

## **Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions**

RAMESHWAR D. GUPTA\*

Department of Mathematics, Statistics and Computer Sciences  
The University of New Brunswick  
Saint John  
Canada

DEBASIS KUNDU

Department of Mathematics  
Indian Institute of Technology Kanpur  
India

### *Summary*

In this article we study some properties of a new family of distributions, namely Exponentiated Exponential distribution, discussed in GUPTA, GUPTA, and GUPTA (1998). The Exponentiated Exponential family has two parameters (scale and shape) similar to a Weibull or a gamma family. It is observed that many properties of this new family are quite similar to those of a Weibull or a gamma family, therefore this distribution can be used as a possible alternative to a Weibull or a gamma distribution. We present two real life data sets, where it is observed that in one data set exponentiated exponential distribution has a better fit compared to Weibull or gamma distribution and in the other data set Weibull has a better fit than exponentiated exponential or gamma distribution. Some numerical experiments are performed to see how the maximum likelihood estimators and their asymptotic results work for finite sample sizes.

*Key words:* Gamma distribution; Weibull distribution; Likelihood ratio ordering; Hazard rate ordering; Stochastic ordering; Fisher Information matrix; Maximum Likelihood Estimator.

### 1. Introduction

Two-parameter gamma and two-parameter Weibull are the most popular distributions for analyzing any lifetime data. Gamma has a long history and it has several desirable properties, see JOHNSON, KOTZ, and BALAKRISHNAN (1994) for

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the different properties of the two-parameter gamma distribution. It has lots of applications in different fields other than lifetime distributions, some of the references can be made to ALEXANDER (1962), JACKSON (1963), KLINKEN (1961) and MASUYAMA and KUROIWA (1952). The two parameters of a gamma distribution represent the scale and the shape parameters and because of the scale and shape parameters, it has quite a bit of flexibility to analyze any positive real data. It has increasing as well as decreasing failure rate depending on the shape parameter, which gives an extra edge over exponential distribution, which has only constant failure rate. Since sum of independent and identically distributed (i.i.d.) gamma random variables has a gamma distribution, it has a nice physical interpretation also. If a system has one component and  $n$ -spare parts and if the component and each spare parts have i.i.d. gamma lifetime distributions, then the lifetime distribution of the system also follows a gamma distribution. Another interesting property of the family of gamma distributions is that it has likelihood ratio ordering, with respect to the shape parameter, when the scale parameter remains constant. It naturally implies the ordering in hazard rate as well as in distribution.

But one major disadvantage of the gamma distribution is that the distribution function or survival function cannot be expressed in a closed form if the shape parameter is not an integer. Since it is in terms of an incomplete gamma function, one needs to obtain the distribution function, survival function or the failure rate by numerical integration. This makes gamma distribution little bit unpopular compared to the Weibull distribution, which has a nice distribution function, survival function and hazard function. Weibull distribution was originally proposed by WEIBULL (1939), a Swedish physicist, and he used it to represent the distribution of the breaking strength of materials. Weibull distribution also has the scale and shape parameters. In recent years the Weibull distribution becoming very popular to analyze lifetime data mainly because in presence of censoring it is much easier to handle, at least numerically, compared to a gamma distribution. It also has increasing and decreasing failure rates depending on the shape parameter. Physically it represents a series system, because the minimum of i.i.d. Weibull distributions also follows a Weibull distribution. Several applications of the Weibull distribution can be found in PLAIT (1962) and JOHNSON (1968) although some of the negative points of the Weibull distribution can be found in GORSKI (1968). One of the disadvantages can be pointed out that the asymptotic convergence to normality for the distribution of the maximum likelihood estimators is very slow (BAIN, 1976). Therefore most of the asymptotic inferences (for example asymptotic unbiasedness or asymptotic confidence interval) may not be very accurate unless the sample size is very large. Some ramifications of this problem can be found in BAIN (1976). It also does not enjoy any ordering properties like gamma distribution.

