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Modified moment estimation for the two-parameter Birnbaum–Saunders distribution

H.K.T. Ng[a], D. Kundu[b], N. Balakrishnan[a,*]

[a] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ont., Canada L8S 4K1*

[b] *Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur 208016, India*

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Abstract

The maximum likelihood estimators and a modification of the moment estimators of a two-parameter Birnbaum–Saunders distribution are discussed. A simple bias-reduction method is proposed to reduce the bias of the maximum likelihood estimators and the modified moment estimators. The jackknife technique is also used to reduce the bias of these estimators. Monte Carlo simulation is used to compare the performance of all these estimators. The probability coverages of confidence intervals based on inferential quantities associated with all these estimators are evaluated using Monte Carlo simulations for small, moderate and large sample sizes. Two illustrative examples and some concluding remarks are finally presented.

Keywords: Birnbaum–Saunders distribution; Jackknife estimate; Bias-reduced estimator; Monte Carlo simulations; Probability coverage; Confidence interval

1. Introduction

The two-parameter Birnbaum–Saunders distribution was originally proposed (Birnbaum and Saunders, 1969a) as a failure time distribution for fatigue failure caused under cyclic loading. It was also assumed that the failure is due to the development and growth of a dominant crack. A more general derivation was provided by Desmond (1985) based on a biological model. Desmond (1985) also strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders (1969a). Desmond (1986) investigated the relationship between the Birnbaum–Saunders distribution and the inverse Gaussian distribution. Some

* Corresponding author.

E-mail address: bala@mcmail.cis.mcmaster.ca (N. Balakrishnan).

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recent works on Birnbaum–Saunders distribution can be found in Chang and Tang (1993, 1994), Dupuis and Mills (1998) and Rieck (1995, 1999), and a review of these developments can be found in Johnson et al. (1995).

The cumulative distribution function (CDF) of a two-parameter Birnbaum–Saunders random variable T can be written as

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{\beta}{t}\right)^{1/2}\right\}\right], \quad 0 < t < \infty, \quad \alpha,\beta > 0, \tag{1}$$

where $\Phi(\cdot)$ is the standard normal CDF. The parameters α and β are the shape and the scale parameters, respectively. It is known that the density function of the Birnbaum–Saunders distribution is unimodal and although the hazard rate is not an increasing function of t but the average hazard rate is nearly a non-decreasing function of t (Mann et al., 1974, p. 155). The maximum likelihood estimators (MLEs) were discussed originally by Birnbaum and Saunders (1969b) and their asymptotic distributions were obtained by Engelhardt et al. (1981).

Although the MLEs have several optimal properties, one still needs to solve a non-linear equation in β to obtain the solution; for this purpose, Birnbaum and Saunders (1969b) suggested some iterative schemes to solve the required non-linear equation. Also, the exact distribution of the MLEs are not available. Therefore, for constructing confidence intervals of the unknown parameters α and β, the asymptotic distributions of the MLEs need to be used. However, it is not known how these asymptotic confidence intervals behave in the case of small sample sizes. Moreover, the conventional moment estimators also have a difficulty in that they may not always exist and even if they do, they may not be unique.

For this reason, modified moment estimators (MMEs) for α and β are first proposed. The MMEs are very easy to compute and they have explicit expressions in terms of the sample observations. Unlike the moment estimators, MMEs always exist uniquely. The asymptotic distributions of the MMEs are derived which are then used to construct confidence intervals for the unknown parameters. The performance of all these estimators is evaluated through simulations. Even though the MLEs and MMEs are asymptotically unbiased, these simulation results reveal that they are highly biased in case of small sample sizes. A simple bias correction technique is proposed which performs quite well even for small sample sizes. Jackknife procedure is also used to reduce the bias of the MLEs and MMEs and is shown to work very well in this case; but, this procedure becomes computationally quite involved in case of large sample sizes.

2. The Birnbaum–Saunders distribution

The CDF of a two-parameter Birnbaum–Saunders random variable T is given by (1) and the corresponding probability density function (PDF) is

$$f_T(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right],$$

$$t > 0, \alpha, \beta > 0. \tag{2}$$

Consider the following monotone transformation:

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right]$$

or

$$T = \beta(1 + 2X^2 + 2X(1 + X^2)^{1/2});$$

then, from (1), we have X to be a normally distributed with mean zero and variance $\frac{1}{4}\alpha^2$. Using the above transformation, the expected value, variance, and coefficients of skewness and kurtosis can be easily obtained as

$$E(T) = \beta(1 + \tfrac{1}{2}\alpha^2), \tag{3}$$

$$Var(T) = (\alpha\beta)^2(1 + \tfrac{5}{4}\alpha^2), \tag{4}$$

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}, \tag{5}$$

$$\beta_2(T) = 3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}. \tag{6}$$

Moreover, if T has a Birnbaum–Saunders distribution with parameters α and β, then T^{-1} also has a Birnbaum–Saunders distribution with the corresponding parameters α and β^{-1}, respectively (Birnbaum and Saunders, 1969a). Therefore, we also readily have

$$E(T^{-1}) = \beta^{-1}(1 + \tfrac{1}{2}\alpha^2) \tag{7}$$

and

$$Var(T^{-1}) = \alpha^2\beta^{-2}(1 + \tfrac{5}{4}\alpha^2). \tag{8}$$

3. Maximum likelihood estimators

Let $\{t_1, t_2, \ldots, t_n\}$ be a random sample of size n from the Birnbaum–Saunders distribution with the PDF as given in (2). The sample arithmetic and harmonic means are defined by

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

Let us further define the harmonic mean function K by

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1} \qquad \text{for } x \geqslant 0,$$

so that $r \equiv K(0)$.

