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GENERALIZED EXPONENTIAL DISTRIBUTIONS

RAMESHWAR D. GUPTA[1*] AND DEBASIS KUNDU[2]

University of New Brunswick and Indian Institute of Technology

Summary

The three-parameter gamma and three-parameter Weibull distributions are commonly used for analysing any lifetime data or skewed data. Both distributions have several desirable properties, and nice physical interpretations. Because of the scale and shape parameters, both have quite a bit of flexibility for analysing different types of lifetime data. They have increasing as well as decreasing hazard rate depending on the shape parameter. Unfortunately both distributions also have certain drawbacks. This paper considers a three-parameter distribution which is a particular case of the exponentiated Weibull distribution originally proposed by Mudholkar, Srivastava & Freimer (1995) when the location parameter is not present. The study examines different properties of this model and observes that this family has some interesting features which are quite similar to those of the gamma family and the Weibull family, and certain distinct properties also. It appears this model can be used as an alternative to the gamma model or the Weibull model in many situations. One dataset is provided where the three-parameter generalized exponential distribution fits better than the three-parameter Weibull distribution or the three-parameter gamma distribution.

Key words: distribution of sum; exponentiated Weibull model; gamma distribution; hazard rate; maximum likelihood estimators; stochastic ordering; uniformly most powerful test; Weibull distribution.

1. Introduction

The three-parameter gamma and three-parameter Weibull are the most popular distributions for analysing lifetime data. Both distributions have been studied in the literature, and both have applications in fields other than lifetime distributions; see e.g. Alexander (1962), Jackson (1969), van Kinken (1961) and Masuyama & Kuroiwa (1952). The three parameters, in both distributions, represent location, scale and shape, and because of them both distributions have quite a bit of flexibility for analysing skewed data. Both distributions allow increasing as well as decreasing hazard rate, depending on the shape parameter, that gives an extra edge over the exponential distribution which has only constant hazard rate. They have certain nice physical interpretations also.

Unfortunately both distributions have drawbacks. One major disadvantage of the gamma distribution is that the distribution function or the survival function cannot be computed easily if the shape parameter is not an integer. One needs to obtain the distribution function, the

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* Author to whom correspondence should be addressed.

[1] Dept of Mathematics, Statistics and Computer Science, The University of New Brunswick, Saint John, PO Box 5050, Canada E2L 4L5.

email: gupta@unbsj.ca

[2] Dept of Mathematics, The Indian Institute of Technology, Kanpur, Pin 208016, Kanpur, India.

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survival function or the hazard function by using mathematical tables or computer software. This makes the gamma distribution unpopular compared to the Weibull distribution, whose distribution function, survival function and hazard function can be computed very easily. The Weibull distribution is often preferred for analysing lifetime data because, in presence of censoring, it is much easier to handle than the gamma distribution. Also in many positive datasets it is observed that the Weibull distribution fits very well. But the Weibull distribution also has disadvantages. For example, Bain (1978) points out that the maximum likelihood estimators of the Weibull parameters may not behave properly for all parameter values even when the location parameter is zero. When the shape parameter is greater than one, the hazard functions of the gamma distribution and Weibull distribution are both increasing functions. However, in the case of the gamma distribution it increases from zero to a finite number (the reciprocal of the scale parameter), whereas in the case of the Weibull distribution it increases from zero to infinity, which may not be appropriate in some situations.

The gamma distribution has likelihood ratio ordering, with respect to the shape parameter, when the scale and location parameters are kept constant. It naturally implies ordering in hazard rate as well as ordering in distribution. It follows that all the moments are an increasing function of the shape parameter if the scale and location parameters remain constant. Also because of the likelihood ratio ordering property, if the scale and location parameters are known, it is possible to obtain the uniformly most powerful test for testing a one-sided hypothesis in the shape parameter. But the family of Weibull distributions does not enjoy any such ordering properties and there does not exist a uniformly most powerful test for testing the one-sided hypothesis on the shape parameter even if the other two parameters are known.

It is well known that, even though the Weibull distribution has a convenient representation of the distribution function, the distribution of the sum of independent and identically distributed (iid) Weibull random variables is not simple to obtain. Therefore the distribution of the mean of a random sample from a Weibull distribution is not easy to compute. On the other hand the distribution of the sum of iid gamma random variables is well known.

Recently, Mudholkar, Srivastava & Freimer (1995) proposed a three-parameter (one scale and two shapes) distribution, the exponentiated Weibull distribution (see also Mudholkar & Srivastava, 1993). Both papers analyse certain datasets and show that the exponentiated Weibull, which has three parameters, has a better fit than the two-parameter (taking location parameter to be zero) Weibull or one-parameter exponential, which are special cases of the exponentiated Weibull. More recently, Gupta & Kundu (1997) considered a special case (exponentiated exponential) of the exponentiated Weibull model assuming the location parameter to be zero, and compared its performances with the two-parameter gamma family and the two-parameter Weibull family, mainly through data analysis and computer simulations. For some of the other exponentiated families (exponentiated Pareto or exponentiated gamma) see Gupta, Gupta & Gupta (1998) who mainly discuss the different hazard rate properties under different situations.

The main aims of the present paper are to introduce the three-parameter 'generalized exponential distribution' (location, scale and shape), to study the theoretical properties of this family and compare them with respect to the well studied properties of the gamma distribution and the Weibull distribution.

The generalized exponential (GE) distribution has increasing or decreasing hazard rate depending on the shape parameter. It has many properties that are quite similar to those of the gamma distribution, but it has a distribution function like that of the Weibull distribution which can be computed simply. The generalized exponential family has likelihood ratio ordering on the shape parameter; so it is possible to construct a uniformly most powerful test for testing a one-sided hypothesis on the shape parameter when the scale and location parameters are known.

