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Smooth estimators for estimating order restricted scale parameters of two gamma distributions

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Abstract. We consider the problem of component-wise estimation of ordered scale parameters of two gamma populations, when it is known apriori which population corresponds to each ordered parameter. Under the scale equivariant squared error loss function, smooth estimators that improve upon the best scale equivariant estimators are derived. These smooth estimators are shown to be generalized Bayes with respect to a non-informative prior. Finally, using Monte Carlo simulations, these improved smooth estimators are compared with the best scale equivariant estimators, their non-smooth improvements obtained in Vijayasree, Misra & Singh (1995), and the restricted maximum likelihood estimators.

Key words: best scale equivariant estimator, mixed estimators, non-informative prior, restricted maximum likelihood estimator, scale equivariant squared error loss function, smooth estimators, unrestricted maximum likelihood estimator

1 Introduction

The problem of estimating ordered parameters, when it is known apriori that they are subject to certain order restrictions, is of considerable interest. Suppose it is desired to estimate the average yields, say, θ_1 and θ_2, under treatments τ_1 and τ_2 respectively, where the treatment τ_1 is using certain fertilizer for the crop, while treatment τ_2 is not using any fertilizer. In this situation, it is reasonable to assume that $\theta_1 \geq \theta_2$. Similarly, in estimating average incomes, say θ_i, $i = 1, \ldots, k$, of k classes of employees in an establishment, it is quite natural to assume an ordering among the $\theta_i s$, according to the grade of employees. In the development of a system, engineering changes are made in stages to correct the design deficiencies and thereby increasing reliability. Thus, if we have k stages in which changes are made, then at each stage we expect the reliability to increase. If θ_i is a measure of the reliability at the i-th stage, then we may assume that $\theta_1 \leq \cdots \leq \theta_k$.

Most of the work on estimating ordered parameters, when it is known apriori that they are subject to order restrictions, is concerned with obtaining their maximum likelihood estimators. For some of the early contributions in this area, one may refer to Eeden (1957 a–d). One may also refer to Barlow et al. (1972) and Robertson et al. (1988) for a detailed discussion of results on maximum likelihood estimation of order restricted parameters.

For simultaneous estimation of ordered probabilities of successes (say, θ_1 and θ_2, with $\underline{\theta} = (\theta_1, \theta_2) \in \Theta_1 = \{\underline{\theta} : 0 < \theta_1 \le \theta_2 \le 1\}$) of two independent binomial distributions, Katz (1963) established that any estimator $\underline{\delta} = (\delta_1, \delta_2)$, which satisfies $P_{\underline{\theta}}(\delta_1 > \delta_2) > 0$, for all $\underline{\theta} \in \Theta_1$, is dominated by the corresponding "mixed estimator" $\underline{\delta_\alpha} = (\delta_{1,\alpha}, \delta_{2,\alpha})$, based on $\underline{\delta} = (\delta_1, \delta_2)$; here $\delta_{1,\alpha} = \min(\delta_1, \alpha\delta_1 + (1-\alpha)\delta_2)$ and $\delta_{2,\alpha} = \max(\delta_2, (1-\alpha)\delta_2 + \alpha\delta_1)$. Katz (1963) also obtained estimators which are minimax among the estimators in the class $\{\underline{\delta_\alpha} : 0 \le \alpha \le 1\}$.

For some later contributions, the reader may refer to Blumenthal & Cohen (1968), Cohen & Sackrowitz (1970), Sackrowitz & Strawderman (1974), Lee (1981), Sackrowitz (1970, 1982), Kumar & Sharma (1988), Kelly (1989), Kushary & Cohen (1989, 1991), Elfessi & Pal (1992), Gupta & Singh (1992), Ghosh & Sarkar (1994), Hwang & Peddada (1994), Misra & Dhariyal (1995) and Misra & van der Meulen (1997).

Because of the applicability of exponential probability models in many real life situations, in the recent years, considerable amount of work has been done for estimating order restricted parameters of exponential distributions.

Kushary & Cohen (1989) showed that the unrestricted best location equivariant estimators of location parameters μ_1 and μ_2 of two independent exponential distributions, having a common known scale parameter, are inadmissible when one assumes $\mu_1 \le \mu_2$ and the sample sizes to be unequal. Jin & Pal (1991) have studied the problem of simultaneous estimation of location parameters of two independent exponential populations, when location and/or scale parameters are ordered. For component-wise estimation of ordered means of two exponential distributions, having known location parameters (hence, taken to be zero, without loss of generality), Kaur & Singh (1991) established that the "unrestricted maximum likelihood estimators" (i.e., the sample means) of the two exponential means are inadmissible and are dominated by their respective "restricted maximum likelihood estimators", derived under the order restriction. Following Katz (1963), for the problem of component-wise and simultaneous estimation of ordered means of two exponential distributions, having known location parameters, Vijayasree & Singh (1991, 1993) considered mixed estimators, based on the sample means, and obtained classes of estimators that are minimal complete for the class(es) of the mixed estimators. Pal & Kushary (1992) dealt with component-wise estimation of location parameters of two independent exponential distributions, when these location parameters are known to be ordered. These authors established that the usual estimators in the unrestricted case are inadmissible, and obtained improved estimators under the squared error loss function. Singh, Gupta & Misra (1993) dealt with estimation of location and scale parameters of an exponential distribution, when the location parameter is known to be bounded above by a known constant. They proposed estimators that are better than usual estimators in the unrestricted case, under the squared error loss function. They also compared these estimators with respect to the Pitman measure of closeness and obtained some interesting and paradoxical results.

The theory developed is then applied to the problem of estimating location and scale parameters of two exponential distributions, when the location parameters are known to satisfy an order restriction. For the problem of component-wise estimation of ordered location parameters of two exponential distributions, having known scale parameters, Misra & Singh (1994) considered mixed estimators, based on best location equivariant estimators in the unrestricted case, and obtained minimal complete classes for mixed estimators. Vijayasree, Misra & Singh (1995) considered component-wise estimation of ordered scale parameters of k (≥ 2) exponential distributions and, using the orbit by orbit improvement technique of Brewster & Zidek (1974), obtained non-smooth estimators that improve upon the "best scale equivariant estimators", under the squared error loss function.

