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Estimation of frequencies in presence of heavy tail errors

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Abstract

In this paper, we consider the problem of estimating the sinusoidal frequencies in presence of additive white noise. The additive white noise has mean zero but it may not have finite variance. We propose to use the least-squares estimators or the approximate least-squares estimators to estimate the unknown parameters. It is observed that the least-squares estimators and the approximate least-squares estimators are asymptotically equivalent and both of them provide consistent estimators of the unknown parameters. We obtain the asymptotic distribution of the least-squares estimators under the assumption that the errors are from a symmetric stable distribution. We propose different methods of constructing confidence intervals and compare their performances through Monte Carlo simulations. We also discuss the properties of the estimators if the errors are correlated and finally we discuss some open problems. © 2002 Published by Elsevier Science B.V.

Keywords: Sinusoidal signals; Consistent estimators; Stable distributions; Confidence intervals

1. Introduction

One of the most important problem of the time series analysis has proved to be the estimation of frequencies in presence of an additive noise. This problem may occur in several discipline in a variety of ways. Suppose we observe a sequence of observations from the following time series model:

$$y(t) = \sum_{j=1}^{P} (A_j \cos(\omega_j t) + B_j \sin(\omega_j t)) + e(t). \tag{1.1}$$

Here $y(t)$'s are observed values at the equidistant time points, for $t = 1, \ldots, n$, ω_j's are unknown frequencies and lying between $(0, \pi)$, A_j's and B_j's are amplitudes and they are unknown real numbers. The additive errors $\{e(t)\}$'s are independent and identically distributed (i.i.d.) random

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variables with mean zero but they may not have finite variances. The problem is to estimate the unknown parameters namely A_j's, B_j's and ω_j's and study their properties assuming p is known.

The estimation of the parameters of the model (1.1) is a fundamental problem in signal processing (Kay, 1988) and time series analysis (Brillinger, 1987). There is a vast amount of literature exists regarding the estimation procedures as well as the theoretical behavior of the different estimators if the error random variables have finite variances or when the errors are from a stationary sequence. The asymptotic theory of the least-squares estimators (LSEs) of this model has a long history. Whittle (1953) obtained some of the earlier results. More recent results are by Hannan (1971), Walker (1971), Rice and Rosenblatt (1988), Kundu (1993, 1997) and Kundu and Mitra (1996). The main aim of this paper is to consider the case when the error random variables have heavier tails. A heavy tail distribution is one whose extreme probabilities approach zero relatively slowly. The non-existence of finite variance is an important criterion for heavy tailedness as noted by Mandelbrot (1963). In fact, Mandelbrot (1963) defined distributions as heavy tailed if and only if the variance is infinite. We are using the same definition of Mandelbrot (1963). It can be shown that under the assumption $E|e(t)|^{1+\delta} < \infty$ for some $\delta > 0$ on the error random variables, the LSEs and the ALSEs both provide consistent estimators of the unknown parameters. Furthermore, if we assume that $e(t)$'s are from a symmetric α stable distribution, then the asymptotic distribution of the LSEs or ALSEs is multivariate stable. Using this asymptotic distribution it is possible to construct asymptotic confidence intervals of the unknown parameters.

The rest of the paper is organized as follows. We introduce the LSEs and the ALSEs in Section 2 and prove the consistency results in this section. The asymptotic distribution of the LSEs or the ALSEs is provided in Section 3. Construction of the confidence intervals are discussed in Section 4. Some numerical results are presented in Section 5. Finally we conclude our paper and discuss some open problems in Section 6.

2. Least-squares estimators and approximate least-squares estimators

In this section, we study the properties of the most intuitive estimators, namely the LSEs and the most used estimators, namely the ALSEs. For brevity, we assume $p = 1$, although the results can be established for any integer p along the same line. In this section, we mainly consider the following model:

$$y(t) = A\cos(\omega t) + B\sin(\omega t) + e(t). \tag{2.1}$$

For the model (2.1), the LSE of $\boldsymbol{\theta} = (A, B, \omega)$, say $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\omega})$ can be obtained by minimizing

$$Q(\boldsymbol{\theta}) = \sum_{t=1}^{n} (y(t) - A\cos(\omega t) - B\sin(\omega t))^2 \tag{2.2}$$

with respect to $\boldsymbol{\theta}$. We represent $\boldsymbol{\theta}_0 = (A_0, B_0, \omega_0)$, the true value of $\boldsymbol{\theta}$ and throughout we assume that $\boldsymbol{\theta}_0$ is an interior point of the parameter space.

The ALSE of ω of the model (2.1) can be obtained by maximizing the periodogram function

$$I(\omega) = \frac{2}{n} \left| \sum_{t=1}^{n} y(t) e^{i\omega t} \right|^2 \tag{2.3}$$

with respect to ω. If $\tilde{\omega}$ maximizes $I(\omega)$, then $\tilde{\omega}$ is called the ALSE of ω. Following the approach of Walker (1971) or Hannan (1971), we define the ALSEs of A and B, say $\tilde{A}$ and $\tilde{B}$, respectively, as

$$\tilde{A} = \frac{2}{n}\sum_{t=1}^{n} y(t)\cos(\tilde{\omega}t), \quad \tilde{B} = \frac{2}{n}\sum_{t=1}^{n} y(t)\sin(\tilde{\omega}t). \tag{2.4}$$

For the motivation of using $\tilde{\boldsymbol{\theta}} = (\tilde{A}, \tilde{B}, \tilde{\omega})$ as estimator of $\boldsymbol{\theta}$, see Hannan (1971) or Walker (1971).

Although in case of i.i.d. errors, the LSEs are the most intuitive estimators, but for the model (1.1), the most popular estimators of the unknown parameters are the ALSEs. Note that the LSEs can be easily defined for the model (1.1) and the ALSEs of ω's can be defined as the local maximums of $I(\omega)$ instead of the global maximum. Once ω_j is estimated the corresponding ALSEs of A_j's and B_j's can be obtained using (2.4). The following two theorems give strong consistency for both the LSE and ALSE. We defer the proofs of these two results to Appendix A, at the end of the paper.

Theorem 1. *If $\hat{\boldsymbol{\theta}}$ is the LSE of $\boldsymbol{\theta}$ of the model* (2.1) *and $e(t)$'s are i.i.d. random variables with mean zero and $E|e(t)|^{1+\delta} < \infty$ for some $0 < \delta < 1$, then $\hat{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}$.*

Theorem 2. *If $\tilde{\boldsymbol{\theta}}$ is the ALSE of $\boldsymbol{\theta}$ of the model* (2.1) *and $e(t)$'s are same as in Theorem* 1, *then $\tilde{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}$.*

3. Asymptotic distributions of the LSEs and ALSEs

In this section, we obtain the asymptotic distributions of the LSEs and ALSEs under the assumption that the errors are from a symmetric stable distribution. Before progressing any further, first we define a symmetric α-stable ($S\alpha S$) distribution as follows:

Definition 1. A symmetric (around 0) random variable X is said to have $S\alpha S$ distribution, with scale parameter σ, and stability index α, if the characteristic function of the random variable X is

$$Ee^{itX} = e^{-\sigma^{\alpha}|t|^{\alpha}}.$$

For detailed treatments of the different $S\alpha S$ distributions the readers are referred to the book of Samorodnitsky and Taqqu (1994). From now on we always take $1 + \delta < \alpha < 2$.