The paper is organized as follows: Section 2 introduces the GE distribution and discusses some of its properties. We obtain the moment generating function and derive several moments in Section 3. In Section 4 we derive the distribution of the sum and the extreme values. Section 5 discusses the maximum likelihood estimators, their asymptotic distributions and some testing of hypothesis questions. Section 6 analyses one dataset where the three-parameter GE fits better than the three-parameter Weibull or the three-parameter gamma.

2. Definition and Some Properties

We say that the random variable X has a generalized exponential distribution if X has the distribution function

$$F(x;\alpha,\lambda,\mu) = (1-e^{-(x-\mu)/\lambda})^{\alpha} \qquad (x>\mu, \alpha>0, \lambda>0). \tag{2.1}$$

If X has the distribution function (2.1), then the corresponding density function is

$$f(x;\alpha,\lambda,\mu) = \frac{\alpha}{\lambda}(1-e^{-(x-\mu)/\lambda})^{\alpha-1}e^{-(x-\mu)/\lambda} \qquad (x>\mu, \alpha>0, \lambda>0). \tag{2.2}$$

Here α is a shape parameter, λ is a scale parameter and μ is a location parameter. We denote the GE distribution with shape parameter α, scale parameter λ and location parameter μ as $\mathrm{GE}(\alpha,\ \lambda,\ \mu)$.

It is interesting to note the similarities of the density and the distribution functions of the GE family with the corresponding gamma family and the Weibull family. If the shape parameter is $\alpha=1$, then all the three distributions coincide with the two-parameter exponential distribution. Therefore all three distributions are extensions or generalizations of the exponential distribution, but in different ways. If X has an exponential distribution with moment generating function (mgf) $M_{\mathrm{E}}(t)$ and distribution function $F_{\mathrm{E}}(x)$ then it is well known that $M_{\mathrm{G}}(t) = (M_{\mathrm{E}}(t))^{\alpha}$ and $F_{\mathrm{W}}(x) = F_{\mathrm{E}}(x^{\alpha})$. The GE distribution is such that $F_{\mathrm{GE}}(x) = (F_{\mathrm{E}}(x))^{\alpha}$. From the form of the density function of the GE distribution we see that, if $\alpha \le 1$, the density function is a strictly decreasing function, whereas if $\alpha > 1$, it is a unimodal skewed density function.

If $X \sim \mathrm{GE}(\alpha,\lambda,\mu)$, the survival function and hazard function are given by

$$S(x;\alpha,\lambda,\mu) = 1-F(x;\alpha,\lambda,\mu) = 1-(1-e^{-(x-\mu)/\lambda})^{\alpha} \qquad (x>\mu) \tag{2.3}$$

$$h(x;\alpha,\lambda,\mu) = \frac{f(x;\alpha,\lambda,\mu)}{S(x;\alpha,\lambda,\mu)} = \frac{\alpha}{\lambda}\,\frac{(1-e^{-(x-\mu)/\lambda})^{\alpha-1}e^{-(x-\mu)/\lambda}}{1-(1-e^{-(x-\mu)/\lambda})^{\alpha}} \qquad (x>\mu). \tag{2.4}$$

If $\alpha = 1$, the hazard function becomes $1/\lambda$, independent of x.

Lemma 1. *The density of GE(α,λ,μ) is log-convex if $\alpha<1$, and log-concave if $\alpha>1$.*

TABLE 1

Behaviour of the hazard functions of the three distributions

Parameter	Gamma	Weibull	GE
$\alpha = 1$	$1/\lambda$	$1/\lambda$	$1/\lambda$
$\alpha > 1$	Increasing from 0 to $1/\lambda$	Increasing from 0 to ∞	Increasing from 0 to $1/\lambda$
$\alpha < 1$	Decreasing from ∞ to $1/\lambda$	Decreasing from ∞ to 0	Decreasing from ∞ to $1/\lambda$

Proof. The proof follows by observing that the second derivative of the logarithm of the density function is

$$\frac{d^2}{dx^2} \log f(x; \alpha, \lambda, \mu) = -(\alpha - 1) \frac{e^{-(x-\mu)/\lambda}}{(1 - e^{-(x-\mu)/\lambda})^2} \,.$$

Theorem 1. *For any* (λ, μ), *the generalized exponential (GE) distribution has an increasing hazard function if* $\alpha > 1$, *it has a decreasing hazard function if* $\alpha < 1$, *and for* $\alpha = 1$ *the hazard function is constant for* $x > \mu$.

Proof. The proof follows using the log convexity of the density function (Barlow & Proschan, 1975).

The hazard function of the GE distribution behaves like the hazard function of the gamma distribution, which is quite different from the hazard function of the Weibull distribution. Table 1 shows the behaviour of the hazard functions of the three distributions.

For the Weibull distribution, if $\alpha > 1$, the hazard function increases from zero to ∞, and if $\alpha < 1$ the hazard function decreases from ∞ to zero. Many authors point out (see Bain, 1978) that because for the gamma distribution (for $\alpha > 1$) the hazard function increases from zero to a finite number, the gamma may be more appropriate as a population model when the items in the population are in a regular maintenance program. The hazard rate may increase initially, but after some time the system reaches a stable condition because of maintenance. The same comments hold for the GE distribution. Therefore, if it is known that the data are from a regular maintenance environment, it may make more sense to fit the gamma distribution or the GE distribution than the Weibull distribution.