In this paper, we consider the component-wise estimation of ordered scale parameters of two gamma distributions. We use the second technique of Brewster & Zidek (1974) to obtain "smooth estimators" that improve upon best scale equivariant estimators. These estimators are derived in Section 2 of this paper. In Section 3, we derive generalized Bayes estimators of ordered scale parameters with respect to a "non-informative prior" and the "scale equivariant squared error loss function". Interestingly, generalized Bayes estimators turn out to be the improved smooth estimators derived in Section 2. Finally, in Section 4, we compare improved estimators, obtained in Section 2, with best scale equivariant estimators, their non-smooth improvements obtained in Vijayasree, Misra & Singh (1995), and restricted maximum likelihood estimators, using Monte Carlo simulations.

2 Smooth estimators

Let X_i, $i = 1, \dots, n_1$, and Y_j, $j = 1, \dots, n_2$, be two mutually independent random samples from two gamma distributions, having scale parameters θ_1 and θ_2, and shape parameters λ_1 and λ_2, respectively. Here, we assume that λ_1 and λ_2 are known positive real constants. Suppose that an ordering between scale parameters, say $\theta_1 \leq \theta_2$, is known apriori, so that the restricted parametric space is:

$$\Theta_R = \{\underline{\theta} : \underline{\theta} = (\theta_1, \theta_2), 0 < \theta_1 \leq \theta_2 < \infty\}. \tag{2.1}$$

We desire to estimate θ_1 and θ_2 in the presence of this prior information, i.e. when it is known apriori that $\underline{\theta} \in \Theta_R$. Define, $X = \sum_{i=1}^{n_1} X_i$, $Y = \sum_{j=1}^{n_2} Y_j$, $m = \lambda_1 n_1$ and $n = \lambda_2 n_2$. It is well known that (X, Y) is a complete and sufficient (and hence minimal sufficient) statistic for $\underline{\theta}$, and that random variables X and Y follow independent gamma distributions with scale parameters θ_1 and θ_2, and shape parameters m and n, respectively. Since $\underline{T} = (X, Y)$ is a minimal sufficient statistic for $\underline{\theta}$, we may restrict ourselves to the class of estimators that depend upon the observations only through $\underline{T}$.

For an estimator δ_i of θ_i, let the risk function be given by

$$R_i(\underline{\theta}, \delta_i) = E_{\underline{\theta}}\left(\frac{\delta_i(\underline{T})}{\theta_i} - 1\right)^2, \quad i = 1, 2, \tag{2.2}$$

i.e. we consider the scale equivariant squared error loss function.

In the unrestricted case, i.e. in the absence of prior information of ordering between θ_1 and θ_2, the maximum likelihood estimator (mle) for $\underline{\theta}$

is $\underline{\delta_{UM}} = (\delta_{1,UM}, \delta_{2,UM})$, where $\delta_{1,UM} = \frac{X}{m}$ and $\delta_{2,UM} = \frac{Y}{n}$, whereas the restricted mle (i.e. the mle derived under the order restriction $\theta_1 \le \theta_2$) is given by $\underline{\delta_{RM}} = (\delta_{1,RM}, \delta_{2,RM})$, where $\delta_{1,RM} = \min\left(\frac{X}{m}, \frac{X+Y}{m+n}\right)$ and $\delta_{2,RM} = \max\left(\frac{Y}{n}, \frac{X+Y}{m+n}\right)$ (cf. Barlow et al. (1972)). Also, best scale equivariant estimators for θ_1 and θ_2, under the squared error loss function, are given by $\delta_{1,BE} = \frac{X}{m+1}$ and $\delta_{2,BE} = \frac{Y}{n+1}$, respectively.

For $\lambda_1 = \lambda_2 = 1, n_1 = n_2$ and under the risk function R_i, as defined in (2.2), Kaur & Singh (1991) established that, for estimating θ_i, the unrestricted mle $\delta_{i,UM}$ is dominated by the restricted mle $\delta_{i,RM}, i = 1, 2$, for all $\underline{\theta} \in \Theta_R$; here Θ_R is as defined in (2.1). For estimating θ_i, Vijayasree, Misra & Singh (1995) established that the best scale equivariant estimator $\delta_{i,BE}$ is inadmissible under the risk function $R_i, i = 1, 2$, when it is known apriori that $\underline{\theta} \in \Theta_R$. Using the orbit by orbit improvement technique of Brewster & Zidek (1974), they showed that estimators $\delta^*_{1,BE} = \min\left(\frac{X}{m+1}, \frac{X+Y}{m+n+1}\right)$ and $\delta^*_{2,BE} = \max\left(\frac{Y}{n+1}, \frac{X+Y}{m+n+1}\right)$ uniformly dominate best scale equivariant estimators $\delta_{1,BE}$ and $\delta_{2,BE}$, respectively.

Since the class of generalized Bayes estimators is essentially complete (cf. Caridi (1983), Berger & Srinivasan (1978), and Brown (1971)), which are generally smooth, it seems that restricted mles and improved estimators, obtained in Vijayasree, Misra & Singh (1995), being non-smooth, are inadmissible. This motivates us to derive smooth estimators that dominate best scale equivariant estimators $\delta_{1,BE}$ and $\delta_{2,BE}$. In order to obtain such estimators, we use the second technique of Brewster & Zidek (1974).

Define $G_{1,m} = \frac{X}{\theta_1}$ and $G_{2,n} = \frac{Y}{\theta_2}$, so that $G_{1,m}$ and $G_{2,n}$ are independently distributed standard (having scale parameters 1) gamma random variables, with shape parameters m and n, respectively. Again define, $B_{1,n,m} = \frac{G_{2,n}}{G_{1,m} + G_{2,n}}$ and $B_{2,n,m} = \frac{G_{2,n}}{G_{1,m}}$, so that $B_{1,n,m}$ and $B_{2,n,m}$ are distributed as beta random variables of the first and the second kind respectively, with parameter (n, m). For positive real numbers r and s, let $\beta(r, s) = \frac{\Gamma r \Gamma s}{\Gamma(r+s)}$ denote the beta function.

Now we state the following lemma, which will be useful in deriving main results of this paper. The proof of the lemma, being straightforward, is omitted.