The MLE of β (denoted by $\hat{\beta}$) can be obtained as the unique positive root of the equation

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0. \tag{9}$$

Once $\hat{\beta}$ is obtained as a solution of (9), the MLE of α (denoted by $\hat{\alpha}$) can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2\right]^{1/2}.$$

Since (9) is a non-linear equation in β, one needs to use an iterative procedure to solve for $\hat{\beta}$. Birnbaum and Saunders (1969b) proposed two iterative procedures (one simple and one complicated) to compute $\hat{\beta}$, but noted that the simple one works very well for small α ($< \frac{1}{2}$) but may not work at all for large α (> 2). The complicated one also does not work in certain range of the sample space. In this paper, Newton–Raphson method is used for the computation of the MLEs.

Engelhardt et al. (1981) showed that the asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{\alpha^2}{2n} & 0 \\ 0 & \frac{\beta^2}{n[0.25+\alpha^{-2}+I(\alpha)]} \end{pmatrix}\right], \tag{10}$$

where

$$I(\alpha) = 2\int_0^\infty \{[1 + g(\alpha x)]^{-1} - 0.5\}^2 \, \mathrm{d}\Phi(x),$$

$$g(y) = 1 + \frac{y^2}{2} + y\left(1 + \frac{y^2}{4}\right)^{1/2}.$$

It is interesting to observe that $\hat{\alpha}$ and $\hat{\beta}$ are asymptotically independent of each other. The asymptotic confidence interval of α can be easily obtained from (10). Moreover, the asymptotic confidence interval of β, for a given α, can also be obtained from (10).

4. Modified moment estimators

For the usual moment estimators in a two-parameter case, the first and second population moments are equated with the corresponding sample moments. In this case, the sample mean and the sample variance can be equated to the right-hand sides of (3) and (4), respectively, and the corresponding moment estimators of α and β can then be obtained as solutions of α and β to these equations. It can be easily seen from these equations that if the sample coefficient of variation is greater than $\sqrt{5}$, then the moment estimators do not exist. If the sample coefficient of variation is less than $\sqrt{5}$, the moment estimators exist; however, the moment estimator of β may not be unique.

Instead of using (3) and (4), we propose to use (3) and (7) and equate them with the corresponding sample estimates to obtain the MMEs. In this case, we have the following two moment equations:

$$s = \beta\left(1 + \tfrac{1}{2}\alpha^2\right), \tag{11}$$

$$r^{-1} = \beta^{-1}(1 + \tfrac{1}{2}\alpha^2). \tag{12}$$

Solving Eqs. (11) and (12) for α and β, we obtain the MMEs for α and β denoted by $\tilde{\alpha}$ and $\tilde{\beta}$ as

$$\tilde{\alpha} = \left\{2\left[\left(\frac{s}{r}\right)^{1/2} - 1\right]\right\}^{1/2},$$

$$\tilde{\beta} = (sr)^{1/2}.$$

The asymptotic joint distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ is bivariate normal and is given by

$$\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\frac{\alpha^2}{2n} & 0\\ 0 & \frac{(\alpha\beta)^2}{n}\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right)\end{pmatrix}\right]. \tag{13}$$

The proof of this result is presented in the appendix. Note that the MMEs $\tilde{\alpha}$ and $\tilde{\beta}$ are also asymptotically independent of each other, just as in the case of the MLEs.

5. Bias-reduced estimators

Based on the results of an extensive Monte Carlo simulation study, we observed that the MLEs and the MMEs performed very similarly in terms of both bias and mean square error, especially for small values of α. Upon inspecting the pattern of the bias of the MLEs and MMEs, we observed that

$$Bias(\hat{\alpha}) \approx Bias(\tilde{\alpha}) \approx -\frac{\alpha}{n},$$

$$Bias(\hat{\beta}) \approx Bias(\tilde{\beta}) \approx \frac{\alpha^2}{4n}.$$

Then, by employing a standard bias reduction method, we can simply construct almost unbiased maximum likelihood estimators (UMLEs, denoted by $\hat{\alpha}^*$ and $\hat{\beta}^*$) and almost unbiased modified moment estimators (UMMEs, denoted by $\tilde{\alpha}^*$ and $\tilde{\beta}^*$) of α and β. These bias-reduced estimators are given by

$$\hat{\alpha}^* = \left(\frac{n}{n-1}\right)\hat{\alpha}, \quad \hat{\beta}^* = \left(1 + \frac{\alpha^2}{4n}\right)^{-1}\hat{\beta} \approx \left(1 + \frac{\hat{\alpha}^{*2}}{4n}\right)^{-1}\hat{\beta}, \tag{14}$$

$$\tilde{\alpha}^* = \left(\frac{n}{n-1}\right)\tilde{\alpha}, \quad \tilde{\beta}^* = \left(1 + \frac{\alpha^2}{4n}\right)^{-1}\tilde{\beta} \approx \left(1 + \frac{\tilde{\alpha}^{*2}}{4n}\right)^{-1}\tilde{\beta}. \tag{15}$$