Ordering of distributions, particularly among the lifetime distributions, plays an important role in statistical literature. Johnson, Kotz & Balakrishnan (1995 Chap. 33) have a major section on the ordering of various lifetime distributions. Pecaric, Proschan & Tong (1992) also provide a detailed treatment of stochastic orderings, highlighting their growing importance and illustrating their usefulness in numerous practical applications. Mosler & Scarsini (1993) document a classical bibliography of stochastic orderings and their applications. It is already known that the gamma family has a likelihood ratio ordering $(<_{\mathrm{LR}})$, which implies that it has ordering in the hazard rate $(<_{\mathrm{HAZ}})$ and also in distribution $(<_{\mathrm{ST}})$. Since the gamma family has likelihood ratio ordering, it has the monotone likelihood ratio property. This implies there exists a uniformly most powerful test (UMP) for any one-sided hypothesis on the shape parameter, when the other parameters are known. The gamma family also has dispersive ordering $(<_{\mathrm{DISP}})$ (see Johnson *et al.*, 1995), therefore it has tail ordering in the sense of Lehmann (1966). The gamma family also has convex ordering $(<_{\mathrm{C}})$ and star shaped $(<_{*})$ ordering. Although the gamma family enjoys several ordering properties, the same thing cannot be said

TABLE 2

Ordering relations within the different families

	$<_{\mathrm{LR}}$	$<_{\mathrm{HAZ}}$	$<_{\mathrm{ST}}$	$<_{\mathrm{C}}$	$<_{*}$	$<_{\mathrm{DISP}}$
gamma	Y	Y	Y	Y	Y	Y
Weibull	N	N	N	Y	Y	N
GE	Y	Y	Y	Y	Y	Y

for the Weibull family. Two Weibull distributions that differ with respect to shape do not have stochastic ordering. This implies that two different Weibull distributions cannot have ordering in hazard rate or in likelihood ratio. The Weibull family has convex ordering and therefore it has star shaped ordering also.

The GE family has ordering properties similar to those of the gamma family. Table 2 highlights the ordering relations within the various families (without detailed proof): 'Y' means ordering exists and 'N' means ordering does not exist.

3. Moments and Other Measures

The results developed in Sections 3 and 4 can be stated for the three-parameter GE model in (2.1). However, for simplicity and clarity here we assume $\mu = 0$ and $\lambda = 1$ and develop the results for $\mathrm{GE}(\alpha) = \mathrm{GE}(\alpha, 1, 0)$, since if $X \sim \mathrm{GE}(\alpha)$ then $\mu + \lambda X \sim \mathrm{GE}(\alpha, \lambda, \mu)$.

If $X \sim \mathrm{GE}(\alpha)$, then the corresponding moment generating function, is given by

$$M(t) = \mathrm{E}(e^{tX}) = \alpha \int_0^\infty (1 - e^{-x})^{\alpha-1} e^{(t-1)x}\, dx. \tag{3.1}$$

Making the substitution $y = e^{-x}$, (3.1) reduces to

$$M(t) = \alpha \int_0^1 (1 - y)^{\alpha-1} y^{-t}\, dy = \frac{\Gamma(\alpha+1)\Gamma(1-t)}{\Gamma(\alpha - t + 1)} \qquad (t < 1). \tag{3.2}$$

Differentiating $\log M(t)$ and evaluating at $t = 0$, we get the mean and the variance of $\mathrm{GE}(\alpha)$ as

$$\mathrm{E}(X) = \psi(\alpha+1) - \psi(1) \quad \text{and} \quad \mathrm{var}(X) = \psi'(1) - \psi'(\alpha+1), \tag{3.3}$$

where $\psi(\cdot)$ is the digamma function and $\psi'(\cdot)$ is its derivative. The higher central moments can be obtained in terms of the polygamma functions, which can be evaluated using mathematical software such as MAPLE. However, the moments can also be obtained in the form of a series which is finite or infinite depending on whether α is an integer or not. Since $0 < e^{-x} < 1$ for $x > 0$, by using the binomial series expansion we have

$$(1 - e^{-x})^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} e^{-jx}, \tag{3.4}$$

and (3.1) can be rewritten as

$$M(t) = \alpha \int_0^\infty \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} e^{-(j+1-t)x}\, dx. \tag{3.5}$$

Since the quantity inside the summation is absolutely integrable, interchanging the summation and integration we have

$$M(t) = \alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{j+1-t} \qquad (t < 1)\,, \tag{3.6}$$

We observe that the infinite series is summable, differentiable and it has only a finite number of terms if α is an integer. Hence by differentiating k times and evaluating at $t = 0$, we get the kth moment as

$$\mu_k = \alpha k! \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{(j+1)^{k+1}}\,.$$

If $X \sim \mathrm{GE}(\alpha)$, then the distribution of X is the same as the distribution of

$$\sum_{j=1}^{[\alpha]} \frac{Y_j}{j+\langle\alpha\rangle} + Z, \tag{3.7}$$

where the Y_j are iid exponential random variables with scale parameter 1 and $Z \sim \mathrm{GE}(\langle\alpha\rangle)$, which is independent of the Y_j. Here $\langle\alpha\rangle$ represents the fractional part and $[\alpha]$ denotes the integer part of α. This result can be established by equating the mgf of (3.7) with (3.2). Therefore if α is an integer, say n, it follows immediately that the distribution of X is same as the distribution of $\sum_{j=1}^{n} Y_j/j$. As a consequence,

$$\mathrm{E}(X) = \sum_{i=1}^{n} \frac{1}{i} \quad \text{and} \quad \mathrm{var}(X) = \sum_{i=1}^{n} \frac{1}{i^2}\,. \tag{3.8}$$

The distribution of X is also same as the distribution of the maximum order statistic, $Y_{(n)}$, of the Y_j. The summands in $\sum_{j=1}^{n} Y_j/j$, represent spacings between successive order statistics of an exponential sample (David, 1970 p. 18). For integer α, from the mean and variance expressions we have the following identities, which may have some independent interest

$$\sum_{i=1}^{n} \frac{1}{i} = n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{(j+1)^2}\,, \tag{3.9}$$

$$\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2 + \sum_{i=1}^{n} \frac{1}{i^2} = 2n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{(j+1)^3}\,. \tag{3.10}$$