In this paper we consider a two-parameter exponentiated exponential distribution and study some of its properties. The two parameters of an exponentiated

## Exponentiated Exponential Density

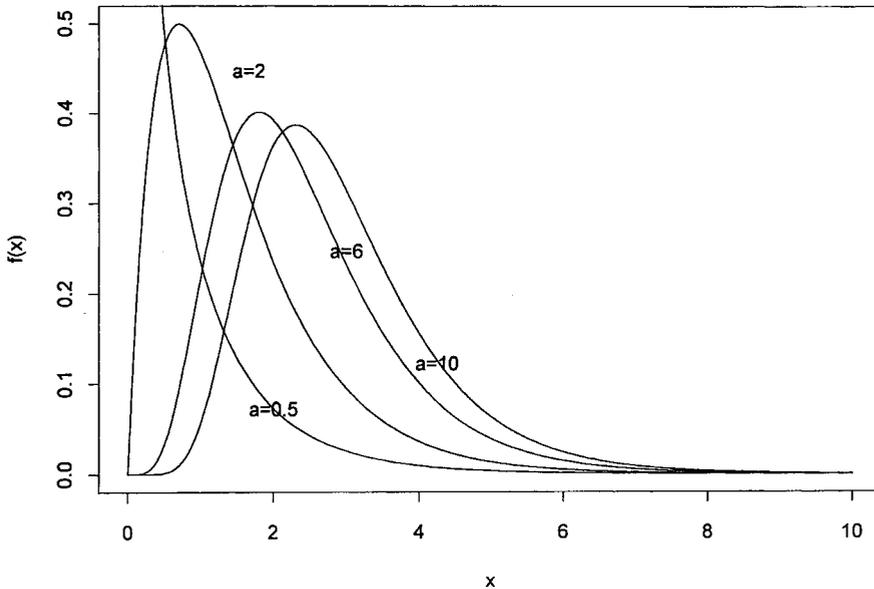


Fig. 1

exponential distribution represent the shape and the scale parameter like a gamma distribution or a Weibull distribution. It also has the increasing or decreasing failure rate depending of the shape parameter. The density function varies significantly depending of the shape parameter (see Figure 1). It is observed that it has lots of properties which are quite similar to those of a gamma distribution but it has an explicit expression of the distribution function or the survival function like a Weibull distribution. It has also likelihood ratio ordering with respect to the shape parameter, when the scale parameter is kept constant. We also observe that for fixed scale and shape parameters there is a stochastic ordering between the three distributions. The main aim of this paper is to introduce a new family of distributions and make comments both positive and negative of this family with respect to a Weibull family and a gamma family and give the practitioner one more option, with a hope that it may have a 'better fit' compared to a Weibull family or a gamma family in certain situations.

The rest of the paper is organized as follows. In Section 2, we introduce the exponentiated exponential distribution and compare its properties with the Weibull and the gamma distributions. Some of the stochastic ordering results are presented in Section 3. The maximum likelihood estimators and their asymptotic properties have been discussed in Section 4. We analyze two data sets in Section 5 and some numerical experimental results are presented in Section 6. Finally we draw conclusions in Section 7.

## 2. Exponentiated Exponential Distribution

The exponentiated exponential (EE) distribution is defined in the following way. The distribution function,  $F_E(x, \alpha, \lambda)$ , of EE is

$$F_E(x, \alpha, \lambda) = (1 - e^{-\lambda x})^\alpha; \quad \alpha, \lambda, x > 0,$$

therefore it has the density function

$$f_E(x, \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}.$$

The corresponding survival function is

$$S_E(x, \alpha, \lambda) = 1 - (1 - e^{-\lambda x})^\alpha,$$

and the hazard function is

$$h_E(x, \alpha, \lambda) = \frac{\alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}}{1 - (1 - e^{-\lambda x})^\alpha}.$$

Here  $\alpha$  is the shape parameter and  $\lambda$  is the scale parameter. When  $\alpha = 1$ , it represents the exponential family. Therefore, all three families, namely gamma, Weibull and EE, are generalization of the exponential family but in different ways. The EE distribution has a nice physical interpretation also. Suppose, there are  $n$ -components in a parallel system and the lifetime distribution of each component is inde-