From the distributional results presented in (10), we readily have the asymptotic joint distribution of $\hat{\alpha}^*$ and $\hat{\beta}^*$ to be bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha}^* \\ \hat{\beta}^* \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{n\alpha^2}{2(n-1)^2} & 0 \\ 0 & \frac{16n\beta^2}{(4n+\alpha^2)^2[0.25+\alpha^{-2}+I(\alpha)]} \end{pmatrix}\right]; \tag{16}$$

similarly, from (13), the asymptotic joint distribution of $\tilde{\alpha}^*$ and $\tilde{\beta}^*$ is bivariate normal and is given by

$$\begin{pmatrix} \tilde{\alpha}^* \\ \tilde{\beta}^* \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{n\alpha^2}{2(n-1)^2} & 0 \\ 0 & \frac{16n(\alpha\beta)^2}{(4n+\alpha^2)^2}\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{pmatrix}\right]. \tag{17}$$

6. Jackknife estimators

Jackknifing is based on sequentially deleting one sample point t_i and recomputing the MLEs and MMEs from the reduced sample of size $n-1$. We remove the point t_j from the data set, and then recompute r and s and also the function K as

$$s_{(j)} = \frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} t_i = \frac{ns - t_j}{n-1},$$

$$r_{(j)} = \left[\frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} t_i^{-1}\right]^{-1} = \frac{nr - t_j^{-1}}{n-1},$$

$$K_{(j)}(x) = \left[\frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} (x+t_i)^{-1}\right]^{-1} = \frac{nK(x) - (x+t_j)^{-1}}{n-1}.$$

Then, we obtain $\hat{\beta}_{(j)}$ as the unique positive root of the equation

$$\beta^2 - \beta[2r_{(j)} + K_{(j)}(\beta)] + r_{(j)}[s_{(j)} + K_{(j)}(\beta)] = 0$$

and

$$\hat{\alpha}_{(j)} = \left[\frac{s_{(j)}}{\hat{\beta}_{(j)}} + \frac{\hat{\beta}_{(j)}}{r_{(j)}} - 2\right]^{1/2};$$

Similarly, we find

$$\tilde{\alpha}_{(j)} = \left\{2\left[\left(\frac{s_{(j)}}{r_{(j)}}\right)^{1/2} - 1\right]\right\}^{1/2},$$

$$\tilde{\beta}_{(j)} = (s_{(j)} r_{(j)})^{1/2}.$$

Let us now define

$$\hat{\alpha}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}_{(j)}, \qquad \hat{\beta}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\hat{\beta}_{(j)},$$

$$\tilde{\alpha}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\tilde{\alpha}_{(j)}, \qquad \tilde{\beta}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\tilde{\beta}_{(j)}.$$

Then, the bias-corrected jackknife maximum likelihood estimates (JMLEs) of α and β (see, for example, Efron, 1982) are given by

$$\hat{\alpha}_J = n\hat{\alpha} - (n-1)\hat{\alpha}_{(\cdot)},$$

$$\hat{\beta}_J = n\hat{\beta} - (n-1)\hat{\beta}_{(\cdot)};$$

similarly, the bias-corrected jackknife modified moment estimates (JMMEs) of α and β are given by

$$\tilde{\alpha}_J = n\tilde{\alpha} - (n-1)\tilde{\alpha}_{(\cdot)},$$

$$\tilde{\beta}_J = n\tilde{\beta} - (n-1)\tilde{\beta}_{(\cdot)}.$$

7. Monte Carlo simulation results

In order to compare the performance of all the above estimators, we performed a simulation study for different sample sizes and for different parameter values. We took the sample size as $n=5, 10, 20, 50, 100$, and the shape parameter as $\alpha=0.10, 0.25, 0.50, 1.00, 2.00$. Since β is the scale parameter, β was kept fixed at 1.0, without loss of any generality. Of course, the values of bias and standard deviation of estimates of β simply need to be multiplied by β (if it is different from 1) while the bias and standard deviation of estimates of α will not be affected. All the results were based on 10,000 Monte Carlo runs. We computed the MLE, MME, UMLE, UMME, JMLE and JMME for each run, and then computed the average estimates and the standard deviations over the 10,000 runs for all these estimators. The results so obtained are reported in Tables 1 and 2.