The GE distribution is unimodal with mode at $\log\alpha$ for $\alpha > 1$ and it is reverse 'J' shaped if $\alpha < 1$. It has median $-\log\left(1-(0.5)^{1/\alpha}\right)$. The mean, median and mode all are non-linear functions of the shape parameter, and as the shape parameter goes to infinity, all of them tend to infinity. For large values of α, the mean, median and mode all are approximately equal to $\log\alpha$. It can be shown that as α tends to ∞, (mean–mode) tends to $\lambda =$ Euler's constant ≈ 0.5772 and (median–mode) tends to $-\log\log 2 \approx 0.3665$. The survival function at the median is always 0.5; the survival function at the mean is $1-(1-e^{\psi(1)-\psi(\alpha+1)})^\alpha$, which is an increasing function of α with limit $(1-e^{-e^{-\gamma}})$; the survival function at the mode is $1-(1-\frac{1}{\alpha})^\alpha$,

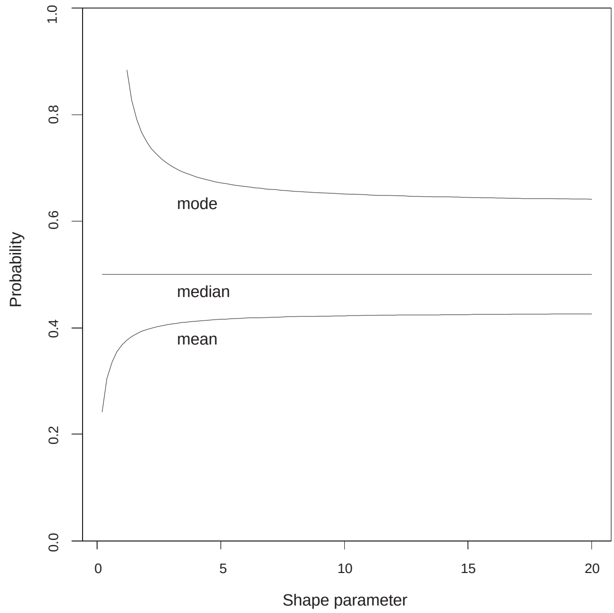

Fig. 1. Survival function at the mean, median and mode

a decreasing function of α, which decreases to $(1-e^{-1})$. Figure 1 graphs the survival function at the mean, median and mode. This indicates that the density function remains skewed even for large values of α.

The variance of GE(α) is an increasing function of α; therefore, from the expression at (3.8), the variance is increasing to a constant, namely $\pi^2/6$. This is different from the gamma distribution or the Weibull distribution. In the case of the gamma distribution the variance tends to infinity as α increases, but in the case of the Weibull distribution the variance is approximately $\pi^2/6\alpha^2$ for large values of α. Skewness and kurtosis can be expressed in terms of polygamma functions and they are decreasing functions of α. Numerical work using MAPLE indicates that the limiting value of the skewness is approximately 1.1395. A table is available on request from the authors for the mean, median, mode, standard deviation, coefficient of variation, skewness and kurtosis for various values of α. Figure 2 is a graph of these measures.

The GE laws are not totally new in the literature. With α regarded as a time index and with $\mu = 0$ and $\lambda = 1$, the GE laws are one-dimensional distributions of the extremal process generated by the standard exponential laws or with the directing measure m, where $m(x,\infty) = -\log(1-e^{-x})$. This connection allows the use of all known structural theory of extremal processes; see Embrechts, Kluppelberg & Mikosch (1997) for a particularly nice account of it.

4. Distribution of the Sum and the Extreme Values

It is well known that the distribution of a sum of independent gamma random variables with the same scale parameter is again a gamma distribution. It can be obtained very simply using the mgf of the gamma random variable. In the case of the Weibull distribution, because the mgf is not in a very convenient form, we cannot readily obtain the distribution of the sum

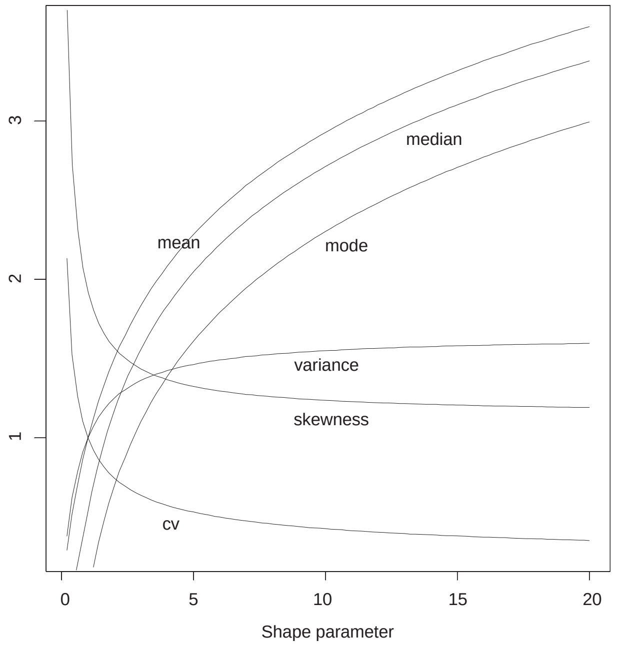

Fig. 2. Mean, median, mode, CV, variance and skewness for the generalized exponential (GE) distribution

of independent Weibull random variables. For the GE distribution also, the mgf is not in a convenient form for obtaining the distribution of the sum of iid GE random variables.

However, the distribution of the sum can be obtained using a transformation to the beta distribution. If $X \sim \text{GE}(\alpha)$, then $U = e^{-X} \sim \text{B}(1, \alpha)$. Therefore, if $X_1, \ldots, X_n$ are independent random variables such that $X_i \sim \text{GE}(\alpha_i)$, then $U_i = e^{-X_i} \sim \text{B}(1, \alpha_i)$ and the U_i are independent for $i = 1, \ldots, n$. Let

$$U = \prod_{i=1}^{n} U_i = e^{-\Sigma_{i=1}^{n} X_i} = e^{-X} . \tag{4.1}$$

It is well known that the product of the independent beta random variables belongs to a family whose densities are particular Meijer G-functions, which are well studied in the literature (see Springer, 1979; Richards, 1978; Bhargava & Khatri, 1981; Mathai, 1993). We state the following result without proof.