Lemma 2.1: (a) *For any* $z > 0$,

$$E[G_{1,m} \mid B_{2,n,m} = z] = \frac{m+n}{1+z},$$

and

$$E[G^2_{1,m} \mid B_{2,n,m} = z] = \frac{(m+n)(m+n+1)}{(1+z)^2}.$$

(b) *For any real valued function* $h(.)$,

$$E[G_{1,m}^k h(G_{1,m})] = \frac{\Gamma(m+k)}{\Gamma m} E[h(G_{1,m+k})],$$

provided expectations exist. Here, the random variable $G_{1,m+k}$ *has standard gamma distribution with shape parameter* $m+k$.

(c) *For* $B_{2,n,m+2} = \dfrac{G_{2,n}}{G_{1,m+1} + G_{1,1}}$, $B_{1,m+1,1} = \dfrac{G_{1,m+1}}{G_{1,m+1} + G_{1,1}}$, *and* $B_{2,n,m+1} = \dfrac{G_{2,n}}{G_{1,m+1}}$, *where* $G_{1,1}, G_{1,m+1}$ *and* $G_{2,n}$ *are independent standard gamma random variables, with shape parameters* $1, m+1$ *and* n, *respectively, we have*

$$B_{2,n,m+2} = B_{2,n,m+1} B_{1,m+1,1},$$

and $B_{1,m+1,1}$ *and* $B_{2,n,m+2}$ *are independently distributed beta random variables of the first and the second kind, with parameters* $(m+1, 1)$ *and* $(n, m+2)$, *respectively.*

In order to improve upon the best scale equivariant estimator $\delta_{1,BE}$, we first consider estimators of the form

$$\phi_{1,1}(c, r, \underline{T}) = \begin{cases} \dfrac{X}{m+1}, & \text{if } S \geq r \\ cX, & \text{if } S < r, \end{cases}$$

where r and c are fixed positive real constants, and $S = \dfrac{Y}{X}$.

Note that $\phi_{1,1}\left(\dfrac{1}{m+1}, r, \underline{T}\right) = \delta_{1,BE}(\underline{T})$, $\forall r > 0$. Let $\sigma = \dfrac{\theta_1}{\theta_2}$, so that $0 < \sigma \leq 1$. Let $I(A)$ denote the indicator function of the set A. Then, for given r and $\underline{\theta} \in \Theta_R$, the risk function $R_1(\underline{\theta}, \phi_{1,1}(c, r, .))$, as defined in (2.2), is minimized at $c \equiv c_{1,r}(\sigma)$, where

$$c_{1,r}(\sigma) = \theta_1 \frac{E_{\underline{\theta}}[XI(S < r)]}{E_{\underline{\theta}}[X^2 I(S < r)]}. \tag{2.3}$$

In the following lemma, we derive properties of the function $c_{1,r}(\sigma)$, $0 < \sigma \leq 1$.

Lemma 2.2: (a) *For fixed* $r > 0$ *and* $\underline{\theta} \in \Theta_R$, $R_1(\underline{\theta}, \phi_{1,1}(c, r, .))$ *is minimized at* $c \equiv c_{1,r}(\sigma)$, *where* $c_{1,r}(\sigma)$ *is as defined in (2.3).*

(b) *For* $r > 0$, $\sup_{\sigma \in (0,1]} c_{1,r}(\sigma) = c_{1,r}(1) = c_1(r)$, *say, where* $c_1(r)$ *is given by*

$$c_1(r) = \frac{E[G_{1,m} I(B_{2,n,m} < r)]}{E[G_{1,m}^2 I(B_{2,n,m} < r)]}$$

$$= \frac{1}{m+1} \left[1 - \frac{\displaystyle\int_0^1 \left\{ \frac{x^{m+n}}{(1+rx)^{m+n+2}} \right\} dx}{\displaystyle\int_0^1 \left\{ \frac{x^{n-1}}{(1+rx)^{m+n+2}} \right\} dx} \right].$$

(c) $c_1(r)$ *is increasing in* $r \in (0, \infty)$, *and,* $\forall r > 0$,

$$c_1(r) < \frac{1}{m+1} = \lim_{r \to \infty} c_1(r).$$

Proof. We have already observed that the first assertion is true. For proving the second assertion, from (2.3), we have

$$c_{1,r}(\sigma) = \theta_1 \frac{E_{\underline{\theta}}[XI(S < r)]}{E_{\underline{\theta}}[X^2 I(S < r)]}$$

$$= \frac{E[G_{1,m} I(B_{2,n,m} < r\sigma)]}{E[G_{1,m}^2 I(B_{2,n,m} < r\sigma)]}.$$

For independent standard gamma random variables $G_{1,1}, G_{1,m}, G_{2,n}, G_{1,m+1}$ and $G_{1,m+2}$, having shape parameters $1, m, n, m+1$ and $m+2$, respectively, on using Lemma 2.1 (b), we get

$$c_{1,r}(\sigma) = \frac{1}{m+1} \frac{P(B_{2,n,m+1} < r\sigma)}{P(B_{2,n,m+2} < r\sigma)},$$

where $B_{2,n,m+1} = \dfrac{G_{2,n}}{G_{1,m+1}}$ and $B_{2,n,m+2} = \dfrac{G_{2,n}}{G_{1,m+2}} = \dfrac{G_{2,n}}{G_{1,m+1} + G_{1,1}}$.

Now, on using Lemma 2.1 (c), we can write $B_{2,n,m+1} = \dfrac{B_{2,n,m+2}}{B_{1,m+1,1}}$, where $B_{2,n,m+2}$ and $B_{1,m+1,1} = \dfrac{G_{1,m+1}}{G_{1,m+1} + G_{1,1}}$ are independently distributed. Therefore,

$$c_{1,r}(\sigma) = \frac{1}{m+1} \frac{P\left(\dfrac{B_{2,n,m+2}}{B_{1,m+1,1}} < r\sigma\right)}{P(B_{2,n,m+2} < r\sigma)}.$$

Define, $\dfrac{B_{2,n,m+2}}{r\sigma} = W_6$. Then,

$$c_{1,r}(\sigma) = \frac{1}{m+1} \frac{P_{\underline{\theta}}\left(\dfrac{W_6}{B_{1,m+1,1}} < 1\right)}{P_{\underline{\theta}}(W_6 < 1)}.$$