We also computed the 90% and 95% probability coverages of confidence intervals based on inferential quantities associated with all these estimators using the asymptotic distributions given earlier. Specifically, the $100(1-\gamma)\%$ confidence intervals for α and β based on the MLEs and UMLEs are given by

$$\left[\hat{\alpha}\left(\frac{z_{\gamma/2}}{\sqrt{2n}}+1\right)^{-1}, \hat{\alpha}\left(\frac{z_{1-\gamma/2}}{\sqrt{2n}}+1\right)^{-1}\right],$$

$$\left[\hat{\beta}\left(\frac{z_{\gamma/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}, \hat{\beta}\left(\frac{z_{1-\gamma/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}\right],$$

Table 1
Means of estimates based on Monte Carlo simulation ($\beta = 1.0$)

n	α	Estimate of α						Estimate of β					
		MLE	MME	UMLE	UMME	JMLE	JMME	MLE	MME	UMLE	UMME	JMLE	JMME
5	0.10	0.0842	0.0842	0.1053	0.1053	0.1015	0.1015	1.0007	1.0007	1.0004	1.0004	0.9997	0.9997
	0.25	0.2102	0.2102	0.2629	0.2629	0.2536	0.2536	1.0054	1.0054	1.0022	1.0022	0.9992	0.9992
	0.50	0.4186	0.4186	0.5236	0.5236	0.5068	0.5068	1.0225	1.0225	1.0081	1.0081	0.9984	0.9984
	1.00	0.8273	0.8270	1.0351	1.0348	1.0115	1.0112	1.0835	1.0832	1.0241	1.0242	0.9927	0.9922
	2.00	1.6199	1.6142	2.0279	2.0210	2.0251	2.0182	1.2524	1.2524	1.0422	1.0444	0.9190	0.9193
10	0.10	0.0924	0.0924	0.1023	0.1023	0.1005	0.1005	1.0002	1.0002	1.0005	1.0005	0.9997	0.9997
	0.25	0.2310	0.2310	0.2557	0.2557	0.2513	0.2513	1.0024	1.0024	1.0020	1.0020	0.9993	0.9993
	0.50	0.4610	0.4610	0.5103	0.5103	0.5024	0.5024	1.0104	1.0104	1.0061	1.0061	0.9985	0.9985
	1.00	0.9168	0.9166	1.0150	1.0148	1.0046	1.0046	1.0380	1.0379	1.0148	1.0150	0.9955	0.9953
	2.00	1.8181	1.8140	2.0134	2.0089	2.0124	2.0110	1.1017	1.1029	1.0025	1.0056	0.9730	0.9738
20	0.10	0.0964	0.0964	0.1011	0.1011	0.1002	0.1002	1.0001	1.0001	1.0000	1.0000	0.9999	0.9999
	0.25	0.2408	0.2408	0.2525	0.2525	0.2506	0.2506	1.0012	1.0012	1.0005	1.0005	0.9997	0.9997
	0.50	0.4811	0.4811	0.5046	0.5046	0.5012	0.5012	1.0053	1.0053	1.0021	1.0021	0.9994	0.9994
	1.00	0.9596	0.9595	1.0064	1.0063	1.0023	1.0023	1.0190	1.0190	1.0059	1.0059	0.9986	0.9986
	2.00	1.9122	1.9099	2.0057	2.0032	2.0056	2.0053	1.0466	1.0479	0.9952	0.9966	0.9937	0.9941
50	0.10	0.0984	0.0984	0.1005	0.1005	0.0999	0.0999	0.9999	0.9999	1.0001	1.0001	0.9998	0.9998
	0.25	0.2459	0.2459	0.2513	0.2513	0.2497	0.2497	1.0000	1.0000	1.0004	1.0004	0.9994	0.9994
	0.50	0.4915	0.4915	0.5023	0.5023	0.4994	0.4994	1.0013	1.0013	1.0012	1.0012	0.9989	0.9989
	1.00	0.9820	0.9820	1.0035	1.0035	0.9987	0.9987	1.0059	1.0059	1.0030	1.0030	0.9980	0.9980
	2.00	1.9615	1.9605	2.0044	2.0034	1.9976	1.9976	1.0151	1.0157	0.9981	0.9987	0.9966	0.9966
100	0.10	0.0992	0.0992	0.1002	0.1002	0.0999	0.0999	0.9999	0.9999	1.0001	1.0001	0.9998	0.9998
	0.25	0.2479	0.2479	0.2504	0.2504	0.2498	0.2498	0.9999	0.9999	1.0004	1.0004	0.9996	0.9996
	0.50	0.4957	0.4957	0.5007	0.5007	0.4996	0.4996	1.0004	1.0004	1.0010	1.0010	0.9992	0.9992
	1.00	0.9909	0.9909	1.0008	1.0008	0.9992	0.9992	1.0025	1.0025	1.0021	1.0021	0.9986	0.9986
	2.00	1.9805	1.9800	2.0003	1.9998	1.9984	1.9984	1.0067	1.0070	0.9996	1.0001	0.9978	0.9978

