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Determination of Discrete Spectrum in a Random Field

Debasis Kundu†

†*Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur - 208016, India.*

Swagata Nandi††

††*Institut für Angewandte Mathematik, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany*

We consider a two dimensional frequency model in a random field, which can be used to model textures and also has wide applications in Statistical Signal Processing. First we consider the usual least squares estimators and obtain the consistency and the asymptotic distribution of the least squares estimators. Next we consider an estimator, which can be obtained by maximizing the periodogram function. It is observed that the least squares estimators and the estimators obtained by maximizing the periodogram function are asymptotically equivalent. Some numerical experiments are performed to see how the results work for finite samples. We apply our results on simulated textures to observe how the different estimators perform in estimating the true textures from a noisy data.

Key Words and Phrases: Strong consistency, frequencies, amplitudes, textures classifications, statistical signal processing.

1 Introduction

In this paper we consider the following two dimensional frequency model

$$y(m,n) = \sum_{k=1}^{p} \left[A_k^0 \cos(m\lambda_k^0 + n\mu_k^0) + B_k^0 \sin(m\lambda_k^0 + n\mu_k^0)\right] + X(m,n) \quad (1)$$

for $m = 1,\ldots,M$ and $n = 1,\ldots,N$. Here A_k^0's and B_k^0's are unknown real numbers, λ_k^0's and μ_k^0's are unknown frequencies and $\lambda_k^0, \mu_k^0 \in (0, \pi)$. $X(m, n)$ is a stationary random field and it satisfies Assumption 1 below. The number of components 'p' is assumed to be known. Given a sample $\{y(m, n); m = 1,\ldots, M, n = 1,\ldots, N\}$, the problem is to estimate A_k^0's, B_k^0's, λ_k^0's and μ_k^0's for $k = 1,\ldots, p$.

†kundu@iitk.ac.in

††nandi@statlab.uni-heidelberg.de

MANDREKAR and ZHANG (1995), KUNDU and GUPTA (1998) considered the following model;

$$y(m,n) = \sum_{k=1}^{p} \left[A_k^0 \cos(m\lambda_k^0 + n\mu_k^0) \right] + X(m,n). \tag{2}$$

It is observed in MANDREKAR and ZHANG (1995) that model (2) can be used quite effectively to model textures. Model (1) is a simple generalization of model (2). It is assumed that $X(m, n)$'s are from a stationary random field with zero mean. Here $y(m, n)$ is a non-stationary random field. The first term in $y(m, n)$ corresponds to the deterministic (regular) textures. So this model represents textures in the noisy environment. Figure 1 represents the two dimensional image plot of $y(m, n)$, whose gray level at (m, n) is proportional to the value of $y(m, n)$ when there is no noise and Figure 2 represents the image plot when it is corrupted by additive noise.

MANDREKAR and ZHANG (1995) proposed consistent estimators of the λ_k^0's and μ_k^0's. Unfortunately the corresponding amplitude estimators of the A_k^0's are not consistent. Their results are mainly based on the work of LAI and WEI (1982) and they did not provide the asymptotic distribution of the estimators. KUNDU and GUPTA (1998) considered the model (2) when $X(m, n)$'s are independent and identically distributed (*i.i.d.*) random variables.

This problem has a special interest in spectrography and it is studied by group theoretic methods by MALLIAVAN (1994a, 1994b). In the particular case when the $X(m, n)$'s are *i.i.d.* random variables this problem can be interpreted as 'signal detection' and it has different applications in Multidimensional Signal Processing. This is a basic model in many fields, such as antenna array processing, geophysical perception, biomedical spectral analysis, etc. See for example the work of BARBIERI and BARONE (1992), CABRERA and BOSE (1993), CHUN and BOSE (1995), HUA (1992), KUNDU and GUPTA (1998) and LANG and MCCLELLAN (1982) for the different estimation procedures and for their properties. But nowhere, at least not to the authors' knowledge, have the properties of the least squares estimators been discussed

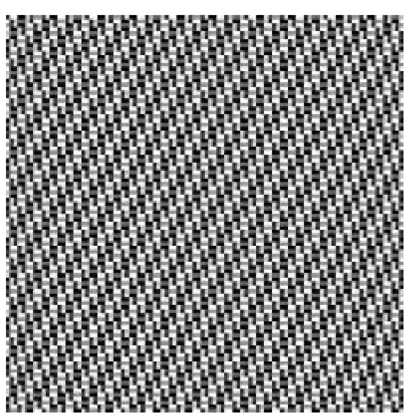

Fig. 1. Image plot of the original data.