Since, the random variable $B_{1,m+1,1}$ has cumulative distribution function given by $H(x) = 0$, if $x < 0$, $= x^{m+1}$, if $0 \le x < 1$ and $= 1$, if $x \ge 1$, and the random variable W_6 is non-negative, on using the independence of W_6 and $B_{1,m+1,1}$, we get

$$c_{1,r}(\sigma) = \frac{1}{m+1} \frac{E_{\underline{\theta}}[(1 - W_6^{m+1}) I(W_6 < 1)]}{P_{\underline{\theta}}(W_6 < 1)}$$

$$= \frac{1 - E_{\underline{\theta}}(W_7^{m+1})}{m+1},$$

where W_7 is a random variable, whose probability density function (pdf) is given by

$$g_7(x,\sigma) = \frac{\dfrac{x^{n-1}}{(1+r\sigma x)^{m+n+2}}}{\displaystyle\int_0^1 \frac{z^{n-1}}{(1+r\sigma z)^{m+n+2}}\,dz}, \quad 0 \le x \le 1, \sigma \in (0,1].$$

It is easy to verify that the family of pdfs $\{g_7(x,\sigma) : 0 \le x \le 1, 0 < \sigma \le 1\}$ has the monotone likelihood ratio (mlr) property in $\left(\frac{1}{\sigma}, x\right)$. Thus, on appealing to Lehmann (1986, Problem 14, Page 115), it follows that $E_{\underline{\theta}}(W_7^{m+1})$ is a decreasing function of σ. Therefore,

$$\begin{aligned}\sup_{\sigma\in(0,1]} c_{1,r}(\sigma) &= c_{1,r}(1)\\ &= \frac{1}{m+1}\left[1 - \frac{\displaystyle\int_0^1 \frac{x^{m+n}}{(1+rx)^{m+n+2}}\,dx}{\displaystyle\int_0^1 \frac{x^{n-1}}{(1+rx)^{m+n+2}}\,dx}\right]\\ &= c_1(r).\end{aligned}$$

The proof of the third assertion follows on repeating the above arguments.

Remark: *On using Lemma 2.1 (b) and (2.3), $c_1(y)$ may be written, alternatively, as:*

$$\begin{aligned}c_1(y) &= c_{1,y}(1)\\ &= \frac{E[G_{1,m} I(B_{2,n,m} < y)]}{E[G_{1,m}^2 I(B_{2,n,m} < y)]}\\ &= \frac{1}{m+1}\,\frac{P(B_{2,n,m+1} < y)}{P(B_{2,n,m+2} < y)},\end{aligned}$$

where, for independent standard gamma random variables $G_{2,n}, G_{1,m+1}$ and $G_{1,m+2}$, having shape parameters $n, m+1$ and $m+2$, respectively, $B_{2,n,m+1} = \frac{G_{2,n}}{G_{1,m+1}}$ and $B_{2,n,m+2} = \frac{G_{2,n}}{G_{1,m+2}}$. Now, on using the relationship between $B_{2,n,m+1}$ and $B_{2,n,m+2}$ with beta random variables of first kind, we have:

$$c_1(y) = \frac{1}{m+1}\,\frac{I_{n,m+1}\left(\dfrac{y}{1+y}\right)}{I_{n,m+2}\left(\dfrac{y}{1+y}\right)}, \tag{2.4}$$

where, for $\alpha > 0$, *and* $\gamma > 0$, $I_{\alpha,\gamma}(x) = \frac{1}{\beta(\alpha,\gamma)} \int_0^x t^{\alpha-1}(1-t)^{\gamma-1}\, dt$ *denotes the incomplete beta function.*

Since, for each fixed $\underline{\theta} \in \Theta_R$ and $r > 0, R_1(\underline{\theta}, \phi_{1,1}(c, r, .))$ is minimized at $c = c_{1,r}(\sigma)$, and $\sup_{\sigma \in (0,1]} c_{1,r}(\sigma) = c_1(r)$, on using the convexity of the risk function $R_1(\underline{\theta}, \phi_{1,1}(c, r, .))$ (as a function of c, for fixed $\underline{\theta} \in \Theta_R$ and $r > 0$), we conclude that, for each fixed $\underline{\theta} \in \Theta_R$ and $r > 0, R_1(\underline{\theta}, \phi_{1,1}(c, r, .))$ is a strictly increasing function of c, for $c \in [c_1(r), \infty)$. Since, $c_1(r) < \frac{1}{m+1}$, $\forall r > 0$, we have the following theorem.

Theorem 2.1: *For estimating* θ_1, *under the scale equivariant squared error loss function, the estimator* $\phi_{1,1}(c_1(r), r, \underline{T})$ *dominates the best scale equivariant estimator* $\delta_{1,BE}(\underline{T}) = \frac{X}{m+1}$.

Further select $0 < r' < r$. Since $c_1(r)$ is an increasing function of r, we have $c_1(r') < c_1(r)$. Now, on considering the class of estimators of the form

$$\phi_{1,2}(c, r', r, \underline{T}) = \begin{cases} cX, & \text{if } S < r' \\ c_1(r)X, & \text{if } r' \le S < r \\ \dfrac{X}{m+1}, & \text{if } S \ge r, \end{cases}$$

and on repeating above arguments, it can be seen that the estimator $\phi_{1,1}(c_1(r), r, \underline{T})$ is further dominated by the following estimator

$$\phi_{1,2}(c_1(r'), r', r, \underline{T}) = \begin{cases} c_1(r')X, & \text{if } S < r' \\ c_1(r)X, & \text{if } r' \le S < r \\ \dfrac{X}{m+1}, & \text{if } S \ge r. \end{cases}$$

Now, on using the idea of Brewster & Zidek (1974), we select a finite partition of $[0, \infty)$, represented by $0 = r_{i,0} < r_{i,1} < \cdots < r_{i,n_i} = \infty$, for each $i = 1, 2, \ldots$, and a corresponding estimator defined by

$$\phi_{1,i} = \begin{cases} c_1(r_{i,1})X, & \text{if } S < r_{i,1} \\ c_1(r_{i,2})X, & \text{if } r_{i,1} \le S < r_{i,2} \\ \vdots & \\ c_1(r_{i,n_i-1})X, & \text{if } r_{i,n_i-2} \le S < r_{i,n_i-1} \\ \dfrac{X}{m+1}, & \text{if } S \ge r_{i,n_i-1}. \end{cases}$$

Then, providing $\max_j |r_{i,j} - r_{i,j-1}| \to 0$ and $r_{i,n_i-1} \to \infty$, as $i \to \infty$, the sequence of estimators $\phi_{1,i}$ will converge point wise to $\delta^{**}_{1,BE}(\underline{T})$, where

$$\delta^{**}_{1,BE}(\underline{T}) = c_1(S)X. \tag{2.5}$$

Now, we have the following theorem.