Consider (2.2), we use the following notations:

$$Q'(\boldsymbol{\theta}) = \left(\frac{\partial Q(\boldsymbol{\theta})}{\partial A}, \frac{\partial Q(\boldsymbol{\theta})}{\partial B}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \omega}\right)$$

and $Q''(\boldsymbol{\theta})$ is the 3×3 matrix of the second derivatives of $Q(\boldsymbol{\theta})$. Now expanding $Q'(\hat{\boldsymbol{\theta}})$ around $\boldsymbol{\theta}_0$ by multivariate Taylor series, we obtain

$$Q'(\hat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}_0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)Q''(\bar{\boldsymbol{\theta}}) \tag{3.1}$$

where $\bar{\boldsymbol{\theta}}$ is a point on the line joining $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$. Suppose $\mathbf{D}_1$ and $\mathbf{D}_2$ are two diagonal matrices of order 3×3 each and they are as follows:

$$\mathbf{D}_1 = \mathrm{diag}\{n^{-1/\alpha}, n^{-1/\alpha}, n^{-(1+\alpha)/\alpha}\} \quad \text{and} \quad \mathbf{D}_2 = \mathrm{diag}\{n^{-(\alpha-1)/\alpha}, n^{-(\alpha-1)/\alpha}, n^{-(2\alpha-1)/\alpha}\}.$$

Since $Q'(\hat{\boldsymbol{\theta}}) = 0$, therefore, (3.1) can be written as

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)\mathbf{D}_2^{-1} = -[Q'(\boldsymbol{\theta}_0)\mathbf{D}_1][\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}})\mathbf{D}_1]^{-1}, \tag{3.2}$$

as $\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}})\mathbf{D}_1$ is invertible almost surely for large n. It can be easily seen that

$$\lim_{n\to\infty}[\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}})\mathbf{D}_1] = \lim_{n\to\infty}[\mathbf{D}_2 Q''(\boldsymbol{\theta}_0)\mathbf{D}_1] = \begin{pmatrix} 1 & 0 & \dfrac{B_0}{2} \\ 0 & 1 & -\dfrac{A_0}{2} \\ \dfrac{B_0}{2} & -\dfrac{A_0}{2} & \dfrac{1}{3}(A_0^2 + B_0^2) \end{pmatrix}.$$

Therefore,

$$\lim_{n\to\infty}[\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}})\mathbf{D}_1]^{-1} = \lim_{n\to\infty}[\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}}_0)\mathbf{D}_1]^{-1} = \frac{1}{A_0^2 + B_0^2}\begin{bmatrix} A_0^2 + 4B_0^2 & -3A_0B_0 & -6B_0 \\ -3A_0B_0 & 4A_0^2 + B_0^2 & 6A_0 \\ -6B_0 & 6A_0 & 12 \end{bmatrix}. \tag{3.3}$$

Now first we show that $[Q'(\boldsymbol{\theta}_0)\mathbf{D}_1]$ converges to a three-dimensional multivariate stable distribution. Let us consider

$$\begin{aligned} Q'(\boldsymbol{\theta}_0)\mathbf{D}_1 &= \Bigg[-\frac{2}{n^{1/\alpha}}\sum_{t=1}^{n} e(t)\cos(\omega_0 t), -\frac{2}{n^{1/\alpha}}\sum_{t=1}^{n} e(t)\sin(\omega_0 t), \\ &\qquad \frac{2}{n^{(1+\alpha)/\alpha}}\sum_{t=1}^{n} te(t)[A_0\sin(\omega_0 t) - B_0\cos(\omega_0 t)]\Bigg] \\ &= (X_n, Y_n, Z_n) \quad \text{(say)}. \end{aligned} \tag{3.4}$$

Therefore, if $\mathbf{t} = (t_1, t_2, t_3)$, then the joint characteristic function of (X_n, Y_n, Z_n) is

$$\phi_n(\mathbf{t}) = E\mathrm{e}^{t_1X_n + t_2Y_n + t_3Z_n} = \mathrm{e}^{-2^\alpha\sigma^\alpha(1/n)\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^\alpha}, \tag{3.5}$$

where

$$K_{\mathbf{t}}(j) = -t_1\cos(\omega_0 j) - t_2\sin(\omega_0 j) + \frac{jt_3}{n}(A_0\sin(\omega_0 j) - B_0\cos(\omega_0 j)).$$

Although we could not prove it theoretically but it is observed by extensive numerical computations that as n tends to ∞, $(1/n)\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^\alpha$ converges. Assuming that $(1/n)\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^\alpha$ converges, it can be proved that (see Appendix B) it converges to a non-zero limit for $\mathbf{t} \neq \mathbf{0}$. Suppose

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^\alpha = \tau_{\mathbf{t}}(A_0, B_0, \omega_0, \alpha), \tag{3.6}$$

then, from (3.5) it is clear that any linear combination of X_n, Y_n and Z_n is a $S\alpha S$ distribution. Also

$$\lim_{n\to\infty} \phi_n(\mathbf{t}) = \mathrm{e}^{-2^\alpha \sigma^\alpha \tau_{\mathbf{t}}(A_0,B_0,\omega_0,\alpha)} \tag{3.7}$$

indicates that any linear combination of X_n, Y_n and Z_n even if $n \to \infty$ is also a $S\alpha S$ distribution. Now using the result (Theorem 2.1.5) of Samorodnitsky and Taqqu (1994) that *a random vector is symmetric stable in* R^3 *if and only if any linear combination is symmetric stable distribution in* R^1, it immediately follows that:

$$\lim_{n\to\infty} [Q'(\boldsymbol{\theta}_0)\mathbf{D}_1][\mathbf{D}_2 Q''(\bar{\boldsymbol{\theta}})\mathbf{D}_1]^{-1}, \tag{3.8}$$

converges to a symmetric stable random vector in R^3, which has the characteristic function

$$\Phi(\mathbf{t}) = \mathrm{e}^{-2^\alpha \sigma^\alpha \tau_{\mathbf{u}}(A_0,B_0,\omega_0,\alpha)} \tag{3.9}$$

and $\tau_{\mathbf{u}}$ is defined through (3.6) replacing $\mathbf{t}$ by $\mathbf{u}$. Here

$$\mathbf{u} = (u_1(t_1,t_2,t_3,A_0,B_0), u_2(t_1,t_2,t_3,A_0,B_0), u_3(t_1,t_2,t_3,A_0,B_0))$$

and

$$u_1(t_1,t_2,t_3,A_0,B_0) = [(A_0^2+4B_0^2)t_1 - 3A_0B_0t_2 - 6B_0t_3]\,\frac{1}{A_0^2+B_0^2},$$

$$u_2(t_1,t_2,t_3,A_0,B_0) = [-3A_0B_0t_1 + (4A_0^2+B_0^2)t_2 + 6A_0t_3]\,\frac{1}{A_0^2+B_0^2},$$

$$u_3(t_1,t_2,t_3,A_0,B_0) = [-6B_0t_1 + 6A_0t_2 + 12t_3]\,\frac{1}{A_0^2+B_0^2}.$$

Therefore, we can put this as the following theorem.