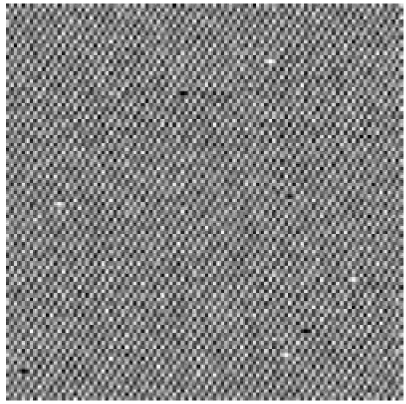

Fig. 2. Image plot of the corrupted data.

of models (1) or (2) under these general set ups. It is important to observe that model (1) is a non-linear regression model and unfortunately it does not satisfy the standard sufficient conditions stated by JENNRICH (1969) or WU (1981) for the least squares estimators (LSE's) to be consistent. It may be noted that when $N = 1$ and $\mu_1^0 = 0$, this model coincides with the standard one dimensional frequency model;

$$y(m) = \sum_{k=1}^{p} \left[A_k^0 \cos(m\lambda_k^0) + B_k^0 \sin(m\lambda_k^0)\right] + e(m), \tag{3}$$

which was discussed quite extensively by HANNAN (1971), WALKER (1971), KUNDU and MITRA (1996b) and KUNDU (1997). It was shown in KUNDU and MITRA (1996b) that even the one dimensional model does not satisfy the sufficient conditions of JENNRICH (1969) or WU (1981) for the LSE's to be consistent. Therefore, the consistency and the asymptotic normality properties of the LSE's of (1) are not immediate in this case.

The main aim of this paper is to consider the model (1) and study different properties of the estimators under the stationary assumptions on $X(m, n)$. The stationary assumption we impose on $X(m, n)$, is a natural extension from the one dimensional case to the two dimensional one. Moreover the estimators we propose are strongly consistent and their asymptotic distributions are also obtained. It gives an idea of the rates of convergence of the proposed estimators and also it can be used to obtain confidence bands of the unknown parameters. We propose mainly two types of estimators. The first one is the usual least squares estimators, which can be obtained by minimizing

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \left(y(m,n) - \sum_{k=1}^{p} [A_k \cos(m\lambda_k + n\mu_k) + B_k \sin(m\lambda_k + n\mu_k)] \right)^2$$

with respect to A_k, B_k, λ_k and μ_k. The second one is the approximate least squares estimators (ALSE's) obtained by maximizing the periodogram function defined as follows:

$$\frac{1}{MN}\left|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-i(\lambda m+n\mu)}\right|^2.$$

Note that once the non-linear frequencies are estimated the linear parameters A_k's and B_k's can be estimated by using the simple linear regression technique. We make the following assumption on $X(m, n)$.

ASSUMPTION 1. *Let us denote the set of positive integers by Z.* $\{X(m, n);\ m, n \in Z\}$ *can be represented as follows:*

$$X(m,n) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a(j,k)e(m-j, n-k)$$

where $a(j, k)$ *are real constants such that*

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |a(j,k)| < \infty$$

and $\{e(m, n);\ m, n \in Z\}$ *is a double array sequence of i.i.d. random variables with mean zero and finite variance* σ^2.

It is observed that the LSE's and the ALSE's are asymptotically equivalent. Moreover under Assumption 1 both of them have the same asymptotic distribution. The rest of the paper is organized as follows. In section 2, we prove the consistency of the LSE's of the parameters of the model (1) and obtain the asymptotic distribution of the LSE's in section 3. In section 4, we consider the ALSE's and study their properties. Numerical results are presented in section 5 and also we use the LSE's and the ALSE's to estimate the textures presented in Figure 1 from the noisy data presented in Figure 2. Finally in section 6 we make connections with our work and some of the related works in signal processing and conclude the paper.

2 Consistency of the LSE's

In this section for brevity first we consider the consistency of the LSE's of the parameters of the following model ($p = 1$ in model (1));

$$y(m,n) = A^0\cos(m\lambda^0 + n\mu^0) + B^0\sin(m\lambda^0 + n\mu^0) + X(m,n). \tag{4}$$

The LSE's of $\boldsymbol{\theta}^0=(A^0, B^0, \lambda^0, \mu^0)$ can be obtained by minimizing

$$Q_{MN}(\boldsymbol{\theta}) = \sum_{m=1}^{M}\sum_{n=1}^{N}(y(m,n) - A\cos(m\lambda + n\mu) - B\sin(m\lambda + n\mu))^2$$

with respect to $\boldsymbol{\theta} = (A, B, \lambda, \mu)$. The LSE of $\boldsymbol{\theta}^0$ will be denoted as $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\lambda}, \hat{\mu})$. We need the following lemmas to prove the necessary results.