Theorem 2.2: *For estimating θ_1, under the squared error loss function, the estimator $\delta^{**}_{1,BE}(\underline{T})$, as defined by (2.4) and (2.5), dominates the best scale equivariant estimator $\delta_{1,BE}(\underline{T}) = \dfrac{X}{m+1}$.*

Proof. Since $\phi_{1,i}$ has uniformly smaller risk than $\delta_{1,BE}$, $\forall i$, the result follows by an application of the Fatou's lemma.

To find an improvement over $\delta_{2,BE}$, the best scale equivariant estimator of θ_2, we first consider estimators of the form

$$\phi_{2,1}(c, r, \underline{T}) = \begin{cases} \dfrac{Y}{n+1}, & \text{if } S \geq r \\ cY, & \text{if } S < r, \end{cases}$$

where r and c are fixed positive real constants, and $S = \dfrac{Y}{X}$.

Note that $\phi_{2,1}\left(\dfrac{1}{n+1}, r, \underline{T}\right) = \delta_{2,BE}(\underline{T})$, $\forall r > 0$. Then, for given r and $\underline{\theta} \in \Theta_R$, the risk function $R_2(\underline{\theta}, \phi_{2,1}(c, r, .))$, as defined in (2.2), is minimized at $c \equiv c_{2,r}(\sigma)$, where

$$c_{2,r}(\sigma) = \theta_2 \frac{E_{\underline{\theta}}[YI(S < r)]}{E_{\underline{\theta}}[Y^2 I(S < r)]}. \tag{2.6}$$

We have the following lemma concerning the properties of the function $c_{2,r}(\sigma)$, $0 < \sigma \leq 1$.

Lemma 2.3: (a) *For fixed $r > 0$ and $\underline{\theta} \in \Theta_R$, $R_2(\underline{\theta}, \phi_{2,1}(c, r, .))$ is minimized at $c \equiv c_{2,r}(\sigma)$, where $c_{2,r}(\sigma)$ is as defined in (2.6).*

(b) *For $r > 0$, $\inf_{\sigma \in (0,1]} c_{2,r}(\sigma) = c_{2,r}(1) = c_2(r)$, say, where*

$$c_2(y) = \frac{1}{n+1} \frac{I_{n+1,m}\left(\dfrac{y}{1+y}\right)}{I_{n+2,m}\left(\dfrac{y}{1+y}\right)}. \tag{2.7}$$

(c) *$c_2(r)$ is decreasing in $r \in (0, \infty)$, and, $\forall r > 0$,*

$$c_2(r) > \frac{1}{n+1} = \lim_{r \to \infty} c_2(r).$$

Proof. We have already observed that the first assertion is true. For proving the second and the third assertions, from (2.6), we have

$$c_{2,r}(\sigma) = \theta_2 \frac{E_{\underline{\theta}}[YI(S<r)]}{E_{\underline{\theta}}[Y^2 I(S<r)]}$$

$$= \frac{E[G_{2,n} I(B_{2,n,m} < r\sigma)]}{E[G_{2,n}^2 I(B_{2,n,m} < r\sigma)]}.$$

Now, on using Lemma 2.1 (a), we get on simplification

$$c_{2,r}(\sigma) = \frac{1}{m+n+1} \frac{E\left[B_{1,n,m} I\left(B_{1,n,m} < \frac{r\sigma}{1+r\sigma}\right)\right]}{E\left[B_{1,n,m}^2 I\left(B_{1,n,m} < \frac{r\sigma}{1+r\sigma}\right)\right]}$$

$$= \frac{1}{m+n+1} \frac{\int_0^{r\sigma/(1+r\sigma)} x^n (1-x)^{m-1}\, dx}{\int_0^{r\sigma/(1+r\sigma)} x^{n+1} (1-x)^{m-1}\, dx}$$

$$= \frac{1}{m+n+1} E_{\underline{\theta}}\left[\frac{1}{W_8}\right],$$

where W_8 is a random variable, whose pdf is given by

$$g_8(x,\sigma) = \frac{x^{n+1}(1-x)^{m-1}}{\int_0^{r\sigma/(1+r\sigma)} z^{n+1}(1-z)^{m-1}\, dz}, \quad 0 \le x \le \frac{r\sigma}{1+r\sigma}, \quad \sigma \in (0,1].$$

It is easy to verify that the family of pdfs $\left\{g_8(x,\sigma) : 0 \le x \le \frac{r\sigma}{1+r\sigma}, 0 < \sigma \le 1\right\}$ has the mlr property in (σ, x). Thus, on appealing to Lehmann (1986, Problem 14, Page 115), it follows that $c_{2,r}(\sigma)$ is a decreasing function of σ. Therefore,

$$\inf_{\sigma \in (0,1]} c_{2,r}(\sigma) = c_{2,r}(1)$$

$$= \frac{1}{m+n+1} \frac{\int_0^{r/(1+r)} x^n (1-x)^{m-1}\, dx}{\int_0^{r/(1+r)} x^{n+1} (1-x)^{m-1}\, dx}$$

$$= c_2(r).$$

Now, on repeating above arguments, we observe that $c_2(r)$ is a decreasing function of r. Hence,

$$c_2(r) > \lim_{r \to \infty} c_2(r)$$

$$= \frac{1}{m+n+1} \frac{\int_0^1 w^n (1-w)^{m-1}\, dw}{\int_0^1 w^{n+1} (1-w)^{m-1}\, dw}$$

$$= \frac{1}{n+1}.$$

Now, on using the convexity of the risk function $R_2(\underline{\theta}, \phi_{2,1}(c, r, .))$ and on repeating arguments, similar to ones preceding Theorem 2.1, we have the following theorem.