Theorem 2. *Let $U_1, \ldots, U_n$ denote independent random variables from $\text{B}(1, \alpha_i)$. Then the density function of $U = \prod_{i=1}^{n} U_i$, is*

$$f_U(u) = \sum_{j=0}^{\infty} C_j f_{\text{B}}(u; 1, \alpha^* + j), \tag{4.2}$$

where $\alpha^* = \sum_{i=1}^{n} \alpha_i$, $f_{\text{B}}(u; 1, \alpha^* + j)$ is the density function of the beta distribution with the corresponding parameters 1 and $\alpha^* + j$, and the constants C_j are defined as follows:

$$C_0 = \frac{\prod_{i=1}^{n} \Gamma(\alpha_j + 1)}{\Gamma(1 + \alpha^*)}, \quad C_j = \frac{C_0 \alpha^*}{(\alpha^* + j)} C_j^{(n)} \qquad (j = 1, \ldots),$$

and

$$C_j^{(2)} = \frac{(\alpha_1)_j(\alpha_2)_j}{j!(\alpha_1+\alpha_2)_j}, \quad C_j^{(k)} = \frac{(\alpha_1+\cdots+\alpha_{k-1})_j}{(\alpha_1+\cdots+\alpha_k)_j}\sum_{i=0}^{j}\frac{(\alpha_k)_i}{i!}C_{j-i}^{(k-1)} \quad (k=3,\ldots,n).$$

Here $(\alpha)_j = \Gamma(\alpha+j)\big/\Gamma(\alpha)$.

Using the transformation (4.1), the density function of $X=\sum_{i=1}^n X_i$ is

$$f_X(x) = \sum_{j=0}^{\infty} C_j(\alpha^*+j)e^{-x}(1-e^{-x})^{\alpha^*+j-1} = \sum_{j=0}^{\infty} C_j f_{\mathrm{GE}}(x;\alpha^*+j). \tag{4.3}$$

Hence we have the following result.

Theorem 3. *Let* $X_1,\ldots,X_n$ *denote independent random variables from GE*(α_i) *for* $i=1,\ldots,n$. *Then the density function of* $X=\sum_{i=1}^n X_i$ *is given by (4.3).*

Since the C_j are all positive and they sum to one, the sum of n independent GE random variables has a mixture distribution, where the individual components are also GE distributions. An immediate corollary of Theorem 3 is the following.

Corollary 1. *If* $X_1,\ldots,X_n$ *are iid random variables from* $\mathrm{GE}(\alpha)$, *the density function of* $X=\sum_{i=1}^n X_i$ *is*

$$f_X(x) = \sum_{j=0}^{\infty} C_j^*(n\alpha+j)e^{-x}(1-e^{-x})^{n\alpha+j-1} = \sum_{j=0}^{\infty} C_j^* f_{\mathrm{GE}}(x;n\alpha+j), \tag{4.4}$$

where the C_j^* *are obtained from* C_j *by putting* $\alpha_1=\cdots=\alpha_n=\alpha$.

Since (4.4) is an infinite series, $f_X(x)$ can be approximated by truncating it after M terms:

$$f_X(x) \approx \sum_{j=0}^{M-1} C_j^* f_{\mathrm{GE}}(x;n\alpha+j). \tag{4.5}$$

It can be shown (Richards, 1978) that the error caused by this truncation cannot exceed

$$\left(1-\sum_{j=0}^{M-1} C_j^*\right)g(x), \tag{4.6}$$

where

$$g(x) = -\frac{e^{-x}}{\log(1-e^{-x})}(1-e^{-x})^{-\left[\frac{1}{\log(1-e^{-x})}+1\right]}.$$

Note that $g(x)\to 0$ as $x\to 0$ and $g(x)\to e^{-1}$ as $x\to\infty$.

From Corollary 1, if $X_1,\ldots,X_n$ are iid random variables from $\mathrm{GE}(\alpha)$, the distribution function $F_X(x)$ of $X=\sum_{i=1}^n X_i$ is

$$F_X(x) = \sum_{j=0}^{\infty} C_j^* F_{\mathrm{GE}}(x;n\alpha+j). \tag{4.7}$$

Once again, based on the first M terms, $F_X(x)$ can be approximated by

$$F_X(x) \approx \sum_{j=0}^{M-1} C_j^*(1-e^{-x})^{n\alpha+j} = \sum_{j=0}^{M-1} C_j^* F_{\text{GE}}(x; n\alpha + j). \tag{4.8}$$

Using the stochastic ordering property of $\text{GE}(\alpha)$ it can be shown that the error caused by this approximation cannot exceed

$$\left(1 - \sum_{j=0}^{M-1} C_j^*\right) F_{\text{GE}}(x; n\alpha + M). \tag{4.9}$$

When $\alpha = k$ is an integer, the distribution of $X = \sum_{i=1}^{n} X_i$ is the same as the distribution of $\sum_{i=1}^{k} Z_i$, where the Z_i are independent gamma random variables with the shape parameter n and the scale parameter $1/i$ $(i = 1, \ldots, k)$.

We give the following result based on the representation of the distribution of the sum of independent gamma random variables given in Gupta & Richards (1979).