LEMMA 1. *Let us denote*

$$S_{C,K} = \{\boldsymbol{\theta}; \boldsymbol{\theta} = (A, B, \lambda, \mu), |\boldsymbol{\theta} - \boldsymbol{\theta}^0| \geqslant C, |A| \leqslant K, |B| \leqslant K\}.$$

If for any $C > 0$ *and for some* $K < \infty$,

$$\underline{\lim} \inf_{\theta \in S_{C,K}} \frac{1}{MN} \left[Q_{MN}(\boldsymbol{\theta}) - Q_{MN}(\boldsymbol{\theta}^0) \right] > 0, \quad a.s. \tag{5}$$

then $\hat{\boldsymbol{\theta}}$, *is a strongly consistent estimator of* $\boldsymbol{\theta}^0$.

PROOF OF LEMMA 1: See the appendix.

LEMMA 2. *Let* $\{X(m, n); \; m, n \in Z\}$ *be a stationary random field satisfying Assumption* 1. *Without loss of generality, let us assume* $\sigma^2 = 1$, *then*

$$\sup_{\alpha,\beta} \left| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} X(m,n) e^{i(m\alpha + n\beta)} \right| \to 0, \quad a.s.$$

as $\min\{M, N\} \to \infty$. *Here a.s. means almost sure convergence.*

COROLLARY OF LEMMA 2:

$$\sup_{\alpha,\beta} \left| \frac{1}{M^{k+1}N} \sum_{m=1}^{M} \sum_{n=1}^{N} m^k X(m,n) e^{i(m\alpha + n\beta)} \right| \to 0, \quad a.s.$$

$$\sup_{\alpha,\beta} \left| \frac{1}{MN^{l+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} n^l X(m,n) e^{i(m\alpha + n\beta)} \right| \to 0, \quad a.s.$$

$$\sup_{\alpha,\beta} \left| \frac{1}{M^{k+1}N^{l+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^k n^l X(m,n) e^{i(m\alpha + n\beta)} \right| \to 0, \quad a.s.$$

for $k, \; l = 1, 2, 3 \ldots$

PROOF OF LEMMA 2: See the appendix.

Note that Lemma 2 is a very strong result. It extends some of the existing one dimensional results of HANNAN (1971), WALKER (1971), RAO and ZHAO (1993), KUNDU (1993), KUNDU and MITRA (1996a), KUNDU and GUPTA (2000) and KUNDU (1997) to the two dimensional case. It also generalizes the two dimensional results of BAI, KUNDU and MITRA (1999), RAO, ZHAO and ZHOU (1994), KUNDU and MITRA (1996a), KUNDU and GUPTA (1998) and NANDI and KUNDU (1999) in certain ways. To establish the consistency results we make the following assumption.

ASSUMPTION 2. *Let* A^0 *and* B^0 *be arbitrary real numbers not identically equal to zero.*

THEOREM 1. *Under the Assumptions* 1 *and* 2, *the LSE's of the parameters of the model* (4) *are strongly consistent.*

PROOF OF THEOREM 1: See the appendix.

Note that the results for the general case (model (1)) follow under the assumption that all the λ_k's and μ_k's are distinct. The proof goes along the same line as the proof of theorem 1 of KUNDU and GUPTA (1998). The same idea can be also used here and therefore the details are avoided for brevity. It is interesting to observe that although the errors are correlated the usual LSE's provide consistent estimators. For the general linear and non-linear models if the errors are correlated, it is well known (RAO, 1973; SEBER and WILD, 1989) that the LSE's are inconsistent. In the correlated case usually the generalized least squares estimators are consistent. On the other hand, Theorem 1 may not be too surprising. It is well known (KUNDU, 1997) that for the one dimensional frequency model (3) even if the errors are correlated, the LSE's are consistent. In this respect one or higher dimensional frequency models are quite different from the usual non-linear models.