Theorem 2.3: *For estimating θ_2, under the scale equivariant squared error loss function, the estimator $\phi_{2,1}(c_2(r), r, \underline{T})$ dominates the best scale equivariant estimator $\delta_{2,BE}(\underline{T}) = \dfrac{Y}{n+1}$.*

Further select $0 < r' < r$. Since $c_2(r)$ is a decreasing function of r, it can be seen that the estimator $\phi_{2,1}(c_2(r), r, \underline{T})$ is further dominated by the following estimator

$$\phi_{2,2}(c_2(r'), r', r, \underline{T}) = \begin{cases} c_2(r')Y, & \text{if } S < r' \\ c_2(r)Y, & \text{if } r' \le S < r \\ \dfrac{Y}{n+1}, & \text{if } S \ge r. \end{cases}$$

Now, proceeding on lines of arguments preceding Theorem 2.2, we conclude the following theorem.

Theorem 2.4: *For estimating θ_2, under the squared error loss function, the estimator $\delta^{**}_{2,BE}(\underline{T}) = c_2(S)Y$, where $c_2(S)$ is as defined by (2.7), dominates the best scale equivariant estimator $\delta_{2,BE}(\underline{T}) = \dfrac{Y}{n+1}$.*

Remark: *It can be easily verified that, for every $s > 0$,*

$$s\frac{\int_0^{s/(1+s)} x^n(1-x)^{m-1}\,dx}{\int_0^{s/(1+s)} x^{n+1}(1-x)^{m-1}\,dx} \ge \frac{\int_0^{s/(1+s)} y^{n-1}(1-y)^{m}\,dy}{\int_0^{s/(1+s)} y^{n-1}(1-y)^{m+1}\,dy},$$

and therefore, with probability one, $\delta^{**}_{1,BE}(\underline{T}) \le \delta^{**}_{2,BE}(\underline{T})$. Thus, it follows that smooth estimators $\delta^{**}_{1,BE}$ and $\delta^{**}_{2,BE}$ satisfy the same inequality that parameters θ_1 and θ_2 are known to satisfy.

3 Smooth estimators as generalized Bayes estimators

In this section, we will show that smooth estimators $\delta^{**}_{1,BE}$ and $\delta^{**}_{2,BE}$, obtained in Section 2, are also generalized Bayes estimators of θ_1 and θ_2, respectively, with respect to the non-informative prior density $\Pi(\theta_1, \theta_2) = \dfrac{1}{\theta_1\theta_2}$, $0 < \theta_1 \le \theta_2 < \infty$.

Theorem 3.1: *Under the risk function $R_i(\underline{\theta}, .)$, $i = 1, 2$, as defined in (2.2), and the non-informative prior density $\Pi(\theta_1, \theta_2) = \dfrac{1}{\theta_1\theta_2}$, $0 < \theta_1 \le \theta_2 < \infty$, smooth estimators $\delta^{**}_{1,BE}$ and $\delta^{**}_{2,BE}$, obtained in Theorems 2.2 and 2.4, are generalized Bayes estimators of θ_1 and θ_2, respectively.*

Proof. For the risk function $R_1(\underline{\theta}, .)$, the Bayes estimator for θ_1 is

$$\hat{\theta}_{1,B}(\underline{T}) = \frac{E\left(\frac{1}{\theta_1} \mid \underline{T}\right)}{E\left(\frac{1}{\theta_1^2} \mid \underline{T}\right)},$$

where the expectation is taken with respect to the posterior density function

$$g^*(\underline{\theta}|\underline{T}) = d(\underline{T}) \frac{1}{\theta_1^{m+1}\theta_2^{n+1}} e^{-(X/\theta_1)} e^{-(Y/\theta_2)}, \quad 0 < \theta_1 \leq \theta_2 < \infty,$$

and $d(\underline{T})$ is the normalizing constant.
Hence,

$$\hat{\theta}_{1,B}(\underline{T}) = \frac{\iint_{R_1} \frac{1}{\theta_1^{m+2}\theta_2^{n+1}} e^{-(X/\theta_1)} e^{-(Y/\theta_2)}\, d\theta_1\, d\theta_2}{\iint_{R_1} \frac{1}{\theta_1^{m+3}\theta_2^{n+1}} e^{-(X/\theta_1)} e^{-(Y/\theta_2)}\, d\theta_1\, d\theta_2},$$

where $R_1 = \{(\theta_1, \theta_2) : 0 < \theta_1 \leq \theta_2 < \infty\}$.
On making the transformation $\frac{X}{\theta_1} = t_1$ and $\frac{Y}{\theta_2} = t_2$, we get

$$\hat{\theta}_{1,B}(\underline{T}) = \frac{\iint_{R_2} t_1^m t_2^{n-1} e^{-(t_1+t_2)}\, dt_1\, dt_2}{\iint_{R_2} t_1^{m+1} t_2^{n-1} e^{-(t_1+t_2)}\, dt_1\, dt_2} X,$$

where $R_2 = \{(t_1, t_2) : 0 < t_2 \leq St_1 < \infty\}$.
For independent standard gamma random variables $G_{1,m+1}$, $G_{1,m+2}$ and $G_{2,n}$, having shape parameters $m+1$, $m+2$ and n respectively, we may write

$$\hat{\theta}_{1,B} = \frac{1}{m+1} \frac{P(G_{2,n} \leq SG_{1,m+1})}{P(G_{2,n} \leq SG_{1,m+2})} X.$$

Now, on using the relationship between random variables $\frac{G_{2,n}}{G_{1,m+1}}$ and $\frac{G_{2,n}}{G_{1,m+2}}$ with beta random variables of first kind, we have

$$\hat{\theta}_{1,B}(\underline{T}) = \frac{1}{m+1} \frac{I_{n,m+1}\left(\frac{S}{1+S}\right)}{I_{n,m+2}\left(\frac{S}{1+S}\right)} X$$

$$= c_1(S)X,$$

as desired.

The proof for the generalized Bayes estimator of θ_2 follows in the similar fashion, and hence is omitted.