Theorem 3. *In model* (2.1), *if* $e(t)$*'s are i.i.d. random variables with mean zero and have* $S\alpha S$ *distribution as defined in Definition* 1, *then* $(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)\mathbf{D}_2^{-1}$ *converges to a multivariate stable distribution, which has a characteristic function as defined in* (3.9).

Now to show that the asymptotic distributions of the LSEs and the ALSEs are same observe the following facts. By simple calculations (similarly as Hannan, 1971 or Walker, 1971) it can be shown that

$$\hat{A}(\omega) = \tilde{A}(\omega) + O_p(n), \quad \hat{B}(\omega) = \tilde{B}(\omega) + O_p(n), \quad \hat{\omega} = \tilde{\omega} + O_p(n^2). \tag{3.10}$$

Here $O_p(m)$ indicates that the term goes to zero in probability and also $mO_p(m)$ is bounded in probability. (3.10) immediately implies that the asymptotic distributions of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)\mathbf{D}_2^{-1}$ and $(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)\mathbf{D}_2^{-1}$ are same. This result can be extended for the general model (1.1). Observe that for the general model, $(\hat{A}_i, \hat{B}_i, \hat{\omega}_i)$ and $(\hat{A}_j, \hat{B}_j, \hat{\omega}_j)$ will be asymptotically independent for $i \neq j$, therefore the joint characteristic function can be easily obtained.

4. Individual confidence intervals

In the previous section, we saw that although we know the characteristic function of the joint distribution of $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\omega})$ but it is not very easy to obtain the joint distribution of $(\hat{A}, \hat{B}, \hat{\omega})$ from that. In this section we obtain the marginal distributions of $\hat{A}$, $\hat{B}$ and $\hat{\omega}$ by inverting the corresponding characteristic functions. The marginal characteristic functions of $n^{(\alpha-1)/\alpha}(\hat{A} - A_0)$, $n^{(\alpha-1)/\alpha}(\hat{B} - B_0)$, and $n^{(2\alpha-1)/\alpha}(\hat{\omega} - \omega_0)$ are

$$\Phi_A(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha (1/n) \sum_{t=1}^n |K_{\mathbf{a}}(t)|^\alpha}, \quad \Phi_B(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha (1/n) \sum_{t=1}^n |K_{\mathbf{b}}(t)|^\alpha}$$

and

$$\Phi_w(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha (1/n) \sum_{t=1}^n |K_{\mathbf{w}}(t)|^\alpha},$$

respectively, where

$$\mathbf{a} = \left(\frac{A_0^2 + 4B_0^2}{A_0^2 + B_0^2}, \frac{-3A_0B_0}{A_0^2 + B_0^2}, \frac{-6B_0}{A_0^2 + B_0^2} \right), \quad \mathbf{b} = \left(\frac{-3A_0B_0}{A_0^2 + B_0^2}, \frac{4A_0^2 + B_0^2}{A_0^2 + B_0^2}, \frac{6A_0}{A_0^2 + B_0^2} \right)$$

and

$$\mathbf{w} = \left(\frac{-6B_0}{A_0^2 + B_0^2}, \frac{6A_0}{A_0^2 + B_0^2}, \frac{12}{A_0^2 + B_0^2} \right).$$

Therefore, the characteristic functions of the limiting distributions of $n^{(\alpha-1)/\alpha}(\hat{A} - A_0)$, $n^{(\alpha-1)/\alpha}(\hat{B} - B_0)$, and $n^{(2\alpha-1)/\alpha}(\hat{\omega} - \omega_0)$ are

$$\lim_{n\to\infty} \Phi_A(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \tau_{\mathbf{a}}(A_0, B_0, \omega_0, \alpha)}, \quad \lim_{n\to\infty} \Phi_B(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \tau_{\mathbf{b}}(A_0, B_0, \omega_0, \alpha)}$$

and

$$\lim_{n\to\infty} \Phi_w(u) = \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \tau_{\mathbf{w}}(A_0, B_0, \omega_0, \alpha)},$$

respectively, where $\tau_{\mathbf{a}}(A_0, B_0, \omega_0, \alpha)$, $\tau_{\mathbf{b}}(A_0, B_0, \omega_0, \alpha)$ and $\tau_{\mathbf{w}}(A_0, B_0, \omega_0, \alpha)$ are defined as (3.6) replacing $\mathbf{t}$ by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{w}$, respectively. Now to construct the asymptotic $100(1-\beta)\%$ confidence intervals of the individual parameters (say for ω_0), we use the inversion formula of the characteristic function (see Chung, 1974). Find x_w, such that

$$1 - \beta = \frac{2}{\pi} \int_0^\infty \frac{\sin(ux)}{u} \mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \tau_{\mathbf{w}}(A_0, B_0, \omega_0, \alpha)} \, \mathrm{d}u = \mu_w(-x_w, x_w) \quad \text{(say)}$$

where $\mu_w(-x, x)$ is the probability measure which corresponds to

$$\mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \tau_{\mathbf{w}}(A_0, B_0, \omega_0, \alpha)}.$$

We do not have the explicit expression of $\tau_{\mathbf{w}}(A_0, B_0, \omega_0, \alpha)$, but numerically we can estimate $\tau_{\mathbf{w}}(\hat{A}, \hat{B}, \hat{\omega}, \alpha)$ using (3.6) say $\hat{\tau}_{\mathbf{w}}(\hat{A}, \hat{B}, \hat{\omega}, \alpha)$ and then we estimate x_w by minimizing $|\hat{\mu}_w(-x, x) - (1-\beta)|$, where $\hat{\mu}_w(-x, x)$ is the probability measure corresponding to the characteristic function

$$\mathrm{e}^{-\sigma^\alpha |u|^\alpha 2^\alpha \hat{\tau}_{\mathbf{w}}(\hat{A}, \hat{B}, \hat{\omega}, \alpha)}.$$

Table 1
The average estimates and the mean absolute deviations of the LSEs and the ALSEs when the sample size is 20.