3 Asymptotic normality of the LSE's

In this section, we obtain the asymptotic distributions of the LSE's of the parameters of the model (4). We use the following notation. The first derivative of $Q_{MN}(\boldsymbol{\theta})$ is a 1×4 vector as

$$Q'_{MN}(\boldsymbol{\theta}) = \left[\frac{\partial Q_{MN}(\boldsymbol{\theta})}{\partial A}, \frac{\partial Q_{MN}(\boldsymbol{\theta})}{\partial B}, \frac{\partial Q_{MN}(\boldsymbol{\theta})}{\partial \lambda}, \frac{\partial Q_{MN}(\boldsymbol{\theta})}{\partial \mu}\right]$$

and $Q''_{MN}(\boldsymbol{\theta})$ is the 4×4 second derivative matrix of $Q_{MN}(\boldsymbol{\theta})$. Therefore, expanding $Q'_{MN}(\hat{\boldsymbol{\theta}})$ around $\boldsymbol{\theta}^0$, we obtain

$$Q'_{MN}(\hat{\boldsymbol{\theta}}) - Q'_{MN}(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) Q''_{MN}(\bar{\boldsymbol{\theta}}) \tag{6}$$

where $\bar{\boldsymbol{\theta}}$ is a point on the line between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Note that $Q'_{MN}(\hat{\boldsymbol{\theta}}) = 0$ and consider the 4×4 diagonal matrix $\mathbf{D}$ as follows:

$$\mathbf{D} = \begin{bmatrix} M^{-\frac{1}{2}}N^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & M^{-\frac{1}{2}}N^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & M^{-\frac{3}{2}}N^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & M^{-\frac{1}{2}}N^{-\frac{3}{2}} \end{bmatrix}.$$

Now (6) can be written as

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 = -Q'_{MN}(\boldsymbol{\theta}^0)\left[Q''_{MN}(\bar{\boldsymbol{\theta}})\right]^{-1}.$$

Note that $Q''_{MN}(\bar{\boldsymbol{\theta}})$ is a full rank matrix *a.s.* for large M and N. Equivalently

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} = -\left[Q'_{MN}(\boldsymbol{\theta}^0)\mathbf{D}\right]\left[\mathbf{D}Q''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}. \tag{7}$$

The different elements of the vector $[Q'_{MN}(\boldsymbol{\theta}^0)]$ are as follows.

$$\frac{\partial Q_{MN}(\boldsymbol{\theta}^0)}{\partial A} = -2\sum_{m=1}^{M}\sum_{n=1}^{N} X(m,n)\cos(m\lambda^0 + n\mu^0),$$

$$\frac{\partial Q_{MN}(\boldsymbol{\theta}^0)}{\partial B} = -2\sum_{m=1}^{M}\sum_{n=1}^{N} X(m,n)\sin(m\lambda^0 + n\mu^0),$$

$$\frac{\partial Q_{MN}(\boldsymbol{\theta}^0)}{\partial \lambda} = 2\sum_{m=1}^{M}\sum_{n=1}^{N} mX(m,n)\left[A^0\sin(m\lambda^0 + n\mu^0) - B^0\cos(m\lambda^0 + n\mu^0)\right],$$

$$\frac{\partial Q_{MN}(\boldsymbol{\theta}^0)}{\partial \mu} = 2\sum_{m=1}^{M}\sum_{n=1}^{N} nX(m,n)\left[A^0\sin(m\lambda^0 + n\mu^0) - B^0\cos(m\lambda^0 + n\mu^0)\right].$$

We use the following trigonometric results (see MANGULIS, 1965) for $\beta \in (0, \pi)$.

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\cos^2(t\beta) = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sin^2(t\beta) = \frac{1}{2}\\
&\lim_{n\to\infty}\frac{1}{n^2}\sum_{t=1}^{n}t\cos^2(t\beta) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{t=1}^{n}t\sin^2(t\beta) = \frac{1}{4}\\
&\lim_{n\to\infty}\frac{1}{n^3}\sum_{t=1}^{n}t^2\cos^2(t\beta) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{t=1}^{n}t^2\sin^2(t\beta) = \frac{1}{6}\\
&\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sin(t\beta)\cos(t\beta) = 0.
\end{aligned} \tag{8}$$

Let us consider the (1,1)-th element of the matrix $[\mathbf{D}Q''_{MN}(\boldsymbol{\theta}^0)\mathbf{D}]$.