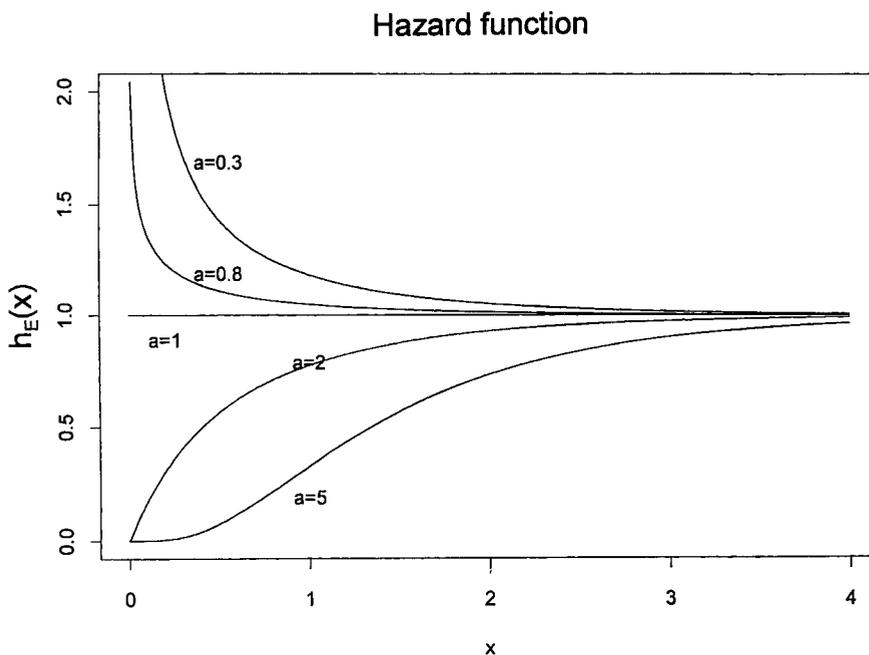


Fig. 2

pendent and identically distributed. If the lifetime distribution of each component is EE, then the lifetime distribution of the system is also EE. As opposed to Weibull distribution, which represents a series system, EE represents a parallel system.

The typical EE density and hazard functions with  $\lambda = 1$ , are shown in Figure 1 and Figure 2 respectively. It is an unimodal density function and for fixed scale parameter as the shape parameter increases it is becoming more and more symmetric. For any  $\lambda$ , the hazard function is a non-decreasing function if  $\alpha > 1$ , and it is a non-increasing function if  $\alpha < 1$ . For  $\alpha = 1$ , it is constant. In this paper we use the following notations of the gamma ( $f_G$ ) distribution and the Weibull ( $f_W$ ) distribution:

$$f_G(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}; \quad \alpha, \lambda, x > 0,$$

$$f_W(x) = \alpha\lambda(x\lambda)^{\alpha-1} e^{-(\lambda x)^\alpha}; \quad \alpha, \lambda, x > 0.$$

Therefore the parameters  $\alpha$  and  $\lambda$  represent the shape and scale parameters respectively in all the three different cases. A comparison of the three different hazard functions are given in Table A below.

Table A  
Hazard Function

Parameters	Gamma	Weibull	EE
$\alpha = 1$	Constant	Constant	Constant
$\alpha > 1$	Increasing from 0 to $\lambda$	Increasing from 0 to $\infty$	Increasing from 0 to $\lambda$
$\alpha < 1$	Decreasing from $\infty$ to $\lambda$	Decreasing from $\infty$ to 0	Decreasing from $\infty$ to $\lambda$

Therefore the hazard function of the EE distribution behaves like the hazard function of the gamma distribution, which is quite different from the hazard function of the Weibull distribution. For the Weibull distribution if  $\alpha > 1$ , the hazard function increases from zero to  $\infty$  and if  $\alpha < 1$ , the hazard function decreases from  $\infty$  to zero. Many authors point out (see BAIN, 1976) that since the hazard function of a gamma distribution (for  $\alpha > 1$ ) increases from zero to a finite constant, the gamma may be more appropriate as a population model when the items in the population are in a regular maintenance program. The hazard rate may increase initially, but after some times the system reaches a stable condition because of maintenance. The same comments hold for the EE distribution also. Therefore, if it is known that the data are from a regular maintenance environment, it may make more sense to fit the gamma distribution or the EE distribution than the Weibull distribution.