Para.	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
	LSE	ALSE	LSE	ALSE	LSE	ALSE	LSE	ALSE
A	1.790 (0.884)	0.858 (1.115)	1.582 (0.421)	1.045 (0.624)	1.501 (0.249)	1.118 (0.446)	1.493 (0.192)	1.212 (0.399)
B	1.990 (0.974)	1.878 (0.858)	1.825 (0.618)	1.788 (0.469)	1.549 (0.235)	1.783 (0.361)	1.509 (0.154)	1.766 (0.293)
ω	2.416 (0.106)	2.459 (0.097)	2.488 (0.030)	2.517 (0.036)	2.501 (0.012)	2.517 (0.025)	2.499 (0.011)	2.521 (0.021)

Here $A = B = 1.5$ and $\omega = 2.5$. In each box, the first row represents the average estimates and the corresponding mean absolute deviations are reported within brackets below.

If $\hat{x}_w$ minimizes $|\hat{\mu}_w(-x,x) - (1-\beta)|$, then the asymptotic $100(1-\beta)\%$ confidence interval of ω is

$$\left(\hat{\omega} - \frac{\hat{x}_w}{n^{(2\alpha-1)/\alpha}}, \hat{\omega} + \frac{\hat{x}_w}{n^{(2\alpha-1)/\alpha}}\right).$$

The confidence intervals of A and B can be obtained exactly in the same manner. We also consider two Bootstrap confidence intervals, percentile Bootstrap (Boot-p) and Bootstrap-t (Boot-t) confidence intervals similarly as Mitra and Kundu (1997). Their performances are compared in the next section.

5. Numerical experiments

In this section, we present some experimental results to see how the LSEs and the ALSEs behave for finite samples. We consider the following model:

$$y(t) = A\cos(\omega t) + B\sin(\omega t) + e(t). \tag{5.1}$$

We took $A = B = 1.5$ and $\omega = 2.5$. We consider $e(t)$'s to be i.i.d. $S\alpha S$ random variable with mean zero and $1 < \alpha < 2$. We want to see how the LSEs and the ALSEs behave for different values of α and for different sample sizes. We consider $\alpha = 1.2$, 1.4, 1.6, 1.8 and $n = 20$, 25 and 30. In all the cases the scale parameter $\sigma = 0.25$. For each combination of α and n, we compute the LSEs and ALSEs of the unknown parameters and obtain the average estimates and the mean absolute deviations (MADs) over 500 replications and the results are reported in Tables 1–3. In each table the first figure represents the average estimates and the corresponding MADs are reported in the bracket. For computing the LSEs and the ALSEs we used the optimization routines of Press et al. (1992).

We also computed the confidence intervals of the different parameters using the asymptotic distribution and also by the two Bootstrap methods. The exact numerical procedures for the different methods are as follows. For a given data set, first we estimate A, B and ω, say $\hat{A}$, $\hat{B}$ and $\hat{\omega}$, respectively. Then compute $\hat{\tau}_{\mathbf{a}}(\hat{A},\hat{B},\hat{\omega},\alpha)$, $\hat{\tau}_{\mathbf{b}}(\hat{A},\hat{B},\hat{\omega},\alpha)$ and $\hat{\tau}_{\mathbf{w}}(\hat{A},\hat{B},\hat{\omega},\alpha)$ using the first 500 terms of the corresponding series as defined in (3.6). Finally we obtained the 95% confidence intervals of A, B

Table 2
The average estimates and the mean absolute deviations of the LSEs and the ALSEs when the sample size is 25.

Para.	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
	LSE	ALSE	LSE	ALSE	LSE	ALSE	LSE	ALSE
A	1.555 (0.592)	1.164 (0.710)	1.459 (0.357)	1.205 (0.447)	1.498 (0.212)	1.216 (0.353)	1.499 (0.159)	1.217 (0.314)
B	1.265 (0.767)	1.712 (0.737)	1.523 (0.295)	1.682 (0.433)	1.505 (0.210)	1.690 (0.305)	1.500 (0.153)	1.691 (0.248)
ω	2.427 (0.096)	2.457 (0.080)	2.479 (0.033)	2.501 (0.031)	2.493 (0.016)	2.508 (0.020)	2.500 (0.007)	2.513 (0.014)

Here $A = B = 1.5$ and $\omega = 2.5$. In each box, the first row represents the average estimates and the corresponding mean absolute deviations are reported within brackets below.

Table 3
The average estimates and the mean absolute deviations of the LSEs and the ALSEs when the sample size is 30.

Para.	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
	LSE	ALSE	LSE	ALSE	LSE	ALSE	LSE	ALSE
A	1.457 (0.506)	1.217 (0.728)	1.476 (0.301)	1.217 (0.436)	1.481 (0.209)	1.246 (0.320)	1.485 (0.154)	1.254 (0.273)
B	1.502 (0.620)	1.624 (0.637)	1.497 (0.306)	1.634 (0.363)	1.491 (0.205)	1.636 (0.266)	1.500 (0.141)	1.646 (0.211)
ω	2.440 (0.080)	2.442 (0.087)	2.478 (0.033)	2.491 (0.033)	2.493 (0.014)	2.502 (0.022)	2.501 (0.006)	2.500 (0.019)

Here $A = B = 1.5$ and $\omega = 2.5$. In each box, the first row represents the average estimates and the corresponding mean absolute deviations are reported within brackets below.

and ω. We also obtained the corresponding Boot-p and Boot-t confidence intervals for the different parameters. We present the results for $\alpha = 1.2, 1.4, 1.6$ and 1.8, $n = 25$ and for ω only. The results for A and B are quite similar in nature therefore they are not provided here. The results are reported in Table 4. In each box the first figure represents the average coverage percentages and in the bracket the average length of the confidence intervals are reported over 500 replications.

Some of the points are very clear from the numerical results. From Tables 1–3 it is observed that as the sample size n increases the biases decrease in general and also the MADs decrease, it verifies the consistency property of both the LSEs and the ALSEs for all the parameters. It is also observed that as α increases the biases and the MADs decrease. It indicates that for heavier tail it is more difficult to estimate the unknown parameters. In all the cases for both the methods, the MADs of the frequencies are significantly smaller than the corresponding MADs of the amplitudes. It verifies that the rate of convergence of the frequencies is more compared to the rate of convergence

Table 4
The coverage percentages and the average confidence lengths of the frequency obtained by different methods when the sample size is 25.

Methods	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
Asymp.	0.90 (0.087)	0.93 (0.059)	0.95 (0.052)	0.97 (0.004)
Boot-p	0.93 (0.601)	0.91 (0.286)	0.90 (0.116)	0.90 (0.060)
Boot-t	0.94 (1.041)	0.92 (0.434)	0.90 (0.183)	0.91 (0.087)

In each box, the first figure represents the coverage percentages and the corresponding average confidence lengths are reported within brackets next to it.

of the amplitudes. Comparing the LSEs and the ALSEs, it seems although they are asymptotically equivalent, LSEs behave marginally better than the ALSEs in terms of the minimum MADs for most of the cases considered and for all the parameters. But computationally ALSEs are much easier to compute than the LSEs at least for large p. Therefore, if p is large we recommend to use the ALSEs but if p is small LSEs are preferable.