$$\begin{aligned}
\frac{1}{MN}\frac{\partial^2 Q_{MN}(\boldsymbol{\theta}^0)}{\partial A^2} &= \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\cos^2(m\lambda^0 + n\mu^0)\\
&= \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\cos^2(m\lambda^0)\cos^2(n\mu^0) + \sin^2(m\lambda^0)\sin^2(n\mu^0)\right.\\
&\quad \left.-2\cos(m\lambda^0)\cos(n\mu^0)\sin(m\lambda^0)\sin(n\mu^0)\right]
\end{aligned}$$

$$= 2\left[\frac{1}{M}\sum_{m=1}^{M}\cos^2(m\lambda^0).\frac{1}{N}\sum_{n=1}^{N}\cos^2(n\mu^0)\right.$$
$$+\frac{1}{M}\sum_{m=1}^{M}\sin^2(m\lambda^0).\frac{1}{N}\sum_{n=1}^{N}\sin^2(n\mu^0)$$
$$\left.-2\frac{1}{M}\sum_{m=1}^{M}\sin(m\lambda^0)\cos(m\lambda^0).\frac{1}{N}\sum_{n=1}^{N}\sin(n\mu^0)\cos(n\mu^0)\right]$$
$$\rightarrow 2\left[\frac{1}{2}.\frac{1}{2}+\frac{1}{2}.\frac{1}{2}-2.0.0\right] = 1 = \Sigma_{11} \quad \text{(say)}.$$

Using similar technique it can be shown that

$$\lim_{M,N\rightarrow\infty}\left[\mathbf{D}Q''_{MN}(\theta^0)\mathbf{D}\right] = \Sigma.$$

where

$$\Sigma = ((\Sigma_{ij})) = \begin{bmatrix} 1 & 0 & \frac{1}{2}B^0 & \frac{1}{2}B^0 \\ 0 & 1 & -\frac{1}{2}A^0 & -\frac{1}{2}A^0 \\ \frac{1}{2}B^0 & -\frac{1}{2}A^0 & \frac{1}{3}(A^{0^2}+B^{0^2}) & \frac{1}{4}(A^{0^2}+B^{0^2}) \\ \frac{1}{2}B^0 & -\frac{1}{2}A^0 & \frac{1}{4}(A^{0^2}+B^{0^2}) & \frac{1}{3}(A^{0^2}+B^{0^2}) \end{bmatrix}.$$

and because of Theorem 1, $\bar{\boldsymbol{\theta}}$ converges to $\boldsymbol{\theta}^0$ *a.s.* Thus

$$\lim_{M,N\rightarrow\infty}\left[\mathbf{D}Q''_{MN}(\bar{\theta})\mathbf{D}\right] = \lim_{M,N\rightarrow\infty}\left[\mathbf{D}Q''_{MN}(\theta^0)\mathbf{D}\right] = \Sigma.$$

Using the central limit theorem of the stochastic process (FULLER, 1976) and using the results of equation (8), it follows that $[Q_{MN}'(\boldsymbol{\theta}^0)\mathbf{D}]$ tends to a 4-variate normal distribution, with mean vector zero and dispersion matrix $2\sigma^2 c\Sigma$, where

$$c = \left|\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a(j,k)e^{-i(j\lambda^0+k\mu^0)}\right|^2, \tag{9}$$

and Σ is as defined above. Therefore, from (7) we have the following theorem.

THEOREM 2. *Under Assumptions* 1 *and* 2, *the limiting distribution of* $\{M^{\frac{1}{2}}N^{\frac{1}{2}}(\hat{\mathrm{A}}-\mathrm{A}^0), M^{\frac{1}{2}}N^{\frac{1}{2}}(\hat{\mathrm{B}}-\mathrm{B}^0), \mathrm{M}^{\frac{3}{2}}\mathrm{N}^{\frac{1}{2}}(\hat{\lambda}-\lambda^0), \mathrm{M}^{\frac{1}{2}}\mathrm{N}^{\frac{3}{2}}(\hat{\mu}-\mu^0)\}$ *as* $min\{M, N\}\rightarrow\infty$ *is a* 4-*variate normal distribution with mean vector zero and the dispersion matrix* 2 $\sigma^2 c\Sigma^{-1}$, *where* Σ^{-1} *has the following structure:*

$$\Sigma^{-1} = \frac{1}{A^{0^2}+B^{0^2}}\begin{bmatrix} A^{0^2}+7B^{0^2} & -6A^0B^0 & -6B^0 & -6B^0 \\ -6A^0B^0 & 7A^{0^2}+B^{0^2} & 6A^0 & 6A^0 \\ -6B^0 & 6A^0 & 12 & 0 \\ -6B^0 & 6A^0 & 0 & 12 \end{bmatrix}.$$