Table 2
Standard deviations of estimates based on Monte Carlo simulation ($\beta = 1.0$)

n	α	Estimate of α						Estimate of β					
		MLE	MME	UMLE	UMME	JMLE	JMME	MLE	MME	UMLE	UMME	JMLE	JMME
5	0.10	0.0306	0.0306	0.0383	0.0383	0.0373	0.0373	0.0449	0.0449	0.0445	0.0445	0.0449	0.0449
	0.25	0.0764	0.0764	0.0957	0.0957	0.0935	0.0935	0.1126	0.1126	0.1112	0.1112	0.1120	0.1120
	0.50	0.1524	0.1524	0.1908	0.1908	0.1883	0.1883	0.2263	0.2263	0.2216	0.2216	0.2225	0.2223
	1.00	0.3038	0.3034	0.3805	0.3800	0.3858	0.3851	0.4583	0.4583	0.4367	0.4368	0.4464	0.4382
	2.00	0.6202	0.6131	0.7764	0.7678	0.8319	0.8164	0.9511	0.9553	0.8657	0.8684	1.0513	0.9397
10	0.10	0.0219	0.0219	0.0246	0.0246	0.0239	0.0239	0.0314	0.0314	0.0315	0.0315	0.0313	0.0313
	0.25	0.0547	0.0547	0.0616	0.0616	0.0597	0.0597	0.0782	0.0782	0.0784	0.0784	0.0780	0.0780
	0.50	0.1092	0.1092	0.1230	0.1230	0.1200	0.1200	0.1550	0.1550	0.1544	0.1544	0.1534	0.1533
	1.00	0.2185	0.2183	0.2459	0.2457	0.2435	0.2434	0.2979	0.2979	0.2912	0.2915	0.2904	0.2882
	2.00	0.4445	0.4410	0.5001	0.4961	0.5060	0.5036	0.5213	0.5261	0.4887	0.4939	0.5177	0.4874
20	0.10	0.0155	0.0155	0.0165	0.0165	0.0162	0.0162	0.0225	0.0225	0.0223	0.0223	0.0225	0.0225
	0.25	0.0388	0.0388	0.0414	0.0414	0.0405	0.0405	0.0560	0.0560	0.0554	0.0554	0.0559	0.0559
	0.50	0.0776	0.0776	0.0827	0.0827	0.0812	0.0812	0.1101	0.1101	0.1088	0.1088	0.1095	0.1095
	1.00	0.1554	0.1553	0.1655	0.1654	0.1635	0.1635	0.2053	0.2056	0.2020	0.2022	0.2019	0.2016
	2.00	0.3139	0.3125	0.3344	0.3329	0.3323	0.3320	0.3280	0.3340	0.3153	0.3206	0.3182	0.3156
50	0.10	0.0099	0.0099	0.0102	0.0102	0.0101	0.0101	0.0142	0.0142	0.0140	0.0140	0.0142	0.0142
	0.25	0.0249	0.0249	0.0255	0.0255	0.0253	0.0253	0.0354	0.0354	0.0349	0.0349	0.0353	0.0353
	0.50	0.0497	0.0497	0.0509	0.0509	0.0505	0.0505	0.0693	0.0693	0.0684	0.0684	0.0691	0.0691
	1.00	0.0994	0.0994	0.1018	0.1018	0.1014	0.1014	0.1273	0.1275	0.1258	0.1258	0.1263	0.1264
	2.00	0.1997	0.1994	0.2044	0.2040	0.2037	0.2037	0.1935	0.1979	0.1904	0.1934	0.1901	0.1930
100	0.10	0.0070	0.0070	0.0072	0.0072	0.0071	0.0071	0.0100	0.0100	0.0099	0.0099	0.0100	0.0100
	0.25	0.0176	0.0176	0.0180	0.0180	0.0177	0.0177	0.0249	0.0249	0.0246	0.0246	0.0249	0.0249
	0.50	0.0351	0.0351	0.0360	0.0360	0.0354	0.0354	0.0487	0.0487	0.0480	0.0480	0.0486	0.0486
	1.00	0.0702	0.0702	0.0720	0.0720	0.0709	0.0709	0.0890	0.0891	0.0880	0.0880	0.0887	0.0887
	2.00	0.1407	0.1406	0.1443	0.1441	0.1421	0.1421	0.1341	0.1366	0.1323	0.1344	0.1330	0.1349

and

$$\left[\hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\gamma/2}}{(n-1)}+1\right)^{-1}, \hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\gamma/2}}{(n-1)}+1\right)^{-1}\right],$$

$$\left[\hat{\beta}^*\left(\sqrt{\frac{n}{h_1(\hat{\alpha})}}\frac{4z_{\gamma/2}}{(4n+\hat{\alpha}^2)}+1\right)^{-1}, \hat{\beta}^*\left(\sqrt{\frac{n}{h_1(\hat{\alpha})}}\frac{4z_{1-\gamma/2}}{(4n+\hat{\alpha}^2)}+1\right)^{-1}\right],$$

where $h_1(x)=0.25+x^{-2}+I(x)$ and z_p is the $100p$th percentile of the standard normal distribution. Similarly, the $100(1-\gamma)\%$ confidence intervals for α and β based on the MMEs and UMMEs are given by