From Table 4, some of the points are very clear. For all the methods as α increases the length of the confidence intervals decrease and also as the sample size increases the length of the confidence intervals decrease (not reported here). For both the Bootstrap methods the coverage percentages gradually decrease but for the asymptotic method the coverage percentages gradually increase as α increases. Between the Bootstrap confidence intervals for fixed α, Boot-t confidence intervals have higher coverage probability compared to Boot-p confidence intervals and also the length of the Boot-t confidence intervals are larger than those of Boot-p. For both the Bootstrap procedures the coverage percentages generally vary between 90% and 94%. On the other hand the coverage percentages of the asymptotic method vary between 90% and 97%, although the length of the confidence intervals are much smaller than the corresponding Bootstrap confidence intervals. Moreover, to compute the asymptotic confidence intervals, we need to know the value of α, which is not required to compute the Bootstrap confidence intervals. Comparing all the points, we recommend to use the Boot-p confidence bounds for the unknown parameters if α is not known and if α is known and it is close to 2, we should use asymptotic confidence bounds. As one of the referee has suggested, we are providing a comparison between the simulation based distribution and the bootstrap distributions based on histograms. The histograms are based on one thousand replications. We are providing the results when $\alpha = 1.8$, others are quite similar in nature so they are not provided here. We provide three histograms, in Figs. 4–6. Fig. 4 represents simulation-based histogram and Figs. 5 and 6 represent the histograms based on Boot-t and Boot-p, respectively. One point is very clear that the shapes are quite similar in nature but the dispersions are less for the bootstrap samples.

We provide a graph (Fig. 1) of a particular realization of the model (5.1) with $\alpha = 1.5$ and $\sigma = 0.5$. From the plot it may not be very clear that the data has infinite variance, but if we look at the plot (Fig. 3) n vs. $\mathrm{Var}\{y(1),\ldots,y(n)\}$, it clearly gives an indication that the variance is not finite. We plot the periodogram function in Fig. 2 and from the periodogram function it is clear that $p = 1$. We estimate the different parameters and also obtain the 95% confidence bounds for all the parameters. They are provided below. $\hat{A} = 1.73579$, $\hat{B} = 1.19572$, $\hat{\omega} = 2.49620$,

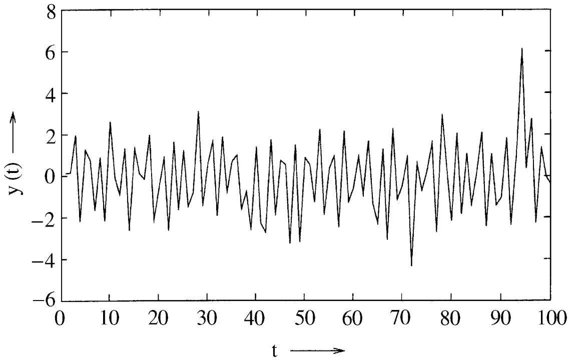

Fig. 1.

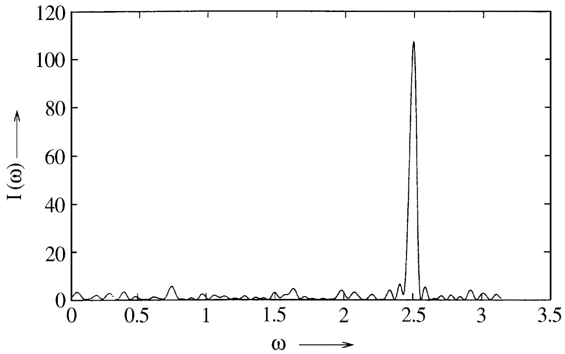

Fig. 2.

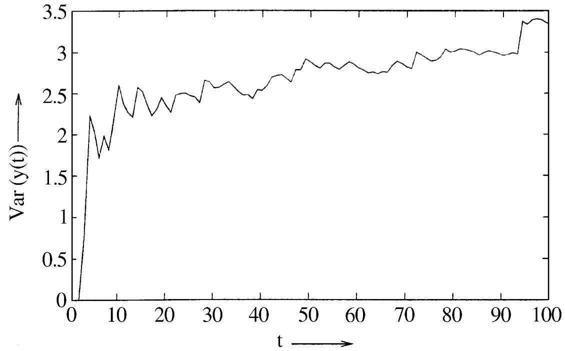

Fig. 3.

$\tilde{A} = 1.70866$, $\tilde{B} = 1.17192$, $\tilde{\omega} = 2.49605$. Using the LSEs the Boot-p confidence intervals for A, B and ω are $(1.3296, 2.1733)$, $(0.6515, 1.7162)$ and $(2.4916, 2.5007)$, respectively. Similarly, using ALSEs the corresponding confidence intervals are $(1.1880, 2.1385)$, $(0.5437, 1.6998)$ and $(2.4908, 2.5009)$.

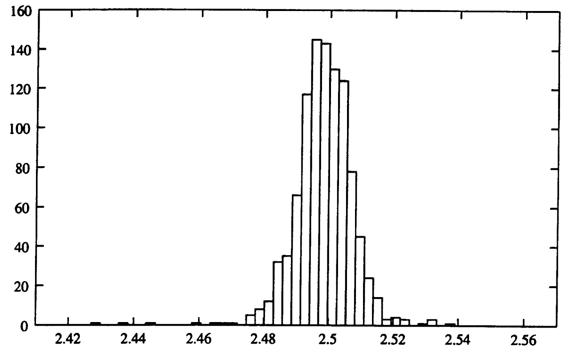

Fig. 4.

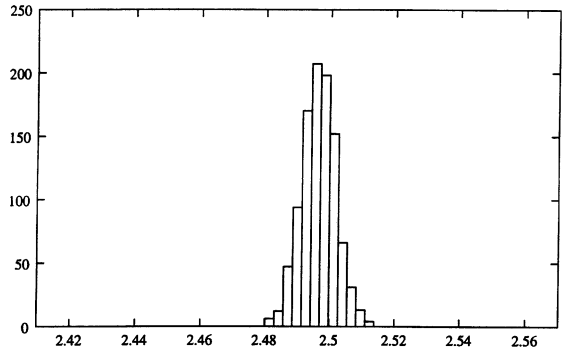

Fig. 5.

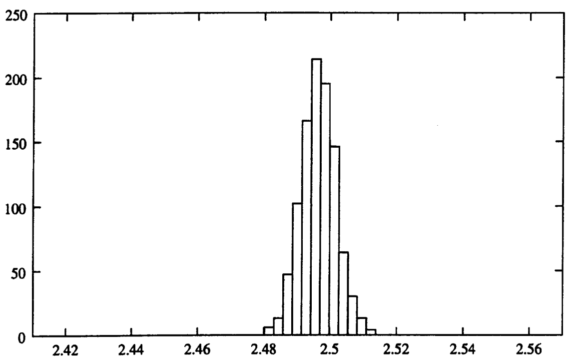

Fig. 6.