Theorem 4. *When $\alpha = k$ is an integer, the distribution function of the sum of a random sample of size n from GE(k) has a mixture representation given by*

$$F_X(x) = \sum_{j=0}^{\infty} d_j \, F_{\text{G}}(kx; nk + j), \tag{4.10}$$

The coefficients d_j are as follows:

$$d_0 = \left(\frac{k!}{k^k}\right)^n \qquad d_j = \frac{d_0 (nk)_j d_j^{(k)}}{k^j} \qquad (j = 1, 2, \ldots),$$

and

$$d_j^{(2)} = \frac{(n)_j}{(2n)_j \, j!}, \qquad d_j^{(m)} = \frac{\big((m-1)n\big)_j}{(mn)_j} \sum_{i=0}^{j} \frac{d_{j-1}^{(m-1)}}{i!} \qquad (m = 3, 4, \ldots, k). \tag{4.11}$$

One can make similar comments on the truncation error if the the right hand side of (4.10) is to be approximated by the first M terms. From (4.10) we have an alternative representation of the distribution of the sum of a random sample from GE, if α is an integer. If α is a small integer $(< n)$, it might be easier to use the latter representation (as a mixture of gamma) and if α is a large integer then the former representation (as a mixture of GE) might be more useful. For $\alpha = 1$, the distribution of the sum reduces to a gamma distribution with shape parameter n. When α = 2, (4.10) becomes:

$$F_X(x) = \sum_{j=0}^{\infty} \frac{(n)_j}{j! 2^{n+j}} F_{\text{G}}(2x; 2n + j). \tag{4.12}$$

The following theorem shows that the distribution of the maximum of n independent GE random variables is also a generalized exponential. It shows that this model can be used to represent a parallel system.

Theorem 5. *If the X_i are independently distributed and X_i is distributed as GE(α_i), for $i = 1, \ldots, n$, then $X_{(n)} = \max(X_1, \ldots, X_n)$ is distributed as GE$(\sum_{i=1}^{n} \alpha_i)$.*

The following theorem gives a characterization of the GE distribution in terms of the maximum.

Theorem 6. *Suppose $X_1, \ldots, X_n$ are iid random variables. Then the X_i are GE random variables if and only if $\max\{X_1, \ldots, X_n\}$ is a GE random variable.*

After proper normalization, the distributions of the maximum and minimum of n iid GE random variables tend to the extreme value distributions of Type I (exponential type) and Type III (Weibull type), respectively. The results can be stated as follows.

Theorem 7. *If $X_{(n)} = \max\{X_1, \ldots, X_n\}$, where the X_i are iid GE(α) random variables, then for all $-\infty < y < \infty$, and $b_n = \log n + \log\alpha$,*

$$\lim_{n\to\infty} \Pr\left\{X_{(n)} - b_n \leq y\right\} = e^{-e^{-y}}.$$

Theorem 8. *If $X_{(1)} = \min\{X_1, \ldots, X_n\}$, where the X_i are iid GE(α) random variables, then for all $y > 0$ and $c_n = n^{-1/\alpha}$,*

$$\lim_{n\to\infty} \Pr\left\{\frac{X_{(1)}}{c_n} \leq y\right\} = 1 - e^{-y^{\alpha}}.$$

5. Inference

5.1. Maximum Likelihood Estimators

In this subsection we discuss how to obtain the maximum likelihood estimators (MLEs) and consider their asymptotic properties. We give the results first for a complete sample and then for Type II censored data. We consider the three-parameter GE model, and for the sake of simplicity we reparametrize $\beta = 1/\lambda$. We denote the MLEs of α, β, μ as $\hat{\alpha}, \hat{\beta}, \hat{\mu}$, respectively. Let $x_1, \ldots, x_n$ be a random sample of size n from GE$(\alpha, 1/\beta, \mu)$; then the log-likelihood function $L(\alpha, \beta, \mu)$ for $\mu < x_{(1)}$ is

$$L(\alpha, \beta, \mu) = n\log(\alpha) + n\log(\beta) - \sum_{i=1}^{n} \beta(x_i - \mu) + (\alpha - 1)\sum_{i=1}^{n} \log(1 - e^{-(x_i-\mu)\beta}). \quad (5.1)$$

If $\alpha > 1$ then the maximum occurs away from the boundary: taking the derivative with respect to α, β and μ and equating to 0, we obtain the normal equations as

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1 - e^{-(x_i-\mu)\beta}) = 0, \quad (5.2)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}(x_i - \mu) + (\alpha - 1)\sum_{i=1}^{n} \frac{(x_i - \mu)e^{-(x_i-\mu)\beta}}{1 - e^{-(x_i-\mu)\beta}} = 0, \quad (5.3)$$

$$\frac{\partial L}{\partial \mu} = n\beta - \beta(\alpha - 1)\sum_{i=1}^{n} \frac{e^{-(x_i-\mu)\beta}}{1 - e^{-(x_i-\mu)\beta}} = 0. \quad (5.4)$$

The three equations can be reduced to two equations if we replace α in terms of β and μ from (5.2). We need to use either the scoring algorithm or the Newton–Raphson algorithm to solve the two non-linear equations simultaneously. Consider the second derivatives of $L(\alpha, \beta, \mu)$:

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2}, \qquad \frac{\partial^2 L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \frac{(x_i - \mu)e^{-(x_i-\mu)\beta}}{1 - e^{-(x_i-\mu)\beta}},$$

$$\frac{\partial^2 L}{\partial \beta^2} = -\frac{n}{\beta^2} - (\alpha - 1)\sum_{i=1}^{n} \frac{(x_i - \mu)^2 e^{-(x_i-\mu)\beta}}{(1 - e^{-(x_i-\mu)\beta})^2}, \qquad \frac{\partial^2 L}{\partial \alpha \partial \mu} = -\beta \sum_{i=1}^{n} \frac{e^{-(x_i-\mu)\beta}}{1 - e^{-(x_i-\mu)\beta}},$$

$$\frac{\partial^2 L}{\partial \beta \partial \mu} = n + (\alpha - 1)\sum_{i=1}^{n} \frac{\big(\beta(x_i - \mu) - 1 + e^{-(x_i-\mu)\beta}\big)e^{-(x_i-\mu)\beta}}{(1 - e^{-(x_i-\mu)\beta})^2},$$

$$\frac{\partial^2 L}{\partial \mu^2} = -\beta^2(\alpha - 1)\sum_{i=1}^{n} \frac{e^{-(x_i-\mu)\beta}}{(1 - e^{-(x_i-\mu)\beta})^2}.$$

