92, .90	3.78, .95,	4.10, .96,	11.47, .97
10	5.19, .94	3.48, .95	6.78, .93	3.35, .95,	3.64, .97,	10.91, .96
50	5.21, .95	3.40, .95	6.76, .95	3.35, .95,	3.64, .97,	7.01, .97
100	5.21, .95	3.40, .95	6.76, .95	3.35, .95,	3.64, .97,	6.39, .97

Table 3a

$\theta = 1$, T = 1.0, R = 10

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
10	3.74, .96	2.21, .96	5.18, .80	1.77, .95,	2.23, .96,	8.04, .97
15	1.92, .95	1.54, .95	2.49, .85	1.40, .95,	1.67, .94,	8.01, .97
50	1.49, .94	1.22, .95	1.93, .90	1.21, .95,	1.43, .97,	4.29, .98
100	1.49, .95	1.22, .95	1.93, .95	1.21, .95,	1.43, .97,	2.81, .97

Table 3b

$\theta = 2$, T = 1.0, R = 10

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
10*	28.80, .96	8.51, .94	38.07, .82	4.70, .94,	5.16, .96,	10.23, .96
15	10.39, .97	5.93, .95	12.59, .89	3.88, .95,	4.15, .97,	10.20, .97
50	2.98, .95	2.45, .95	3.42, .95	2.41, .94,	2.41, .95,	7.60, .96
100	2.98, .95	2.45, .95	3.42, .95	2.41, .95,	2.23, .95,	5.30, .97

* MLE did not exist 14 times.

Table 4a

$\theta = 1$, T = 5.0, R = 10

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
10	1.52, .94	1.27, .95	1.98, .89	1.22, .94,	1.45, .95,	7.11, .97
15	1.50, .95	1.26, .95	1.95, .89	1.22, .94,	1.44, .95,	6.82, .97
50	1.49, .94	1.24, .95	1.93, .90	1.21, .95,	1.43, .97,	3.99, .97
100	1.49, .95	1.24, .95	1.93, .90	1.21, .95,	1.43, .97,	2.89, .97

Table 4b

$\theta = 2$, T = 5.0, R = 10

n	FMD	True	Bayes 1	Bayes 2	Bayes 3	Asymp
10	3.47, .94	2.88, .95	3.94, .90	2.65, .94,	2.77, .95,	8.31, .97
15	3.01, .95	2.53, .97	3.46, .93	2.44, .94,	2.55, .96,	8.22, .96
50	2.98, .94	2.45, .95	3.42, .95	2.41, .95,	2.52, .95,	6.19, .96
100	2.98, .95	2.45, .95	3.42, .95	2.41, .95,	2.52, .95,	4.63, .97

exact distribution to form the confidence intervals although computationally it is more expensive than the rest. If we have a good prior knowledge about θ, it is advisable to use the Bayes procedure.

5. DATA ANALYSIS

In this section we consider one data set and apply different methods discussed in the previous section. We use the following data set.

Data Set: (Bain; 1978) Suppose 20 items from an exponential population are put on life-test and observed for 150 hours. During that period 13 items fail with the following lifetime, measured in hours: 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109 and 138.

First let's consider this problem as a Hybrid censoring problem with $n = 20$ and $R = 13$. We obtain $S = 1677$ and $\hat{\theta} = 129.00$. We report the confidence intervals obtained by different methods in Table A.

Since it is observed that the asymptotic approximation does not work well in this case, we have not used the asymptotic method to construct the confidence interval. We consider two different priors for the Bayes credible intervals. One is the non-informative prior (Bayes 1) and the other case we have taken the prior mean to be 100 (Bayes 2).

Now suppose we have the following setup of the same experiment, $n = 20$, $R = 10$ and $T = 150$. Then we have the following data;, 3, 19, 23, 26, 37, 38, 41, 45, 58 and 84. In this case, we have $S = 1214$ and $\hat{\theta} = 121.4$. We also construct the different intervals exactly as before. We obtain the following Table B.

From Tables A and B, it is interesting to observe that all the above methods behave quite similarly. Bayes 2 gives the minimum credible length and the length of the confidence interval obtained by using the exact distribution of $\hat{\theta}$ follows next. In the actual data analysis it is observed that the FMD confidence intervals have the maximum lengths. The length of the confidence intervals in Table B are larger than the corresponding lengths in Table A. It is not very surprising because we have more information in the second case than the first one.

Table A

Methods	Lower Limit	Upper Limit	Length
FMD	75.437	242.273	166.836
Bayes 1	80.003	242.272	162.269
Bayes 2	75.650	211.663	136.013
True	68.011	229.733	161.722

Table B

Methods	Lower Limit	Upper Limit	Length
FMD	66.013	253.160	187.147
Bayes 1	71.057	253.160	182.103
Bayes 2	66.761	211.915	145.154
True	57.193	208.363	151.170

6. CONCLUSIONS

In this paper we consider the Hybrid censoring schemes in presence of exponential failure distributions. We observe that Type I and Type II censoring schemes are the particular cases of the Hybrid censoring scheme. We propose two-sided confidence intervals of θ, using the approach of Chen and Bhattacharya (1988). We compare different confidence intervals through computer simulations and also apply the different methods on a particular data set. It is observed that Bayes estimator works quite well when the prior mean of θ is close to $\hat{\theta}$, which may not be in the practical situation. Unfortunately, the asymptotic approximation does not work well in this case. The confidence interval obtained using the exact distribution of $\hat{\theta}$ works well although it turns out to be more expensive computationally. It is quite clear that since the exact distributional results are very complicated, one may not use it unless one had extensive tables, a good approximation or a computer algorithm. It seems more work is needed in that direction.