Now let us consider the different moments of the EE distribution. Suppose  $X$  denote the EE random variable with parameter  $\alpha$  and  $\lambda$ , then

$$E(X^k) = \alpha\lambda \int_0^{\infty} x^k (1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x} dx.$$

Now since  $0 < e^{-\lambda x} < 1$ , for  $\lambda > 0$  and  $x > 0$ , therefore by using the series representation (finite or infinite) of  $(1 - e^{-\lambda x})^{\alpha-1}$

$$(1 - e^{-\lambda x})^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i c(\alpha - 1, i) e^{-i\lambda x},$$

where  $c(\alpha - 1, i) = \frac{(\alpha - 1) \dots (\alpha - i)}{i!}$ , we obtain

$$E(X^k) = \frac{\alpha\Gamma(k+1)}{\lambda^k} \sum_{i=0}^{\infty} (-1)^i c(\alpha - 1, i) \frac{1}{(i+1)^{k+1}}. \quad (2.1)$$

Since (2.1) is a convergent series for any  $k \geq 0$ , therefore all the moments exist and for integer values of  $\alpha$ , (2.1) can be represented as a finite series representation. Therefore putting  $k = 1$ , we obtain the mean as

$$E(X) = \frac{\alpha}{\lambda} \sum_{i=0}^{\infty} (-1)^i c(\alpha - 1, i) \frac{1}{(i+1)^2},$$

and putting  $k = 2$ , we obtain the second moment as

$$E(X^2) = \frac{2\alpha}{\lambda^2} \sum_{i=0}^{\infty} (-1)^i c(\alpha - 1, i) \frac{1}{(i+1)^3}.$$

It is also possible to express the moment generating function in terms of the gamma function, which in turn can be used to obtain different moments. The moment generating function,  $M(t)$ , of  $X$  for  $0 < t < \lambda$  can be written as

$$M(t) = E(e^{tX}) = \alpha\lambda \int_0^{\infty} (1 - e^{-\lambda x})^{\alpha-1} e^{(t-\lambda)x} dx. \quad (2.2)$$

Making the substitution  $y = e^{-\lambda x}$ , (2.2) reduces to

$$M(t) = \alpha \int_0^1 (1-y)^{\alpha-1} y^{-\frac{t}{\lambda}} dy = \frac{\Gamma(\alpha+1) \Gamma\left(1 - \frac{t}{\lambda}\right)}{\Gamma\left(\alpha - \frac{t}{\lambda} + 1\right)}. \quad (2.3)$$

Differentiating  $\ln(M(t))$  and evaluating at  $t = 0$ , we get the mean and the variance of  $X$  as

$$E(X) = \frac{1}{\lambda} (\psi(\alpha+1) - \psi(1)) \quad \text{and} \quad \text{var}(X) = \frac{1}{\lambda^2} (\psi'(1) - \psi'(\alpha+1)), \quad (2.4)$$

where  $\psi(\cdot)$  is the digamma function and  $\psi'(\cdot)$  is its derivative. The higher central moments can be obtained in terms of the polygamma functions.

### 3. Some Ordering Properties

Ordering of distributions, particularly among the lifetime distributions, plays an important role in statistical literature. JOHNSON, KOTZ, and BALAKRISHNAN (1995, Chap 33) have a major section on the ordering of various positive valued distributions. PECARIC, PROSCHAN, and TONG (1992) also provide a detailed treatment of stochastic ordering, highlighting their growing importance and illustrating their usefulness in numerous practical applications. It might be useful to obtain the bounds in survival functions, hazard functions or on the moments depending on the circumstances. In this section we discuss some of the ordering properties within each family of distribution and between the three families also. In this section we take the scale parameter to be one throughout.