For $\alpha > 2$,

$$\begin{aligned}
\mathrm{E}\left(\frac{\partial^2 L}{\partial \alpha^2}\right) &= -\frac{n}{\alpha^2}, \\
\mathrm{E}\left(\frac{\partial^2 L}{\partial \alpha \partial \beta}\right) &= n\mathrm{E}\left(\frac{(X-\mu)e^{-(X-\mu)\beta}}{1 - e^{-(X-\mu)\beta}}\right) \\
&= n\Big[\frac{\alpha}{\beta(\alpha-1)}\big(\psi(\alpha) - \psi(1)\big) - \frac{1}{\beta}\big(\psi(\alpha+1) - \psi(1)\big)\Big], \\
\mathrm{E}\left(\frac{\partial^2 L}{\partial \beta^2}\right) &= -n\left[\frac{1}{\beta^2} + (\alpha - 1)\mathrm{E}\left(\frac{(X-\mu)^2 e^{-(X-\mu)\beta}}{(1 - e^{-(X-\mu)\beta})^2}\right)\right] \\
&= \frac{-n}{\beta^2} - \frac{n\alpha(\alpha-1)}{(\alpha-2)\beta^2}\big(\psi'(1) - \psi'(\alpha-1) + \big(\psi(\alpha-1) - \psi(1)\big)^2\big) \\
&\quad + \frac{n\alpha}{\beta^2}\big(\psi'(1) - \psi(\alpha) + \big(\psi(\alpha) - \psi(1)\big)^2\big), \\
\mathrm{E}\left(\frac{\partial^2 L}{\partial \alpha \partial \mu}\right) &= -\beta n\mathrm{E}\left(\frac{e^{-(X-\mu)\beta}}{1 - e^{-(X-\mu)\beta}}\right) = -\beta n\frac{1}{\alpha - 1}, \\
\mathrm{E}\left(\frac{\partial^2 L}{\partial \beta \partial \mu}\right) &= n\left[1 + (\alpha - 1)\mathrm{E}\left(\frac{e^{-(X-\mu)\beta}\big(\beta(X-\mu) - 1 + e^{-(X-\mu)\beta}\big)}{(1 - e^{-(X-\mu)\beta})^2}\right)\right] \\
&= \frac{n\alpha(\alpha-1)}{\alpha-2}\big(\psi(\alpha-1) - \psi(1)\big) - n\alpha\big(\psi(\alpha) - \psi(1)\big), \\
\mathrm{E}\left(\frac{\partial^2 L}{\partial \mu^2}\right) &= -n\beta^2(\alpha - 1)\mathrm{E}\left(\frac{e^{-(X-\mu)\beta}}{(1 - e^{-(X-\mu)\beta})^2}\right) = -n\beta^2\frac{\alpha}{\alpha - 2}.
\end{aligned}$$

For $\alpha > 2$, the GE family satisfies all the regularity conditions (see Bain, 1978 pp.86–87) in a way similar to the gamma family or the Weibull family, and therefore we have the result in Theorem 9.

Theorem 9. *For $\alpha > 2$, the maximum likelihood estimators $(\hat\alpha, \hat\beta, \hat\mu)$ of (α, β, μ) are consistent estimators, and $\sqrt{n}(\hat\alpha - \alpha, \hat\beta - \beta, \hat\mu - \mu)$ is asymptotically normal with mean vector $\mathbf{0}$ and the variance covariance matrix $\mathbf{I}^{-1}$, where*

$$\mathbf{I} = -\frac{1}{n}\begin{bmatrix} \mathrm{E}\big(\frac{\partial^2 L}{\partial\alpha^2}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\alpha\partial\beta}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\alpha\partial\mu}\big) \\ \mathrm{E}\big(\frac{\partial^2 L}{\partial\beta\partial\alpha}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\beta^2}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\beta\partial\mu}\big) \\ \mathrm{E}\big(\frac{\partial^2 L}{\partial\mu\partial\alpha}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\mu\partial\beta}\big) & \mathrm{E}\big(\frac{\partial^2 L}{\partial\mu^2}\big) \end{bmatrix}.$$

Even for the three-parameter Weibull distribution or the three-parameter gamma distribution the regularity conditions are not met for $\alpha \le 2$ (see Bain, 1978 p. 334; Johnson, Kotz & Balakrishnan, 1994 p. 656). The asymptotic results are not known for $\alpha \le 2$ although some special cases have been investigated numerically by Harter (1969) for the gamma distribution. More work is needed along this direction for the GE family.

Since lifetime data are often censored, we derive normal equations for Type II censored data. The log likelihood function for the first r ordered observations from a sample of size n can be written as:

$$L(\alpha,\beta,\mu) = C + r\big(\log\alpha + \log\beta\big) + (\alpha-1)\sum_{i=1}^{r}\log(1 - e^{-(x_{(i)}-\mu)\beta}) - \sum_{i=1}^{r}(x_{(i)}-\mu)\beta + \alpha(n-r)\log(1 - e^{-(x_{(r)}-\mu)\beta}).$$

The normal equations become

$$\frac{r}{\alpha} + \sum_{i=1}^{r}\log(1 - e^{-(x_{(i)}-\mu)\beta}) + (n-r)\log(1 - e^{-(x_{(r)}-\mu)\beta}) = 0, \tag{5.5}$$

$$\frac{r}{\beta} - \sum_{i=1}^{r}(x_{(i)}-\mu) + (\alpha-1)\sum_{i=1}^{r}\frac{(x_{(i)}-\mu)e^{-(x_{(i)}-\mu)\beta}}{1 - e^{-(x_{(i)}-\mu)\beta}} + \alpha(n-r)\frac{(x_{(r)}-\mu)e^{-(x_{(r)}-\mu)\beta}}{1 - e^{-(x_{(r)}-\mu)\beta}} = 0, \tag{5.6}$$

$$r\beta - \beta(\alpha-1)\sum_{i=1}^{r}\frac{e^{-(x_{(i)}-\mu)\beta}}{1 - e^{-(x_{(i)}-\mu)\beta}} - \beta(n-r)\frac{e^{-(x_{(r)}-\mu)\beta}}{1 - e^{-(x_{(r)}-\mu)\beta}} = 0. \tag{5.7}$$