Bain and Engelhardt (1991) discuss the inferences about the mean of the exponential distribution in case of the Type I censored data. Two approaches are discussed, (a) inferences based solely on D^* and (b) inferences based on the asymptotic distribution of the MLE. Defining $p = F(t) =$ the distribution of S, obviously D^* is binomial with parameters n and p. Because, p, is a simple monotonic function of the mean of the exponential distribution, it is possible to convert inferences about p to inferences about the mean. Inferences based on the MLE can be made (with approximate confidence levels) using the asymptotic normal distribution derived by Bartholomew (1963). It is also noted that method (a) is highly efficient for small p and method (b) gives a good approximation when p is large. Thus it is recommended to use (a) for small p and (b) for large p. This may not be esthetically pleasing, but it is quite effective. Strictly speaking, the most sensible formulation of this approach is when the initial inferences is a test of hypothesis about the exponential mean (or equivalently about p), possibly with subsequent inversion to obtain a confidence interval. One wonders, if there is some approach like this which would work for the Hybrid censoring situation. Unfortunately, because of the complicated nature of the asymptotic variance it is quite difficult to implement. If some good approximations are available, it might be useful.

Another important point, which is typically ignored is the case when $D^* = 0$. Since the MLE does not exist even when the data is Type I censored, the problem becomes complicated under the Hybrid censoring scheme. Nice discussions are available in Bain and Engelhardt (1991) for this case under the Type I censoring scheme. In this situation ($D^* = 0$) for the Hybrid censoring scheme we can use the result of Fairbanks *et al.* (1982) to obtain the $(1-\alpha)100\%$ confidence interval. Their results work in this case because, 'with replacement' and 'without replacement' are equivalent when $D^* = 0$.

APPENDIX

The proof of the identity

$$\frac{n!}{(R-1)!(n-R)!}\sum_{k=0}^{R-1}(-1)^k(k+n-R+1)^{-1}\binom{R-1}{k} = 1.$$

Since

$$(1-x)^{R-1}x^{n-R} = \sum_{k=0}^{R-1}(-1)^k\binom{R-1}{k}x^{k+n-R},$$

therefore

$$\int_0^1 (1-x)^{R-1} x^{n-R} dx = Beta(R, n-R+1) = \sum_{k=0}^{R-1} (-1)^k \binom{R-1}{k} \int_0^1 x^{k+n-R} dx,$$

and the identity follows.

ACKNOWLEDGMENTS

The authors thank the referee and one Associate Editor for suggestions which led to better presentation of the paper. The part of the work of the first author was supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. OGP-0004850. The part of the work of the second author was supported by a grant (No. SR/OY/M-06/93) of the Department of Science and Technology, Government of India.

BIBLIOGRAPHY

Bain, L.J. (1978). *Statistical Analysis of Reliability and Life-Testing Models*, Marcel and Dekker Inc., New York.

Bain, L.J. and Engelhardt, M. (1991). *Statistical Analysis of Reliability and Life-Testing Models, Theory and Methods*, 2^{nd} edition, vol. 115, Marcel and Dekker, New York.

Bain, L.J., Engelhardt, M. and Wright, F.T. (1977). "Inferential procedures for the truncated exponential distribution", *Communications in Statistics, Theory and Methods*, 6(2), 103-111.

Bain, L.J. and Weeks, D.L.(1964). "A note on the truncated exponential distribution", *Annals of Mathematical Statistics*, 35, 1366-1367.

Barlow, R.E. and Proschan, F. (1965). *Statistical Theory of Reliability and Life-Testing*, Holt, Rinehart and Winston, New York.

Bartholmew, D.J. (1963). "The sampling distribution of an estimate arising in life-testing", *Technometrics*, 5, 361-372.

Bartoszewicz, J. (1974). "Estimation of reliability in the exponential case (I), *Zastosowania Matematyki Applicationes Mathematicae*, XIV(2), 185-194.

Chen, S.M. and Bhattacharya, G.K. (1988). "Exact confidence bounds for an exponential parameter Hybrid censoring", *Communications in Statistics, Theory and Methods*, 17(6), 1858-1870.

Draper, N. and Guttman, T. (1987). "Bayesian analysis of Hybrid life-test with exponential failure times", *Annals of Institute of Statistical Mathematics A*, 39, 219-225.

Epstein, B. (1954). "Truncated life-tests in the exponential case", *Annals of Mathematical Statistics*, 25, 555-564.

Epstein, B. (1960). "Estimation from life-test data", *Technometrics*, 2, 447-454.

Fairbanks, K., Madsan, R. and Dykstra, R. (1982). "A confidence interval for an exponential parameter from hybrid life-test", *Journal of American Statistical Association*, 77(1), 137-140.

Govindarajulu, Z. (1963). "Recurrence relations for the inverse moments of the positive binomial variable", *Journal of American Statistical Association*, 468-473.

Harter, H.L. (1978). MTBF confidence bounds based on MIL-STD-781C fixed length test results", *Journal of Quality Technology*, 10, 164-169.

Hoem, J.M. (1969). "The sample distribution of an estimator arising in connection with the truncated exponential distribution", *Annals of Mathematical Statistics*, 40 (2), 702-703.

Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data*, John Wiley, New York.

Mendanhall, W. and Lehman, E.H. (1960). "An approximation to the negative moments of the positive binomial useful in life testing", *Technometrics*, 2, 227-242.

MIL-STD-781C (1977). *Reliability Design Qualification and Production Acceptance Tests: Exponential Distribution*, Washington, D.C., U.S. Government Printing Office.

Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. (1986), *Numerical Recipes*, Cambridge University Press, United Kingdom.

Received January, 1998; Revised June, 1998.