6. Conclusions

In this paper, we consider the sum of sinusoidal model under the assumptions of additive heavy tail i.i.d. errors. Although, we considered only i.i.d random variables but the results can be extended even when the errors are moving average type. One important question we did not address in this paper, namely estimation of p, the number of sinusoidal components. We may need to use some information theoretic criteria to estimate p. More work is needed in this direction.

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Appendix A

To prove Theorem 1, we need the following lemmas.

Lemma 1. *Let us denote* $S_{c,M} = \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \omega), |\boldsymbol{\theta} - \boldsymbol{\theta}_0| \geqslant c, |A| \leqslant M, |B| \leqslant M\}$. *Suppose* $e(t)$*'s are i.i.d. random variables with mean zero, if for any* $c > 0$ *and for some* $M < \infty$,

$$\underline{\lim} \inf_{\theta \in S_{c,M}} \frac{1}{n}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}_0)] > 0 \quad a.s.$$

then $\hat{\boldsymbol{\theta}}$ *is a strongly consistent estimator of* $\boldsymbol{\theta}_0$.

Proof. The proof is a simple extension of the result of Wu (1981), so it is omitted.

Lemma 2. *If* $X_1, X_2 \ldots$ *are i.i.d. random variables with mean zero and* $E|X_i|^{1+\delta} < \infty$ *for* $0 < \delta < 1$, *then*

$$\lim_{n\to\infty} \sup_{0\leqslant\theta\leqslant 2\pi} \frac{1}{n}\sum_{t=1}^{n} X_t \cos(t\theta) = \lim_{n\to\infty} \sup_{0\leqslant\theta\leqslant 2\pi} \frac{1}{n}\sum_{t=1}^{n} X_t \sin(t\theta) = 0 \quad a.s.$$

Proof. We prove the result for $\cos(t\theta)$, the result for $\sin(t\theta)$ follows similarly. Let $Z_t = X_t I_{[|X_t| \leqslant t^{1/(1+\delta)}]}$. Then

$$\sum_{t=1}^{\infty} P[Z_t \neq X_t] = \sum_{t=1}^{\infty} P[|X_t| > [t^{1/(1+\delta)}] = \sum_{t=1}^{\infty} \sum_{2^{t-1} \leqslant n < 2^t} P[|X_1| > n^{1/(1+\delta)}]$$

$$\leqslant \sum_{t=1}^{\infty} 2^t P[2^{(t-1)/(1+\delta)} \leqslant |X_1|] \leqslant \sum_{t=1}^{\infty} 2^t \sum_{j=t}^{\infty} P[2^{(j-1)/(1+\delta)} \leqslant |X_1| < 2^{j/(1+\delta)}]$$

$$\leqslant \sum_{j=1}^{\infty} P[2^{(j-1)/(1+\delta)} \leqslant |X_1| < 2^{j/(1+\delta)}] \sum_{t=1}^{j} 2^t$$

$$\leqslant C\sum_{j=1}^{\infty}2^{j-1}P[2^{(j-1)/(1+\delta)} \leqslant |X_1| < 2^{j/(1+\delta)}]$$

$$\leqslant C\sum_{j=1}^{\infty} E|X_1|^{1+\delta}I_{[2^{(j-1)/(1+\delta)} \leqslant |X_1| < 2^{j/(1+\delta)}]} \leqslant CE|X_1|^{1+\delta} < \infty.$$

Therefore $P[Z_t \neq X_t \text{ i.o.}] = 0$. Thus,

$$\sup_{0\leqslant\theta\leqslant 2\pi}\frac{1}{n}\sum_{t=1}^{n}X_t\cos(t\theta) \to 0 \quad \text{a.s} \quad \Leftrightarrow \quad \sup_{0\leqslant\theta\leqslant 2\pi}\frac{1}{n}\sum_{t=1}^{n}Z_t\cos(t\theta) \to 0 \quad \text{a.s.}$$

Let $U_t = Z_t - E(Z_t)$, note that

$$\sup_{0\leqslant\theta\leqslant 2\pi}\left|\frac{1}{n}\sum_{t=1}^{n}E(Z_t)\cos(t\theta)\right| \leqslant \frac{1}{n}\sum_{t=1}^{n}|E(Z_t)| = \frac{1}{n}\sum_{t=1}^{n}\left|\int_{-t^{1/(1+\delta)}}^{t^{1/(1+\delta)}} x\,\mathrm{d}F(x)\right| \to 0.$$

Thus, we only need to show that

$$\sup_{0\leqslant\theta\leqslant 2\pi}\frac{1}{n}\sum_{t=1}^{n}U_t\cos(t\theta) \to 0 \quad \text{a.s.} \tag{A.1}$$

For any fixed θ and $\varepsilon > 0$, let $0 \leqslant h \leqslant \frac{1}{2n^{1/(1+\delta)}}$, then we have

$$P\left\{\left|\frac{1}{n}\sum_{t=1}^{n}U_t\cos(t\theta)\right| \geqslant \varepsilon\right\} \leqslant 2\mathrm{e}^{-hn\varepsilon}\prod_{t=1}^{n}E\mathrm{e}^{hU_t\cos(t\theta)} \leqslant 2\mathrm{e}^{-hn\varepsilon}\prod_{t=1}^{n}(1+2Ch^{1+\delta}).$$

Since $|hU_t\cos(t\theta)| \leqslant \frac{1}{2}$, $\mathrm{e}^x \leqslant 1 + x + 2|x|^{1+\delta}$ for $|x| \leqslant \frac{1}{2}$ and $E|U_t|^{1+\delta} < C$ for some $C > 0$. Clearly,

$$2\mathrm{e}^{-hn\varepsilon}\prod_{t=1}^{n}(1+2Ch^{1+\delta}) \leqslant 2\mathrm{e}^{-hn\varepsilon+2nCh^{1+\delta}}.$$

Choose $h = \frac{1}{2n^{1/(1+\delta)}}$, then for large n,

$$P\left\{\left|\frac{1}{n}\sum_{t=1}^{n}U_t\cos(t\theta)\right| \geqslant \varepsilon\right\} \leqslant 2\mathrm{e}^{-(\varepsilon/2)n^{\delta/(1+\delta)}+C} \leqslant C\mathrm{e}^{-(\varepsilon/2)n^{\delta/(1+\delta)}}.$$

Let $K = n^2$, choose $\theta_1, \ldots, \theta_K$, such that for each $\theta \in (0, 2\pi)$, we have a θ_j satisfying $|\theta_j - \theta| \leqslant 2\pi/n^2$. Note that

$$\left|\frac{1}{n}\sum_{t=1}^{n}U_t(\cos(t\theta) - \cos(t\theta_j))\right| \leqslant C\frac{1}{n}\sum_{t=1}^{n}t^{1/(1+\delta)}t\left(\frac{2\pi}{n^2}\right) \leqslant C\pi n^{-\delta/(1+\delta)} \to 0.$$

Therefore, for large n, we have

$$P\left\{\sup_{0\leqslant\theta\leqslant 2\pi}\left|\frac{1}{n}\sum_{t=1}^{n}U_t\cos(t\theta)\right|\geqslant 2\varepsilon\right\}\leqslant P\left\{\max_{j\leqslant n^2}\left|\frac{1}{n}\sum_{t=1}^{n}U_t\cos(t\theta_j)\right|\geqslant\varepsilon\right\}\leqslant Cn^2\mathrm{e}^{-(\varepsilon/2)n^{\delta/(1+\delta)}}.$$

Since $\sum_{n=1}^{\infty}n^2\mathrm{e}^{-(\varepsilon/2)n^{\delta/(1+\delta)}}<\infty$, therefore (A.1) is proved by using Borel Cantelli lemma. □

Now with the help of Lemmas 1 and 2, we will prove Theorem 1.