$$\left[\tilde{\alpha}\left(\frac{z_{\gamma/2}}{\sqrt{2n}}+1\right)^{-1}, \tilde{\alpha}\left(\frac{z_{1-\gamma/2}}{\sqrt{2n}}+1\right)^{-1}\right],$$

$$\left[\tilde{\beta}\left(\frac{z_{\gamma/2}}{\sqrt{nh_2(\tilde{\alpha})}}+1\right)^{-1}, \tilde{\beta}\left(\frac{z_{1-\gamma/2}}{\sqrt{nh_2(\tilde{\alpha})}}+1\right)^{-1}\right],$$

and

$$\left[\tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\gamma/2}}{(n-1)}+1\right)^{-1}, \tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\gamma/2}}{(n-1)}+1\right)^{-1}\right],$$

$$\left[\tilde{\beta}^*\left(\sqrt{\frac{n}{h_2(\tilde{\alpha})}}\frac{4z_{\gamma/2}}{(4n+\tilde{\alpha}^2)}+1\right)^{-1}, \tilde{\beta}^*\left(\sqrt{\frac{n}{h_2(\tilde{\alpha})}}\frac{4z_{1-\gamma/2}}{(4n+\tilde{\alpha}^2)}+1\right)^{-1}\right],$$

where

$$h_2(x)=\frac{1+\frac{3}{4}x^2}{(1+\frac{1}{2}x^2)^2}.$$

These results are reported in Tables 3 and 4, respectively.

From the simulation results, it is clear that the performance of the MLEs and MMEs are almost identical for different sample sizes and if the shape parameter α is not too large. The average estimates of the MLEs and MMEs and their standard deviations coincide upto four decimal places if α is less than 0.5. It is also evident from these results that the MLEs and the MMEs are both highly biased if n is small and α is large. The bias reduction method works very well in this case for both the parameters even for small samples; but as expected, it increases the corresponding standard deviation of the estimators. The performance of the JMLE and JMME are almost identical at least for small values of α, and they perform better than UMLE and UMME in terms of bias. However, JMLE and JMME possess larger standard deviations than UMLE and UMME. In addition, UMLE and UMME also are computationally simpler while the jackknifed estimators will demand considerably high computational time in case of large sample sizes.

The asymptotic confidence intervals do not work very well when the sample size is very small as the coverage probabilities are much lower than the corresponding nominal

Table 3
Probability coverages of 90% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

n	α	Probability coverages for α				Probability coverages for β			
		MLE	MME	UMLE	UMME	MLE	MME	UMLE	UMME
5	0.10	86.25	86.25	91.12	91.12	78.57	78.57	86.11	86.11
	0.25	86.25	86.25	91.14	91.14	78.45	78.45	86.03	86.03
	0.50	86.15	86.15	91.32	91.32	78.33	78.36	85.64	85.64
	1.00	85.57	85.58	91.45	91.49	78.20	78.16	84.25	84.32
	2.00	83.53	83.72	91.02	91.41	77.40	77.57	80.52	81.74
10	0.10	88.69	88.69	90.43	90.43	84.91	84.91	88.14	88.14
	0.25	88.73	88.73	90.45	90.45	85.00	85.00	88.13	88.13
	0.50	88.72	88.72	90.56	90.56	84.78	84.76	88.05	88.03
	1.00	88.35	88.36	90.73	90.75	84.72	84.87	87.30	87.38
	2.00	87.53	87.70	90.13	90.34	83.96	84.35	85.39	85.99
20	0.10	89.87	89.87	90.31	90.31	87.09	87.09	89.29	89.29
	0.25	89.89	89.89	90.32	90.32	87.09	87.09	89.27	89.27
	0.50	89.83	89.83	90.39	90.39	87.14	87.18	89.07	89.08
	1.00	89.62	89.64	90.42	90.45	86.97	87.03	88.75	88.73
	2.00	89.16	89.21	89.97	90.18	86.66	86.74	87.63	88.22
50	0.10	89.33	89.33	90.14	90.14	88.58	88.58	89.96	89.96
	0.25	89.33	89.33	90.14	90.14	88.66	88.66	89.84	89.84
	0.50	89.28	89.28	90.19	90.19	88.78	88.79	89.72	89.68
	1.00	89.25	89.24	90.11	90.11	88.67	88.80	89.41	89.56
	2.00	89.04	89.10	89.93	89.98	88.18	88.53	88.81	89.24
100	0.10	90.05	90.05	89.65	89.65	89.43	89.43	90.54	90.54
	0.25	90.06	90.06	89.67	89.67	89.45	89.44	90.51	90.52
	0.50	90.04	90.04	89.64	89.64	89.36	89.38	90.43	90.47
	1.00	90.03	90.03	89.64	89.64	89.21	89.40	90.11	90.26
	2.00	89.96	89.98	89.69	89.70	88.95	89.33	89.46	90.01

levels. But for sample sizes 20 or more, the performances are quite satisfactory for confidence intervals for both α and β. The bias reduction technique definitely helps to improve the coverage probabilities in both cases to a certain extent.