It is well known that gamma family has increasing likelihood ratio ordering in the shape parameter for fixed scale parameter so it has the ordering in hazard rate as well as in distribution functions. Since the gamma family has the likelihood ratio ordering, it has the monotone likelihood ratio property. This implies there exists a uniformly most powerful test (UMP) for any one-sided hypothesis or uniformly most powerful unbiased test (UMPU) for any two-sided hypothesis on the shape parameter if the scale parameter is known. Unfortunately the Weibull family does not have the ordering even in distribution, so naturally it does not have the ordering in hazard rate or in likelihood ratio. It can be easily checked that for EE family it has the ordering in likelihood ratio, so it has the ordering in hazard rate as well as in distribution function similarly as the gamma family. Therefore, for EE family also if the scale parameter is known, there exists a UMP test for any one-sided hypothesis or UMPU test for any two-sided hypothesis on the shape parameter.

Now consider some ordering properties between the families, when the shape parameter is kept at a constant value. Since  $f_G(x)/f_E(x)$  is an increasing function for  $\alpha \geq 1$  and a decreasing function for  $\alpha \leq 1$ , therefore we can say that gamma is larger (smaller) than EE in terms of likelihood ratio ordering if  $\alpha \geq 1$  ( $\leq 1$ ) and they are equal when  $\alpha = 1$ . It is interesting to observe that  $\alpha = 1$  plays an important role. When  $\alpha = 1$  all the three distributions become equal to the exponential distribution. It can be easily seen that there is no likelihood ratio ordering between Weibull and gamma or between Weibull and EE. But the following can be easily observed for all values of  $x$ ;

$$\begin{aligned} h_W(x) - h_E(x) &\geq 0 & \text{if } \alpha > 1, \\ h_W(x) - h_E(x) &\leq 0 & \text{if } \alpha < 1. \end{aligned}$$

Since there is a hazard rate ordering between the gamma and EE, we immediately obtain the following

$$\begin{aligned} h_W(x) &\geq h_E(x) \geq h_G(x) & \text{if } \alpha > 1, \\ h_W(x) &\leq h_E(x) \leq h_G(x) & \text{if } \alpha < 1. \end{aligned}$$

Therefore we have an ordering in distribution also between the three as follows;

$$\begin{aligned} F_W(x) &\geq F_E(x) \geq F_G(x) & \text{for } \alpha > 1, \\ F_W(x) &\leq F_E(x) \leq F_G(x) & \text{for } \alpha < 1. \end{aligned}$$

#### 4. Maximum Likelihood Estimators and the Fisher Information Matrix

In this section we discuss the maximum likelihood estimators (MLE's) of a two-parameter EE distribution and their asymptotic properties. Let  $x_1, \dots, x_n$  be a random sample from EE, then the log likelihood function can be written as:

$$L(\alpha, \lambda) = n \ln \alpha + n \ln \lambda + (\alpha - 1) \sum_{i=1}^n \ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^n x_i. \quad (4.1)$$

Therefore, to obtain the MLE's of  $\alpha$  and  $\lambda$ , either we can maximize (4.1) directly with respect to  $\alpha$  and  $\lambda$  or we can solve the non-linear normal equations which are as follows:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln(1 - e^{-\lambda x_i}) = 0, \quad (4.2)$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} - \sum_{i=1}^n x_i = 0. \quad (4.3)$$

From (4.2), we obtain the MLE of  $\alpha$  as a function of  $\lambda$ , say  $\hat{\alpha}(\lambda)$ , as

$$\hat{\alpha}(\lambda) = - \frac{n}{\sum_{i=1}^n \ln(1 - e^{-\lambda x_i})}. \quad (4.4)$$

Therefore, if the scale parameter is known, the MLE of the shape parameter,  $\hat{\alpha}$ , can be obtained directly from (4.4). If both the parameters are unknown, first the estimate of the scale parameter can be obtained by maximizing directly

$$\begin{aligned} g(\lambda) = L(\hat{\alpha}(\lambda), \lambda) &= C - n \ln \left( - \sum_{i=1}^n \ln(1 - e^{-\lambda x_i}) \right) \\ &\quad + n \ln(\lambda) - \sum_{i=1}^n \ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^n x_i. \end{aligned} \quad (4.5)$$

with respect to  $\lambda$ . Here  $C$  is a constant independent of  $\lambda$ . Once  $\hat{\lambda}$  is obtained,  $\hat{\alpha}$  can be obtained from (4.4) as  $\hat{\alpha}(\hat{\lambda})$ . Therefore it reduces the two dimensional problem to a one dimensional problem which is relatively easier to solve.