Proof of Theorem 1. In this proof only we denote $\hat{\boldsymbol\theta}$ by $\hat{\boldsymbol\theta}_n=(\hat A_n,\hat B_n,\hat\omega_n)$ to emphasize that $\hat{\boldsymbol\theta}$ depends on the sample size. If $\hat{\boldsymbol\theta}_n$ is not consistent for $\boldsymbol\theta_0$, then either

Case I: for all sub-sequences $\{n_k\}$ of $\{n\}, |\hat A_{n_k}|+|\hat B_{n_k}|\to\infty$. Then

$$\frac{1}{n_k}\left[Q(\hat{\boldsymbol\theta}_{n_k})-Q(\boldsymbol\theta_0)\right]\to\infty.$$

But as $\hat{\boldsymbol\theta}_{n_k}$ is the LSE of $\boldsymbol\theta_0$ at $n=n_k$,

$$Q(\hat{\boldsymbol\theta}_{n_k})-Q(\boldsymbol\theta_0)<0$$

which leads to a contradiction. So $\hat{\boldsymbol\theta}_n$ is consistent estimator for $\boldsymbol\theta_0$.

Case II: for at least one sub-sequence $\{n_k\}$ of $\{n\}$, $\hat{\boldsymbol\theta}_{n_k}\in S_{c,M}$ for some $c>0$ and a $0<M<\infty$. Let us write $S_{c,M}$, as defined in Lemma 1 as follows:

$$S_{c,M}=\{\boldsymbol\theta:\boldsymbol\theta=(A,B,\omega),|\boldsymbol\theta-\boldsymbol\theta_0|\geqslant 3c,|A|\leqslant M,|B|\leqslant M\}=A_c\cup B_c\cup W_c,$$

where

$$A_c=\{\boldsymbol\theta:\boldsymbol\theta=(A,B,\omega),|A-A_0|\geqslant c,|A|\leqslant M,|B|\leqslant M\},$$

$$B_c=\{\boldsymbol\theta:\boldsymbol\theta=(A,B,\omega),|B-B_0|\geqslant c,|A|\leqslant M,|B|\leqslant M\},$$

$$W_c=\{\boldsymbol\theta:\boldsymbol\theta=(A,B,\omega),|\omega-\omega_0|\geqslant c,|A|\leqslant M,|B|\leqslant M\}.$$

Consider

$$\begin{aligned}\frac{1}{n}[Q(\boldsymbol\theta)-Q(\boldsymbol\theta_0)]&=\frac{1}{n}\sum_{t=1}^{n}[\{(y(t)-A\cos(\omega t)-B\sin(\omega t))\}^2-e(t)^2]\\&=\frac{1}{n}\sum_{t=1}^{n}\{A_0\cos(\omega_0 t)+B_0\sin(\omega_0 t)-A\cos(\omega t)-B\sin(\omega t)\}^2\\&\quad+\frac{2}{n}\sum_{t=1}^{n}e(t)\{A_0\cos(\omega_0 t)+B_0\sin(\omega_0 t)-A\cos(\omega t)-B\sin(\omega t)\}\\&=f_n(\boldsymbol\theta)+g_n(\boldsymbol\theta)\quad\text{(say)}.\end{aligned}\tag{A.2}$$

Using Lemma 2, we get that

$$\lim_{n\to\infty} \sup_{\theta\in S_{c,M}} g_n(\boldsymbol{\theta}) = 0 \quad \text{a.s.} \tag{A.3}$$

Now for any $c > 0$ and a fixed $0 < M < \infty$,

$$\begin{aligned}
\underline{\lim} \inf_{\theta\in A_c} f_n(\boldsymbol{\theta}) &= \underline{\lim} \inf_{\theta\in A_c} \frac{1}{n}\sum_{t=1}^{n}(A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) - A\cos(\omega t) - B\sin(\omega t))^2 \\
&= \underline{\lim} \inf_{\theta\in A_c} \frac{1}{n}\sum_{t=1}^{n}[(A_0\cos(\omega_0 t) - A\cos(\omega t))^2 + (B_0\sin(\omega_0 t) - B\sin(\omega t))^2 \\
&\quad + 2(A_0\cos(\omega_0 t) - A\cos(\omega t))(B_0\sin(\omega_0 t) - B\sin(\omega t))] \\
&= \underline{\lim} \inf_{|A-A_0|>c} \frac{1}{n}\sum_{t=1}^{n}(A_0\cos(\omega_0 t) - A\cos(\omega_0 t))^2 \\
&= \inf_{|A-A_0|>c} \tfrac{1}{2}(A_0 - A)^2 \geqslant \tfrac{1}{2}c^2 > 0 \quad \text{a.s.}
\end{aligned} \tag{A.4}$$

Similarly, it can be proved for B_c and W_c also. Thus, we have

$$\underline{\lim} \inf_{\theta\in S_{c,M}} f_n(\boldsymbol{\theta}) > 0 \quad \text{a.s.} \tag{A.5}$$

Now using Eqs. (A.3) and (A.5) in Eq. (A.2) and using Lemma 1, Theorem 1 follows immediately.

We need the following lemmas to prove Theorem 2.

Lemma 3. *Let $\tilde{\omega}$ be an estimator defined in Section 2 and let for any $\varepsilon > 0$, $S_\varepsilon = \{\omega : |\omega - \omega_0| \geqslant \varepsilon\}$ for some fixed $\omega_0 \in (0, 2\pi)$. If for any $\varepsilon > 0$*

$$\overline{\lim} \sup_{S_\varepsilon} \frac{1}{n}[I_n(\omega) - I_n(\omega_0)] < 0 \quad a.s. \tag{A.6}$$

then $\tilde{\omega} \to \omega_0$ a.s. as $n \to \infty$.

Proof. Lemma 3 can be obtained similarly as Lemma 1.