Note that the Fisher Information matrix is of the same type as the complete sample case, but now it involves the order statistics. The distribution of the ith order statistic $X_{(i)}$ is given by

$$\frac{n!}{(i-1)!(n-r)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\,\mathrm{GE}\big(\alpha(i+j), 1/\beta, \mu\big).$$

TABLE 3

Estimates, log-likelihood, chi-squared and Kolmogrov–Smirnov statistics

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	LL	χ^2	K–S
gamma	2.7316	0.0441	10.2583	–112.8501	0.950	0.107
Weibull	1.5979	0.0156	14.8479	–112.9767	1.321	0.118
GE	4.1658	0.0314	4.7476	–112.7666	0.675	0.103

Therefore the Fisher Information matrix can be computed using this distribution. It becomes very messy, just as in the three-parameter gamma or three-parameter Weibull case, so is not reported here; it can be obtained from the authors on request.

If the location parameter is known, the MLEs of α and β can be obtained by solving equations (5.2) and (5.3). It is possible to write $\hat{\alpha}$ in terms of $\hat{\beta}$; therefore it reduces to solving one equation only. If the location parameter is known, then without loss of generality we can assume it to be zero. In that case, the MLEs always exist and the $\mathrm{GE}(\alpha, 1/\beta)$ family satisfies the regularity conditions for all possible values of α and β. If $\alpha > 2$, the asymptotic variance covariance matrix of $\sqrt{n}(\hat{\alpha}, \hat{\beta})$ is the inverse of the first 2×2 submatrix of $\mathbf{I}$, and for $0 < \alpha \leq 2$, it is the inverse of the symmetric 2×2 matrix $A = (a_{ij})$, where

$$a_{11} = \frac{1}{\alpha^2}, \quad a_{12} = a_{21} = -\frac{\alpha}{\beta}\int_0^\infty x e^{-2x}(1-e^{-x})^{\alpha-2}\,dx,$$

$$a_{22} = \frac{1}{\beta^2} + \frac{(\alpha-1)\alpha}{\beta^2}\int_0^\infty x^2 e^{-2x}(1-e^{-x})^{\alpha-3}\,dx.$$

For all $\alpha > 0$, the above integrals exist.

5.2. Testing of Hypotheses

In this subsection we address the testing of hypotheses on the shape parameter, when the other two parameters are known. Without loss of generality it can be assumed that $\lambda = 1$ and $\mu = 0$. We consider testing H_0: $\alpha = \alpha_0$ vs. H_1: $\alpha > \alpha_0$. The likelihood ratio test indicates that one should reject the null hypothesis if

$$\prod_{i=1}^{n}(1-e^{-x_i}) > c. \tag{5.8}$$

To get c, we need to obtain the distribution of $\prod_{i=1}^{n}(1-e^{-X_i})$ under the null hypothesis. Note that $(1-e^{-X_i})$ $(i = 1, \ldots, n)$ are iid beta random variables, so the distribution of $\prod_{i=1}^{n}(1-e^{-X_i})$ under the null hypothesis can be obtained as in Theorem 2. Hence the constant c and the power function can obtained explicitly. Since the family has the monotone likelihood ratio property, this is the uniformly most powerful test. In the same way we can get the uniformly most powerful test against H_1: $\alpha < \alpha_0$, and uniformly most powerful unbiased test against H_1: $\alpha \neq \alpha_0$. These results can be used to test for exponentiality $(\alpha = 1)$.

6. Data Analysis

In this section we consider one real dataset from Lawless (1982 p. 228). The data arose in tests on the endurance of deep groove ball bearings. They were originally discussed by

TABLE 4

Observed and expected frequencies

Intervals	Observed	Gamma	Weibull	GE
0–40	3	4.493	4.738	4.322
40–80	12	10.658	10.016	10.913
80–120	5	5.423	5.627	5.303
120–160	2	1.805	1.992	1.739
160–200	1	0.621	0.627	0.723

Lieblein & Zelen (1956) and also by Gupta & Kundu (1997). The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are as follows: 17.88 28.92 33.00 41.52 42.12 45.60 48.40 51.84 51.96 54.12 55.56 67.80 68.64 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.40.

We have fitted all three distributions, namely three-parameter gamma, three-parameter Weibull and three-parameter GE. We present the estimates, the log likelihood (LL), the observed and the expected values, the Kolmogrov–Smirnov (K–S) statistics and the chi-squared statistics in Table 3, with observed and expected frequencies in Table 4.

We observe that the three-parameter GE fits marginally better than the three-parameter Weibull or three-parameter gamma, in this case.

7. Conclusions

This paper offers a new family of distributions, the three-parameter generalized exponential distributions, as possible alternatives to the Weibull and gamma families for analysing lifetime data. It may be argued that the GE distributions seem to be less flexible than the other two families for graduating tail thickness. For example, the tail function decays as αe^{-x} for the standard GE family, like $x^{\alpha-1}e^{-x}$ for the standard gamma family, and like $\exp(-x^{\alpha})$ for the standard Weibull family. However, the GE distributions have the same flexibility as gamma or Weibull distributions for small x. For moderate x, the GE distributions are more flexible than gamma and as flexible as Weibull. When fitting any lifetime data, the range is finite and because of the shifting (location parameter) and scaling (scale parameter) of the data the three-parameter GE is always as flexible as the three-parameter Weibull or gamma distributions.

We have studied various properties of the model and observed those many of the properties are quite similar to those of the gamma family or the Weibull family. The GE model has certain features which are distinct from the other two families. Since the distribution function is in a closed form, the inference based on the censored data can be handled more easily than with the gamma family. The GE model can be used as a possible alternative for analysing any skewed dataset.

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