4 Comparisons of estimators

In this section, we will compare smooth estimators, obtained in Section 2, with other natural estimators, empirically. Estimators of θ_1, which are chosen for the comparison study, are the best scale equivariant estimator $\delta_{1,BE}(\underline{T}) = \dfrac{X}{m+1}$, the restricted mle $\delta_{1,RM}(\underline{T}) = \min\left(\dfrac{X}{m}, \dfrac{X+Y}{m+n}\right)$, the non-smooth estimator $\delta^*_{1,BE}(\underline{T}) = \min\left(\dfrac{X}{m+1}, \dfrac{X+Y}{m+n+1}\right)$, obtained in Vijayasree, Misra & Singh (1995), which is known to dominate the best scale equivariant estimator $\delta_{1,BE}(\underline{T})$, and the smooth estimator $\delta^{**}_{1,BE} = c_1(S)X$, as given in Theorem 2.2. Similarly, estimators of θ_2, which are chosen for the comparison study, are the best scale equivariant estimator $\delta_{2,BE}(\underline{T}) = \dfrac{Y}{n+1}$, the restricted mle $\delta_{2,RM}(\underline{T}) = \max\left(\dfrac{Y}{n}, \dfrac{X+Y}{m+n}\right)$, the non-smooth estimator $\delta^*_{2,BE}(\underline{T}) = \max\left(\dfrac{Y}{n+1}, \dfrac{X+Y}{m+n+1}\right)$, obtained in Vijayasree, Misra & Singh (1995), which is known to dominate the best scale equivariant estimator $\delta_{2,BE}(\underline{T})$, and the smooth estimator $\delta^{**}_{2,BE} = c_2(S)Y$, as given in Theorem 2.4.

Since expressions for risk functions of smooth estimators $\delta^{**}_{i,BE}, i = 1,2$ can not be expressed in a simple closed form, theoretical comparisons of smooth estimators with other estimators, considered in the present study, seems difficult. We, therefore, use Monte Carlo simulations to compare these estimators. Although, expressions for risk functions of some of estimators considered in the present study are known to have a simple closed form, for uniformity, we have evaluated risk functions of even these estimators using Monte Carlo simulations. Since risk functions of various estimators, considered in this study, depend on $\underline{\theta}$ only through $\sigma = \dfrac{\theta_1}{\theta_2}$, comparisons are made for different combinations of σ, m and n. For the computation of the risk function R_i of an estimator d_i, we generated observations from suitable gamma distributions and, for each combination of parameters, $\left(\dfrac{d_i}{\theta_i} - 1\right)^2$ $(i = 1,2)$ is computed. The procedure was then repeated 50,000 times to approximate R_i by averages of $\left(\dfrac{d_i}{\theta_i} - 1\right)^2$ $(i = 1,2)$. For different combinations of (m,n) and σ, values of the R_is, so obtained, are tabulated in Tables 1–8. Following conclusions are evident from Tables 1–8.

Conclusions for Estimation of θ_1:

(i) For small values of shape parameters (say, $m = 1$, $n = 1$), estimators $\delta^*_{1,BE}$ and $\delta^{**}_{1,BE}$ perform better than estimators $\delta_{1,BE}$ and $\delta_{1,RM}$. Interestingly, $\delta_{1,BE}$ performs uniformly better than $\delta_{1,RM}$. One may justify this observation by arguing that $\delta_{1,BE}$ was designed for the risk function used here and that is more important than the order restriction for small sample sizes, and especially if $\sigma < 1$. Among estimators $\delta^*_{1,BE}$ and $\delta^{**}_{1,BE}$, estimator $\delta^{**}_{1,BE}$ performs better than $\delta^*_{1,BE}$, if $\sigma \leq .4$, and this trend is reversed for $\sigma \geq .5$.

Table 1. Values of $R_1(\underline{\theta}, \delta_{1,\cdot})$ and $R_2(\underline{\theta}, \delta_{2,\cdot})$ for $(m, n) = (1, 1)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.499	.868	.495	.485	.498	1.003	.489	.454
.20	.498	.779	.489	.481	.503	.979	.473	.422
.30	.504	.725	.493	.489	.503	.973	.451	.401
.40	.503	.660	.487	.487	.500	.936	.422	.378
.50	.498	.604	.481	.488	.500	.941	.402	.375
.60	.504	.586	.485	.493	.499	.940	.384	.383
.70	.498	.543	.481	.494	.501	.934	.370	.393
.80	.503	.535	.483	.497	.499	.934	.360	.419
.90	.502	.518	.482	.498	.501	.958	.355	.457
1.00	.500	.503	.483	.501	.501	1.015	.354	.505

Table 2. Values of $R_1(\underline{\theta}, \delta_{1,\cdot})$ and $R_2(\underline{\theta}, \delta_{2,\cdot})$ for $(m, n) = (5, 5)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.167	.201	.167	.165	.165	.196	.165	.162
.20	.168	.201	.167	.159	.166	.197	.166	.154
.30	.166	.189	.164	.153	.166	.197	.162	.142
.40	.166	.181	.161	.150	.167	.193	.156	.132
.50	.167	.170	.158	.149	.167	.188	.148	.124
.60	.167	.156	.154	.152	.166	.179	.136	.120
.70	.167	.146	.151	.155	.165	.171	.125	.121
.80	.167	.137	.148	.158	.166	.170	.118	.131
.90	.168	.131	.148	.163	.166	.171	.113	.146
1.00	.165	.124	.144	.165	.167	.175	.111	.167

(ii) For moderate and large values of shape parameters (say, $m \geq 5$, $n \geq 5$), the estimator $\delta^{**}_{1,BE}$ performs better than other estimators if $\sigma \leq .6$, while, for $\sigma \geq .7$, the estimator $\delta_{1,RM}$ performs better than other estimators.

(iii) For $m \geq 10$ and $n \geq 10$, the estimator $\delta^{**}_{1,BE}$ performs better than other estimators if $\sigma \leq .7$, while, for $\sigma \geq .8$, the estimator $\delta_{1,RM}$ performs better than other estimators.