Lemma 4. *Let $\{e(t)\}$ be i.i.d. random variables with mean zero and $E|e(t)|^{1+\delta} < \infty$ for some $0 < \delta < 1$, then the estimator $\tilde{\omega}$ of ω_0 as obtained by maximizing* (2.3) *is a strongly consistent estimator of ω_0.*

Proof. Consider

$$\begin{aligned}
&\overline{\lim} \sup_{\omega\in S_\varepsilon} \frac{1}{n}[I_n(\omega) - I_n(\omega_0)] \\
&= \overline{\lim} \sup_{\omega\in S_\varepsilon} \frac{1}{n}\left[\frac{2}{n}\left|\sum_{t=1}^{n} y(t)\mathrm{e}^{-\mathrm{i}\omega t}\right|^2 - \frac{2}{n}\left|\sum_{t=1}^{n} y(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}\right|^2\right]
\end{aligned}$$

$$=2\overline{\lim}\sup_{\omega\in S_\varepsilon}\left[\left(\frac{A_0}{n}\sum_{t=1}^{n}\cos(\omega_0 t)\cos(\omega t)\right)^2+\left(\frac{B_0}{n}\sum_{t=1}^{n}\sin(\omega_0 t)\sin(\omega t)\right)^2\right.$$

$$\left.-\left(\frac{A_0}{n}\sum_{t=1}^{n}\cos^2(\omega_0 t)\right)^2-\left(\frac{B_0}{n}\sum_{t=1}^{n}\sin^2(\omega_0 t)\right)^2\right]$$

[using the trigonometric identity, the first two components go to zero].

$$=2A_0^2\left(-\frac{1}{4}-\frac{1}{4}\right)=-\frac{1}{2}A_0^2<0 \quad \text{a.s}$$

Therefore, because of Lemma 3, the result follows. □

Lemma 5. *If $X_1, X_2, \ldots$ are i.i.d. random variables with mean zero and $E|X_1|^{1+\delta}<\infty$ for some $0<\delta<1$, then*

$$\lim_{n\to\infty}\sup_{0\leqslant\theta\leqslant 2\pi}\frac{1}{n^{k+1}}\sum_{t=1}^{n}t^k X_t\cos(t\theta)=\lim_{n\to\infty}\sup_{0\leqslant\theta\leqslant 2\pi}\frac{1}{n^{k+1}}\sum_{t=1}^{n}t^k X_t\sin(t\theta)=0 \quad a.s.$$

for $k=1,2,\ldots$.

Proof. Lemma 5 can be obtained similarly as Lemma 2.

Lemma 6. *Under the same conditions as of Lemma 4, $n(\tilde{\omega}-\omega_0)\to 0$, a.s.*

Proof. Let $I_n'(\omega)$ and $I_n''(\omega)$ be the first and second derivatives of $I_n(\omega)$. Expanding $I_n'(\tilde{\omega})$ around ω_0 by Taylor Series

$$I_n'(\tilde{\omega})-I_n'(\omega_0)=(\tilde{\omega}-\omega_0)I_n''(\bar{\omega}),$$

where $\bar{\omega}$ is a point between $\tilde{\omega}$ and ω_0. As $\tilde{\omega}$ maximizes $I_n(\omega)$, so $I_n'(\tilde{\omega})=0$ which implies

$$(\tilde{\omega}-\omega_0)=-\frac{I_n'(\omega_0)}{I_n''(\bar{\omega})}\Rightarrow n(\tilde{\omega}-\omega_0)=-\frac{(1/n^2)I_n'(\omega_0)}{(1/n^3)I_n''(\bar{\omega})}. \tag{A.7}$$

Note that

$$\lim_{n\to\infty}\frac{1}{n^3}I_n''(\omega_0)<0 \quad a.s. \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n^2}I_n'(\omega_0)=0 \quad a.s. \tag{A.8}$$

Therefore, using (A.8) in (A.7), along with the fact that $\tilde{\omega}\to\omega^0$ a.s., the lemma follows. □

Lemma 7. *Under the same conditions as of Lemma 4, $\tilde{A}$ and $\tilde{B}$ are strongly consistent estimators of A_0 and B_0, respectively.*

Proof. Let us expand using Taylor series, $\cos(\tilde{\omega}t)$ around ω_0 up to the first order term. Suppose $\bar{\omega}$ is a point such $\bar{\omega}t$ lies between ωt and $\tilde{\omega}t$. Note that $\bar{\omega}$ may depend on t. Now $\tilde{A}=\frac{2}{n}\sum_{t=1}^{n}y(t)\cos(\tilde{\omega}t)$

can be written as

$$\tilde{A} = \frac{2}{n}\sum_{t=1}^{n}\{A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) + e(t)\}\{\cos(\omega_0 t) + t(\tilde{\omega} - \omega_0)\sin(\bar{\omega} t)\}$$

$$= \frac{2A_0}{n}\sum_{t=1}^{n}\cos^2(\omega_0 t) - 2A_0 n(\tilde{\omega} - \omega_0)\left[\frac{1}{n^2}\sum_{t=1}^{n} t\cos(\omega_0 t)\sin(\bar{\omega} t)\right]$$

$$+ \frac{2B_0}{n}\sum_{t=1}^{n}\sin(\omega_0 t)\cos(\omega_0 t) - 2B_0 n(\tilde{\omega} - \omega_0)\left[\frac{1}{n^2}\sum_{t=1}^{n} t\sin(\omega_0 t)\sin(\bar{\omega} t)\right]$$

$$+ \frac{2}{n}\sum_{t=1}^{n} e(t)\cos(\omega_0 t) - 2n(\tilde{\omega} - \omega_0)\left[\frac{1}{n^2}\sum_{t=1}^{n} te(t)\sin(\bar{\omega} t)\right] \to A_0 \quad \text{a.s.} \tag{A.9}$$

Note that the second, fourth and sixth terms of (A.9) converge to zero using Lemmas 6 and 5, third term vanishes because of the trigonometries identity and the fifth term vanishes because of Lemma 2. Similarly it can be shown that $\tilde{B}$ is a consistent estimator of B. □

Proof of Theorem 2. Combining Lemmas 4 and 7, the result follows.

Appendix B

The proof that $\frac{1}{n}\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^{\alpha}$ converges to a non-zero limit for $\mathbf{t} \neq \mathbf{0}$. Note that

$$|K_{\mathbf{t}}(j)| \leqslant |t_1| + |t_2| + |t_3|(A_0 + B_0) = M \quad (say)$$

for all j and n, $1 \leqslant j \leqslant n$, $n = 1, 2, \ldots$. Thus $|K_{\mathbf{t}}(j)/M| \leqslant 1$, hence $|K_{\mathbf{t}}(j)|^{\alpha} \geqslant (M^{\alpha}/M^2)|K_{\mathbf{t}}(j)|^2$ for $0 < \alpha \leqslant 2$ and for all $j = 1, 2 \ldots$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^{\alpha} \geqslant \lim_{n\to\infty}\frac{M^{\alpha-2}}{n}\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^2.$$

Using

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\cos^2(j\omega_0) = \frac{1}{2} > 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\cos(j\omega_0) = 0,$$

it easily follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|K_{\mathbf{t}}(j)|^2 > 0.$$

It proves the result.

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