The result for the general model (1) can be stated as follows;

THEOREM 3. *Under the same consistency assumptions of the LSE's, the limiting distribution of* $\{M^{\frac{1}{2}}N^{\frac{1}{2}}(\hat{A}_k - A_k^0),\ M^{\frac{1}{2}}N^{\frac{1}{2}}(\hat{B}_k - B_k^0),\ M^{\frac{3}{2}}N^{\frac{1}{2}}(\hat{\lambda}_k - \lambda_k^0),\ M^{\frac{1}{2}}N^{\frac{3}{2}}(\hat{\mu}_k - \mu_k^0)\}$ *as* $min\{M,N\} \to \infty$ *is a* 4*-variate normal distribution with mean vector zero and the dispersion matrix* $2\,\sigma^2 c_k \Sigma_k^{-1}$, *where the expressions for* c_k *and* Σ_k^{-1} *can be obtained from* c *and* Σ^{-1} *by replacing* A^0, B^0, λ^0 *and* μ^0 *with* $A_k^0, B_k^0, \lambda_k^0$ *and* μ_k^0 *respectively. Moreover* $(\hat{A}_k, \hat{B}_k, \hat{\lambda}_k, \hat{\mu}_k)$ *and* $(\hat{A}_m, \hat{B}_m, \hat{\lambda}_m, \hat{\mu}_m)$ *are asymptotically independent if* $k \neq m$.

PROOF OF THEOREM 3. Note that the proof of Theorem 3 follows exactly along the same way as the proof of Theorem 2. Expanding $Q'_{MN}(.)$ by Taylor series similarly as in (6), an equivalent to (7) can be obtained for the general case also. For the general case the left hand side is a $1 \times 4p$ random vector whereas the right hand side is a product of a $1 \times 4p$ random vector and a $4p \times 4p$ random matrix. Consider the right hand side as before. The $4p \times 4p$ random matrix converges almost surely to a block diagonal matrix with the k^{th} block as Σ_k. To prove this other than (8), we mainly need to use the following facts several times. For $\beta_1, \beta_2 \in (0, \pi)$ and $\beta_1 \neq \beta_2$:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin(t\beta_1)\cos(t\beta_2) = 0, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin(t\beta_1)\sin(t\beta_2) = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos(t\beta_1)\cos(t\beta_2) = 0. \tag{10}$$

Using (8), (10) and similar techniques as the proof of Theorem 2, it also follows that the $1 \times 4p$ random vector converges to a $4p-$variate normal distribution with mean vector zero and the dispersion matrix having a block diagonal form with the k^{th} block as $2\sigma^2 c_k \Sigma_k$. Therefore, the proof follows.

4 Consistency and asymptotic distribution of the ALSE's

In this section we consider the approximate least squares estimators of the different unknown parameters. First consider the model (4). The ALSE's of λ and μ of the model (4) can be obtained by maximizing the periodogram function;

$$I(\lambda, \mu) = \frac{1}{MN}\left|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-i(m\lambda+n\mu)}\right|^2$$

with respect to λ and μ. If $\tilde{\lambda}$ and $\tilde{\mu}$ maximize $I(\lambda, \mu)$, then $\tilde{\lambda}$ and $\tilde{\mu}$ are called the ALSE's of λ^0 and μ^0 respectively. Once we obtain $\tilde{\lambda}$ and $\tilde{\mu}$, the ALSE's of A and B can be obtained as

$$\tilde{A} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\cos(m\tilde{\lambda} + n\tilde{\mu}),$$

$$\tilde{B} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\sin(m\tilde{\lambda} + n\tilde{\mu}). \tag{11}$$

We need the following lemma to prove the necessary results.

LEMMA 3. *Let us denote*

$$S_c = \{(\lambda,\mu); |\lambda - \lambda^0| > c \ or \ |\mu - \mu^0| > c\},$$

for any $c > 0$. *If there exists* a $c > 0$, *such that*

$$\overline{\lim} \sup_{S_c} \frac{1}{MN}\left[I(\lambda,\mu) - I(\lambda^0,\mu^0)\right] < 0 \quad a.s.,$$

then $(\tilde{\lambda}, \tilde{\mu})$ *converges to* (λ^0, μ^0) *a.s. as* $min\{M,N\} \to \infty$.