In this situation we use the asymptotic normality results to obtain the asymptotic confidence interval. We can state the results as follows:

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow N_2(0, I^{-1}(\theta)) \tag{4.6}$$

where  $I(\theta)$  is the Fisher Information matrix, i.e.

$$I(\theta) = -\frac{1}{n} \begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 L}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) \end{bmatrix}$$

and  $\hat{\theta} = (\hat{\alpha}, \hat{\lambda})$ ,  $\theta = (\alpha, \lambda)$ . Since for  $\alpha > 0$ , the EE family satisfies all the regularity conditions (see BAIN, 1976), therefore (4.6) holds. Now, we provide the elements of the negative Fisher Information matrix, which might be useful in practice. For  $\alpha > 2$

$$\begin{aligned} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) &= -\frac{n}{\alpha^2}, \\ E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) &= \frac{n}{\lambda} \left[ \frac{\alpha}{(\alpha - 1)} (\psi(\alpha) - \psi(1)) - \psi(\alpha + 1) - \psi(1) \right] \\ E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) &= -\frac{n}{\lambda^2} \left[ 1 + \frac{\alpha(\alpha - 1)}{(\alpha - 2)} (\psi'(1) - \psi'(\alpha - 1) + (\psi(\alpha - 1) - \psi(1))^2) \right] \\ &\quad - \frac{n\alpha}{\lambda^2} [(\psi'(1) - \psi(\alpha) + (\psi(\alpha) - \psi(1))^2)] \end{aligned}$$

and for  $0 < \alpha \leq 2$ ,

$$\begin{aligned} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) &= -\frac{n}{\alpha^2}, \quad E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) = \frac{n\alpha}{\lambda} \int_0^\infty x e^{-2x} (1 - e^{-x})^{\alpha-2} dx < \infty, \\ E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) &= -\frac{n}{\lambda^2} - \frac{n\alpha(\alpha - 1)}{\lambda^2} \int_0^\infty x e^{-2x} (1 - e^{-x})^{\alpha-2} dx < \infty. \end{aligned}$$

Since  $\theta$  is unknown in (4.6),  $I^{-1}(\theta)$  is estimated by  $I^{-1}(\hat{\theta})$  and this can be used to obtain the asymptotic confidence intervals of  $\alpha$  and  $\lambda$ . In presence of Type I or Type II censoring the results can be suitably modified. It may be mentioned that for Type I censored data the Fisher Information matrix also can be obtained along the same line as the complete sample case but unfortunately in case of Type II censoring it is not possible to obtain the Fisher Information matrix in a closed form.

## 5. Data Analysis

In this section we use two uncensored data sets and fit the three models namely; gamma, Weibull and Exponentiated Exponential.

**Data Set 1:** The first data set is as follows; (LAWLESS, 1986 page 228). The data given here arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40.

We have fitted Gamma, Weibull and EE to this data set. We present the estimates, the Log-likelihood (LL), the observed and the expected values and the  $\chi^2$  statistics. The results are as follows.

For Gamma distribution

$$\hat{\lambda} = 0.0556, \quad \hat{\alpha} = 4.0196, \quad LL = -113.0274, \quad \chi^2 = 1.040.$$

For Weibull distribution

$$\hat{\lambda} = 0.0122, \quad \hat{\alpha} = 2.1050, \quad LL = -113.6887, \quad \chi^2 = 1.791.$$

For EE distribution

$$\hat{\lambda} = 0.0314, \quad \hat{\alpha} = 5.2589, \quad LL = -112.9763, \quad \chi^2 = 0.783.$$

The observed and the expected frequencies are as given below;