8. Illustrative examples

Practical application of the above estimators is illustrated here with two examples with one involving a large sample and the other with a small sample.

Example 1. The data set is given by Birnbaum and Saunders (1969b) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data are presented in Table 5.

Table 4
Probability coverages of 95% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

n	α	Probability coverages for α				Probability coverages for β			
		MLE	MME	UMLE	UMME	MLE	MME	UMLE	UMME
5	0.10	93.86	93.86	95.60	95.60	84.89	84.89	90.56	90.56
	0.25	93.87	93.87	95.68	95.68	84.75	84.75	90.43	90.43
	0.50	93.79	93.79	95.70	95.70	84.38	84.38	89.81	89.81
	1.00	93.65	93.67	95.94	96.05	83.37	83.34	88.48	88.51
	2.00	92.24	92.38	95.91	96.10	81.54	81.89	86.15	86.63
10	0.10	94.46	94.46	95.43	95.43	90.33	90.33	93.15	93.15
	0.25	94.46	94.46	95.46	95.46	90.36	90.36	93.17	93.17
	0.50	94.46	94.46	95.50	95.50	90.25	90.27	92.79	92.80
	1.00	94.25	94.25	95.45	95.50	89.60	89.72	92.09	92.12
	2.00	93.45	93.56	95.17	95.34	88.49	88.68	90.54	90.99
20	0.10	95.19	95.19	95.32	95.32	92.90	92.90	94.12	94.12
	0.25	95.17	95.17	95.34	95.34	92.78	92.78	94.16	94.15
	0.50	95.16	95.16	95.41	95.41	92.59	92.55	94.23	94.21
	1.00	95.06	95.07	95.41	95.42	92.32	92.36	93.68	93.71
	2.00	94.73	94.80	95.17	95.26	91.44	91.73	92.69	93.02
50	0.10	94.72	94.72	95.08	95.08	93.81	93.81	94.65	94.65
	0.25	94.70	94.70	95.07	95.07	93.82	93.82	94.61	94.61
	0.50	94.72	94.72	95.04	95.04	93.77	93.80	94.57	94.54
	1.00	94.59	94.59	95.04	95.05	93.56	93.63	94.20	94.26
	2.00	94.45	94.47	94.98	95.03	93.35	93.33	93.88	93.88
100	0.10	95.00	95.00	94.70	94.70	94.44	94.44	95.01	95.01
	0.25	94.99	94.99	94.71	94.71	94.46	94.46	94.93	94.94
	0.50	94.98	94.98	94.71	94.71	94.44	94.42	95.02	95.00
	1.00	94.96	94.96	94.73	94.73	94.46	94.40	94.70	94.85
	2.00	94.94	94.95	94.71	94.73	94.21	94.39	94.27	94.67

Table 5
Fatigue lifetime data presented by Birnbaum and Saunders (1969b)

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

In summary, we have in this case $n = 101$, $s = 133.73267$, and $r = 129.93321$. For this example, the point estimates of α and β obtained by all the methods are summarized in Table 6.

Table 6
Point estimates of α and β for Example 1

Estimator	α	β
MLE	0.170385	131.818792
MME	0.170385	131.819255
UMLE	0.172089	131.809130
UMME	0.172089	131.809593
JMLE	0.172006	131.798227
JMME	0.172006	131.798661

Table 7
Standard deviations of estimates and interval estimates of α and β for Example 1

Estimator	α			β		
	SD	90% CI	95% CI	SD	90% CI	95% CI
MLE	0.0120	(0.1527,0.1927)	(0.1497,0.1976)	2.2267	(128.2552,135.5861)	(127.5944,136.3325)
MME	0.0120	(0.1527,0.1927)	(0.1497,0.1976)	2.2267	(128.2556,135.5866)	(127.5948,136.3330)
UMLE	0.0122	(0.1541,0.1949)	(0.1511,0.1999)	2.2487	(128.2116,135.6143)	(127.5448,136.3685)
UMME	0.0122	(0.1541,0.1949)	(0.1511,0.1999)	2.2487	(128.2121,135.6148)	(127.5452,136.3690)

Table 8
Fatigue lifetime data presented by McCool (1974)

152.7	172.0	172.5	173.3	193.0
204.7	216.5	234.9	262.6	422.6

From Eqs. (10), (13), (16) and (17), the asymptotic variance of the estimators can be readily obtained, and also the confidence intervals for α and β based on the MLEs, MMEs, UMLEs and UMMEs can be readily constructed using the asymptotic normality. The results so obtained are presented in Table 7.

Example 2. This example is from McCool (1974) on the fatigue life in hours of 10 bearings of a certain type. These data were used as an illustrative example for the three-parameter Weibull distribution by Cohen et al. (1984). The data are presented in Table 8.

In this case, we find $n = 10$, $s = 220.48$ and $r = 203.8853$. The point estimates and interval estimates obtained from this data are summarized in Tables 9 and 10, respectively.