PROOF OF LEMMA 3. It follows along the same line as Lemma 1, so details are omitted.

LEMMA 4. $(\tilde{\lambda}, \tilde{\mu})$ *converges to* (λ^0, μ^0) *almost surely.*

PROOF OF LEMMA 4. Consider

$$\begin{aligned}
&\frac{1}{MN}\left[I(\lambda,\mu) - I(\lambda^0,\mu^0)\right] \\
&= \frac{1}{(MN)^2}\left[\left|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-i(m\lambda+n\mu)}\right|^2 - \left|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-i(m\lambda^0+n\mu^0)}\right|^2\right] \\
&= \frac{1}{(MN)^2}\left[\left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\cos(m\lambda+n\mu)\right\}^2 + \left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\sin(m\lambda+n\mu)\right\}^2\right. \\
&\quad \left. - \left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\cos(m\lambda^0+n\mu^0)\right\}^2 - \left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\sin(m\lambda^0+n\mu^0)\right\}^2\right] \\
&= \left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(A^0\cos(m\lambda^0+n\mu^0) + B^0\sin(m\lambda^0+n\mu^0) + X(m,n)\right)\cos(m\lambda+n\mu)\right\}^2 \\
&\quad + \left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(A^0\cos(m\lambda^0+n\mu^0) + B^0\sin(m\lambda^0+n\mu^0) + X(m,n)\right)\sin(m\lambda+n\mu)\right\}^2
\end{aligned}$$

$$-\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\big(A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)+X(m,n)\big)\cos(m\lambda^0+n\mu^0)\right\}^2$$
$$-\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\big(A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)+X(m,n)\big)\sin(m\lambda^0+n\mu^0)\right\}^2.$$

Write $S_c = L_c \cup M_c$, where

$$L_c = \{(\lambda,\mu) : |\lambda-\lambda^0| > c\} \text{ and } M_c = \{(\lambda,\mu) : |\mu-\mu^0| > c\}.$$

Now using Lemma 2 and the trigonometric identities stated in equation (8), it can be shown that for some $c > 0$,

$$\begin{aligned}
&\overline{\lim}\sup_{(\lambda,\mu)\in L_c}\frac{1}{MN}\left[I(\lambda,\mu)-I(\lambda^0,\mu^0)\right]\\
&= -\lim_{M,N\to\infty}\left[\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}A^0\cos^2(m\lambda^0+n\mu^0)\right\}^2\right.\\
&\quad\left.-\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}B^0\sin^2(m\lambda^0+n\mu^0)\right\}^2\right]\\
&= -\frac{1}{4}(A^{0^2}+B^{0^2}) < 0 \quad a.s.
\end{aligned}$$

Similarly it can be proved for M_c also. Thus

$$\overline{\lim}\sup_{(\lambda,\mu)\in S_c}\frac{1}{MN}\left[I(\lambda,\mu)-I(\lambda^0,\mu^0)\right] < 0 \quad a.s.$$

Therefore, the result follows immediately from Lemma 3.

Now to prove the consistency of the linear parameters we need the following lemma.

LEMMA 5. *Suppose $\tilde{\lambda}$ and $\tilde{\mu}$ are ALSE's of λ^0 and μ^0 respectively. Let us define $\tilde{\boldsymbol{\omega}} = (\tilde{\lambda},\tilde{\mu})$, $\boldsymbol{\omega}^0 = (\lambda^0,\mu^0)$ and $\mathbf{D}_1 = diag\{\frac{1}{M},\frac{1}{N}\}$, then*

$$(\tilde{\boldsymbol{\omega}}-\boldsymbol{\omega}^0)\mathbf{D}_1^{-1} \to \mathbf{0} \quad a.s.$$

PROOF OF LEMMA 5. See the appendix.

LEMMA 6. *$\tilde{A}$ and $\tilde{B}$ as defined in* (11) *are strongly consistent estimators of A^0 and B^0 respectively.*

PROOF OF LEMMA 6. Note that

$$\tilde{A}=\frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\{A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)+X(m,n)\}\cos(m\tilde{\lambda}+n\tilde{\mu}).$$

Using Lemma 2

$$\frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}X(m,n)\cos(m\tilde{\lambda}+n\tilde{\mu})\to 0.$$