(iv) For small values of shape parameters, the estimator $\delta^*_{1,BE}$ dominates other estimators over a large portion of the parametric space. For moderate

Table 3. Values of $R_1(\underline{\theta}, \delta_{1,.})$ and $R_2(\underline{\theta}, \delta_{2,.})$ for $(m, n) = (10, 10)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.091	.101	.091	.091	.091	.100	.091	.091
.20	.091	.101	.091	.090	.091	.100	.091	.090
.30	.090	.099	.090	.086	.091	.101	.091	.085
.40	.090	.096	.089	.082	.090	.098	.089	.078
.50	.091	.093	.088	.080	.090	.095	.085	.071
.60	.092	.087	.086	.080	.091	.094	.081	.068
.70	.091	.080	.082	.080	.091	.089	.074	.067
.80	.091	.074	.079	.082	.091	.085	.068	.071
.90	.090	.069	.077	.086	.092	.083	.064	.078
1.00	.091	.066	.077	.091	.090	.083	.062	.090

Table 4. Values of $R_1(\underline{\theta}, \delta_{1,.})$ and $R_2(\underline{\theta}, \delta_{2,.})$ for $(m, n) = (20, 20)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.048	.050	.048	.048	.047	.050	.047	.047
.20	.048	.050	.048	.048	.047	.050	.047	.047
.30	.048	.050	.048	.047	.048	.050	.048	.047
.40	.048	.050	.048	.046	.048	.050	.048	.045
.50	.047	.049	.047	.043	.047	.049	.047	.041
.60	.048	.047	.046	.041	.048	.049	.046	.039
.70	.048	.044	.044	.040	.048	.047	.042	.036
.80	.048	.040	.042	.042	.048	.044	.039	.037
.90	.048	.036	.040	.044	.048	.041	.035	.040
1.00	.048	.034	.039	.047	.047	.040	.033	.048

and large values of shape parameters, the estimator $\delta^{**}_{1,BE}$ dominates other estimators over a large portion of the parametric space.

(v) In view of (i)–(iv), above, we recommend the estimator $\delta^*_{1,BE}$ for small values of shape parameters, while the estimator $\delta^{**}_{1,BE}$ is recommended for moderate and large values of shape parameters. If one has an idea about the size of σ, one may choose between $\delta^*_{1,BE}, \delta^{**}_{1,BE}$ and $\delta_{1,RM}$ based on observations reported in (i)–(iii).

Table 5. Values of $R_1(\underline{\theta},\delta_{1,\cdot})$ and $R_2(\underline{\theta},\delta_{2,\cdot})$ for $(m,n)=(5,10)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.168	.201	.168	.167	.091	.100	.091	.090
.20	.168	.201	.168	.161	.091	.100	.091	.088
.30	.169	.198	.168	.153	.091	.100	.091	.084
.40	.167	.186	.164	.146	.090	.098	.088	.080
.50	.166	.169	.158	.143	.091	.096	.086	.077
.60	.168	.154	.153	.145	.090	.093	.082	.075
.70	.166	.135	.146	.147	.090	.091	.078	.075
.80	.166	.123	.142	.153	.090	.090	.074	.078
.90	.166	.113	.138	.159	.091	.090	.073	.084
1.00	.165	.107	.137	.166	.091	.091	.071	.090

Table 6. Values of $R_1(\underline{\theta},\delta_{1,\cdot})$ and $R_2(\underline{\theta},\delta_{2,\cdot})$ for $(m,n)=(5,20)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.167	.201	.167	.167	.048	.050	.048	.048
.20	.166	.198	.166	.161	.048	.051	.048	.047
.30	.166	.196	.165	.152	.048	.050	.048	.046
.40	.168	.190	.165	.144	.048	.050	.047	.045
.50	.167	.172	.160	.140	.048	.049	.046	.044
.60	.167	.150	.152	.139	.048	.048	.045	.043
.70	.167	.129	.143	.141	.047	.047	.044	.043
.80	.167	.112	.137	.148	.048	.047	.043	.044
.90	.166	.101	.132	.156	.048	.047	.042	.046
1.00	.168	.095	.130	.167	.048	.048	.042	.048

Conclusions for Estimation of θ_2:

(i) For all values of shape parameters under study, the estimator $\delta^{**}_{2,BE}$ dominates other estimators over a large portion of the parametric space.
(ii) The region of dominance of the estimator $\delta^{**}_{2,BE}$, over other estimators, enlarges with the increase in values of shape parameters, and, for moderate and large values of shape parameters, the region of dominance of $\delta^{**}_{2,BE}$ over other estimators is $\{\underline{\theta}=(\theta_1,\theta_2):\sigma\leq .7\}$.

Table 7. Values of $R_1(\underline{\theta}, \delta_{1,.})$ and $R_2(\underline{\theta}, \delta_{2,.})$ for $(m, n) = (10, 5)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.091	.100	.091	.091	.167	.201	.167	.166
.20	.091	.100	.091	.089	.168	.202	.167	.158
.30	.091	.098	.090	.086	.166	.197	.162	.140
.40	.091	.095	.089	.084	.167	.192	.156	.124
.50	.091	.092	.088	.083	.164	.184	.142	.109
.60	.090	.086	.085	.083	.166	.176	.130	.102
.70	.092	.084	.085	.085	.167	.168	.116	.104
.80	.090	.080	.082	.086	.167	.159	.103	.114
.90	.091	.077	.082	.089	.167	.157	.095	.135
1.00	.091	.075	.082	.091	.166	.156	.090	.164

Table 8. Values of $R_1(\underline{\theta}, \delta_{1,.})$ and $R_2(\underline{\theta}, \delta_{2,.})$ for $(m, n) = (20, 5)$

σ	$\delta_{1,BE}$	$\delta_{1,RM}$	$\delta^*_{1,BE}$	$\delta^{**}_{1,BE}$	$\delta_{2,BE}$	$\delta_{2,RM}$	$\delta^*_{2,BE}$	$\delta^{**}_{2,BE}$
.10	.047	.050	.047	.047	.167	.201	.167	.166
.20	.048	.050	.048	.047	.167	.201	.167	.159
.30	.048	.050	.048	.046	.167	.198	.163	.141
.40	.048	.049	.047	.045	.166	.191	.154	.118
.50	.048	.048	.047	.045	.167	.187	.143	.101
.60	.047	.046	.045	.044	.167	.175	.125	.088
.70	.047	.045	.045	.045	.166	.164	.107	.088
.80	.047	.044	.044	.045	.167	.153	.091	.100
.90	.048	.044	.045	.047	.167	.147	.080	.127
1.00	.047	.042	.044	.047	.167	.144	.074	.166

(iii) Interestingly, for small values of shape parameters, the estimator $\delta_{1,BE}$ uniformly dominates the estimator $\delta_{1,RM}$.
(iv) In view of (i)–(iii), above, we recommend the estimator $\delta^{**}_{2,BE}$.

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