Table 1

Intervals	Observed	EE	Weibull	Gamma
0– 35	3	2.94	3.01	3.54
35– 55	7	5.70	5.31	4.54
55– 80	5	6.55	6.52	6.05
80–100	3	3.39	3.62	3.86
100–	5	4.43	4.54	5.02

**Data set 2:** (Linhart and Zucchini (1986, page 69). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

In this case also we have fitted all the three distributions. The results are as follows:

For Gamma distribution

$$\hat{\lambda} = 0.0136, \quad \hat{\alpha} = 0.8134, \quad LL = -152.2312, \quad \chi^2 = 3.302.$$

For Weibull distribution

$$\hat{\lambda} = 0.0183, \quad \hat{\alpha} = 0.8554, \quad LL = -152.007, \quad \chi^2 = 3.056.$$

For EE distribution,

$$\hat{\lambda} = 0.0145, \quad \hat{\alpha} = 0.8130, \quad LL = -152.264, \quad \chi^2 = 3.383.$$

The observed and the expected frequencies are as given below;

Table 2

Intervals	Observed	EE	Weibull	Gamma
0– 15	11	7.97	8.45	8.07
15– 30	5	4.91	5.06	4.94
30– 60	3	6.44	6.33	6.43
60–100	6	4.84	4.55	4.80
100–	5	5.84	5.62	5.77

It is observed that EE fits the best in the first data set whereas Weibull fits the best in the second data in terms of likelihood and in terms of Chi-square. Therefore, it is not guaranteed the EE will behave always better than Weibull or gamma but at least it can be said in certain circumstances EE might work better than Weibull or gamma.

### 6. Numerical Experiments and Discussions

In this section we perform some numerical experiments to see how the MLE’s and their asymptotic results work for finite sample. All the numerical works are performed on PC-486 using the random deviate generator by PRESS et al. (1994). We consider the following different model parameters:

- Model 1:  $\alpha = 2.0, \lambda = 0.1,$       Model 2:  $\alpha = 1.0, \lambda = 0.1,$
- Model 3:  $\alpha = 0.5, \lambda = 0.1,$       Model 4:  $\alpha = 2.0, \lambda = 0.2,$
- Model 5:  $\alpha = 1.0, \lambda = 0.2,$       Model 6:  $\alpha = 0.5, \lambda = 0.2,$

We consider the following sample size (SS),  $n = 10, 15$  (small),  $40, 50$  (moderate), and  $100$  (large). For each model parameters and for each sample size, we compute the MLE’s of  $\alpha$  and  $\lambda$ , we also compute the asymptotic confidence interval in each replications. We repeat this process 1000 times and compute the average estimators (AE), the square root of the mean squared errors (SMSE) and the coverage probabilities (CP). The results are reported in Tables 3–4.

Some of the points are very clear from the numerical experiments. It is observed that for all the parametric values the MSE’s and the biases decrease as the sample size increaes. It verifies the consistency properties of the MLE’s as mentioned in (4.6). For fixed  $\lambda$  as  $\alpha$  increases the MSE’s and the biases of  $\hat{\alpha}$  increase

Table 3

SS	Par	$\alpha = 2.0, \lambda = 0.1$			$\alpha = 1.0, \lambda = 0.1$			$\alpha = 0.5, \lambda = 0.1$		
		AE	SMSE	CP	AP	SMSE	CP	AE	SMSE	CP
10	$\alpha$	2.720	1.703	0.97	1.261	0.667	0.97	0.599	0.277	0.97
	$\lambda$	0.112	0.033	0.98	0.113	0.037	0.98	0.113	0.037	0.98
15	$\alpha$	2.470	1.181	0.95	1.171	0.470	0.96	0.565	0.120	0.96
	$\lambda$	0.109	0.029	0.97	0.109	0.031	0.97	0.110	0.034	0.97
40	$\alpha$	2.160	0.579	0.96	1.061	0.238	0.96	0.524	0.103	0.96
	$\lambda$	0.104	0.019	0.95	0.104	0.021	0.96	0.105	0.025	0.96
50	$\alpha$	2.130	0.501	0.96	1.051	0.210	0.96	0.520	0.092	0.96
	$\lambda$	0.103	0.016	0.95	0.104	0.019	0.96	0.105	0.023	0.96
100	$\alpha$	2.053	0.303	0.95	1.024	0.130	0.96	0.509	0.058	0.96
	$\lambda$	0.101	0.011	0.95	0.102	0.013	0.95	0.102	0.016	0.95