It is heartening to observe in both these examples that the MMEs are very nearly the same as the MLEs and also the corresponding confidence intervals. Also, while the unbiased estimators correct for the bias, they do result in a larger standard deviation.

Table 9
Point estimates of α and β for Example 2

Estimator	α	β
MLE	0.282489	212.049084
MME	0.282489	212.020378
UMLE	0.313877	211.528097
UMME	0.313877	211.499462
JMLE	0.323068	211.470377
JMME	0.323068	211.446452

Table 10
Standard deviations of estimates and interval estimates of α and β for Example 2

Estimator	α			β		
	SD	90% CI	95% CI	SD	90% CI	95% CI
MLE	0.0632	(0.2065,0.4468)	(0.1964,0.5029)	18.7527	(185.1207,248.1452)	(180.7241,256.5101)
MME	0.0632	(0.2065,0.4468)	(0.1964,0.5029)	18.7504	(185.0954,248.1121)	(180.6993,256.4760)
UMLE	0.0780	(0.2228,0.5308)	(0.2111,0.6118)	20.7356	(182.2191,252.0726)	(177.5074,261.6814)
UMME	0.0780	(0.2228,0.5308)	(0.2111,0.6118)	20.7332	(182.1940,252.0394)	(177.4828,261.6472)

9. Concluding remarks

Since the maximum likelihood estimators and the modified moment estimators behave very similarly in almost all cases considered, we recommend the use of the modified moment estimators because they are explicit estimators and are very easy to compute. Although jackknife estimators also work very well, they cannot be recommended for large sample sizes. If the sample size is very small (say, less than 10), then bias corrected modified moment estimators or bias corrected jackknife estimators should be used as in this case the original estimators can be highly biased. The asymptotic confidence intervals behave very well for large sample sizes (at least 20), but not for small sample sizes. In the latter case, one may rely on simulated percentage points rather than on the asymptotic normality.

Appendix. Derivation of the asymptotic distribution of the modified moment estimators

Let $T, T_1, \ldots, T_n$ be independent and identically distributed Birnbaum–Saunders random variables with density function as in (2). Let us define the random variables

$$S = \frac{1}{n}\sum_{i=1}^{n} T_i \quad \text{and} \quad R = \left[\frac{1}{n}\sum_{i=1}^{n} T_i^{-1}\right]^{-1}.$$

From the strong law of large numbers, it is known that S and R^{-1} converge almost surely to $E(T)$ and $E(T^{-1})$, respectively. Also from the central limit theorem, we readily observe that S and R^{-1} are asymptotically normally distributed; furthermore, any linear combination of the form

$$aS + bR^{-1} = \frac{1}{n}\sum_{i=1}^{n}[aT_i + bT_i^{-1}]$$

is also asymptotically normally distributed for all a and b. Therefore, $(S\ R^{-1})'$ is asymptotically distributed as bivariate normal. Specially, we have

$$\sqrt{n}\begin{pmatrix} S - E(T) \\ R^{-1} - E(T^{-1}) \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{\Sigma}\right],$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix},$$

$$\sigma_{11} = Var(T) = (\alpha\beta)^2(1 + \tfrac{5}{4}\alpha^2),$$

$$\sigma_{12} = \sigma_{21} = Cov(T, T^{-1}) = E(1) - E(T)E(T^{-1}) = 1 - (1 + \tfrac{1}{2}\alpha^2)^2,$$

$$\sigma_{22} = Var(T^{-1}) = \alpha^2\beta^{-2}(1 + \tfrac{5}{4}\alpha^2).$$

We now need to find the asymptotic joint distribution of

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{bmatrix} f_1(S, R) \\ f_2(S, R) \end{bmatrix},$$

where

$$f_1(x, y) = \left\{2\left[\left(\frac{x}{y}\right)^{1/2} - 1\right]\right\}^{1/2}$$

and

$$f_2(x, y) = (xy)^{1/2}.$$

By using the Taylor series expansion, we obtain

$$\sqrt{n}\begin{pmatrix} \tilde{\alpha} - \alpha \\ \tilde{\beta} - \beta \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tilde{\mathbf{\Sigma}}\right],$$

where

$$\tilde{\mathbf{\Sigma}} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \mathbf{\Sigma} \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^{\mathrm{T}} \Bigg|_{x=E(T),\, y=E(T^{-1})}$$

$$= \begin{bmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & (\alpha\beta)^2\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{bmatrix}.$$

Of course, we are using here the following expressions:

$$\left.\frac{\partial f_1}{\partial x}\right|_{x=E(T),\,y=E(T^{-1})} = \frac{1}{2\alpha\beta},$$

$$\left.\frac{\partial f_1}{\partial y}\right|_{x=E(T),\,y=E(T^{-1})} = \frac{\beta}{2\alpha},$$

$$\left.\frac{\partial f_2}{\partial x}\right|_{x=E(T),\,y=E(T^{-1})} = \frac{1}{2+\alpha^2},$$

$$\left.\frac{\partial f_2}{\partial y}\right|_{x=E(T),\,y=E(T^{-1})} = -\frac{\beta^2}{2+\alpha^2}.$$

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