Table 4

SS	Par	$\alpha = 2.0, \lambda = 0.2$			$\alpha = 1.0, \lambda = 0.2$			$\alpha = 0.5, \lambda = 0.2$		
		AE	SMSE	CP	AP	SMSE	CP	AE	SMSE	CP
10	$\alpha$	2.896	2.004	0.97	1.315	0.750	0.97	0.616	0.300	0.97
	$\lambda$	0.233	0.080	0.98	0.236	0.087	0.98	0.241	0.096	0.99
15	$\alpha$	2.558	1.375	0.95	1.201	0.520	0.96	0.575	0.211	0.96
	$\lambda$	0.221	0.066	0.97	0.225	0.074	0.97	0.230	0.084	0.97
40	$\alpha$	2.167	0.604	0.95	1.065	0.248	0.96	0.526	0.107	0.96
	$\lambda$	0.208	0.039	0.95	0.210	0.046	0.95	0.213	0.056	0.96
50	$\alpha$	2.131	0.504	0.96	1.052	0.213	0.96	0.521	0.093	0.96
	$\lambda$	0.206	0.034	0.95	0.208	0.040	0.95	0.212	0.050	0.96
100	$\alpha$	2.053	0.303	0.95	1.021	0.130	0.96	0.509	0.058	0.96
	$\lambda$	0.203	0.022	0.95	0.203	0.026	0.95	0.205	0.033	0.96

where as the corresponding MSE's and the biases of  $\hat{\lambda}$  decrease for all the sample sizes. Therefore, estimation of  $\alpha$  becomes better as  $\alpha$  decreases where as the estimation of  $\lambda$  becomes more accurate as  $\alpha$  increases. On the other hand for fixed  $\alpha$  as  $\lambda$  increases the MSE's and the biases of both  $\hat{\alpha}$  and  $\hat{\lambda}$  increase. Note that for large sample sizes  $\frac{\lambda}{\alpha}$  remains constant for all  $\alpha$ . It is not very surprising because  $\lambda$  is the scale parameter and it also follows from (4.6). Interestingly for moderate or large sample sizes it is observed that for fixed  $\alpha$  the MLE's of  $\alpha$  and the corresponding MSE's remain constant for different  $\lambda$ . It is clear that the MLE's of  $\alpha$  and  $\lambda$  are positively biased although biases go to zero as sample size increases. It is also interesting to observe that the asymptotic confidence interval maintains the nominal coverage probabilities even for small sample sizes. Therefore, the MLE's

and their the asymptotic results can be used for estimation and for constructing confidence intervals even for small sample sizes.

## 7. Conclusions

In this article we consider EE family of distributions. It is observed that the two-parameter EE family are quite similar in nature to the other two-parameter family like Weibull family or gamma family. It is observed that most of the properties of a EE distribution are quite similar in nature to those of a gamma distribution but computationally it is quite similar to that of a Weibull distribution. Therefore, it can be used as an alternative to a Weibull distribution or a gamma distribution and it is expected that in some situations it might work better (in terms of fitting) than a Weibull distribution or a gamma distribution although it can not be guaranteed. We present two real life data sets, where in one data set it is observed that EE has a better fit compare to Weibull or gamma but in the other the Weibull has a better fit than EE or gamma. Moreover it is well known that gamma has certain advantages compare to Weibull in terms of the faster convergence of the MLE's. It is expected that EE also should enjoy those properties. Extensive simulations are required to compare the rate of convergences of the MLE's of the different distributions. More work is needed in that direction. Primary numerical experiments confirm that for EE family asymptotic results can be used even for small sample sizes for different  $\alpha$ 's and  $\lambda$ 's.

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DEBASIS KUNDU  
Department of Mathematics  
Indian Institute of Technology Kanpur  
Kanpur, Pin 208009  
India  
E-mail: kundu@iitk.ac.in

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