Now expanding $\cos(m\tilde{\lambda}+n\tilde{\mu})$ by Taylor series around (λ^0, μ^0), we have

$$\begin{aligned}\tilde{A}=&\frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\{A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0))\}\\&\times\Big[\Big\{\cos(m\lambda^0)-m(\tilde{\lambda}-\lambda^0)\sin(m\bar{\lambda})\Big\}\{\cos(n\mu^0)-n(\tilde{\mu}-\mu^0)\sin(n\bar{\mu})\}\\&-\Big\{\sin(m\lambda^0)+m(\tilde{\lambda}-\lambda^0)\cos(m\bar{\lambda})\Big\}\{\sin(n\mu^0)+n(\tilde{\mu}-\mu^0)\cos(n\bar{\mu})\}\Big].\end{aligned}$$

Now using Lemma 5 and (8), it follows that $\tilde{A}\to A^0$. Similarly it can be shown that $\tilde{B}\to B^0$ almost surely.

Now we would like to obtain the asymptotic distributions of the ALSE's. It has been observed that ALSE's have the same distribution as the LSE's and can be stated in the following theorem.

THEOREM 4. *Under the same consistency assumption of the ALSE's, the limiting distribution of* $\{M^{\frac{1}{2}}N^{\frac{1}{2}}(\tilde{A}-A^0), M^{\frac{1}{2}}N^{\frac{1}{2}}(\tilde{B}-B^0), M^{\frac{3}{2}}N^{\frac{1}{2}}(\tilde{\lambda}-\lambda^0), M^{\frac{1}{2}}N^{\frac{3}{2}}(\tilde{\mu}-\mu^0)\}$ *as* $\min\{M, N\}\to\infty$ *is same as that of the LSE's.*

PROOF OF THEOREM 4. See in appendix.

The results for the general case can be obtained along the same line.

5 Numerical experiments

In this section we present some simulation results to compare the performances of the LSE's and the ALSE's for finite sample. All the computations are performed at the Indian Institute of Technology Kanpur, on Silicon Graphics Machine using FORTRAN 77 and using the random number generator of PRESS et al. (1992). We consider the following two dimensional model:

$$y(m,n)=4.0\cos(1.886m+1.100n)+4.0\sin(1.886m+1.100n)+X(m,n), \tag{12}$$

and $X(m, n)$ has the following structure:

$$X(m,n) = e(m,n) + 0.25e(m-1,n) + 0.25e(m+1,n) \\ + 0.25e(m,n-1) + 0.25e(m,n+1).$$

Here $e(m, n)$'s are *i.i.d.* Gaussian random variables with mean zero and finite variance σ^2 for $M = N = 40$. The stationary random field $X(m, n)$ has that particular structure indicates that the error at the point (m, n) is equally influenced by the four equidistant points from the point (m, n). We consider $\sigma = 0.25$, 0.50, 0.75 and 1.00. For $M = N = 40$ and for each σ we generate a data set from the model (12) and compute the LSE's and the ALSE's of A,B,λ and μ. We replicate the process and compute the average estimates and mean squared errors (MSE's) of all the parameters over five hundred replications. The asymptotic variances (AVAR's) in each case are also reported for comparison purposes. The results are presented in Table 1. The first row in Table 1 in the box $\sigma = 0.25$ represents the average estimates of the LSE's and the second row represents the corresponding MSE's. The third row and the fourth row represent the corresponding results of the ALSE's. Finally the fifth row provides the asymptotic variances for the different parameters. Similarly the other boxes have the results for different σ's. We also compute the 95% individual confidence intervals for all the parameters. Note that to compute the confidence intervals of the different parameters we need to estimate σ^2 and c, as defined in (9). Although we can not estimate σ^2 and c separately, it is possible to estimate $\sigma^2 c$, which is needed to compute the confidence intervals of the different parameters (see Theorem 2). By straight forward but lengthy calculation, it can be shown that

$$\sigma^2 c = E\left(\frac{1}{MN}\left|\sum_{m=1}^{M}\sum_{n=1}^{N} X(m,n)e^{-i(m\lambda^0 + n\mu^0)}\right|^2\right).$$

We estimate $\sigma^2 c$ by averaging the periodogram function over a window $(-5, 5) \times (-5, 5)$ across the point estimate of (λ,μ) as suggested by Quinn and Thompson (1991) similarly for one dimensional model. For each parameter we compute the 95% confidence interval and also the confidence length and we report the results of the coverage percentages and the average confidence lengths over five hundred replications. The results are reported in Table 2. In Table 2 corresponding to the box $\sigma = 0.25$, the first row represents the coverage percentages of the LSE's (LSEC) and the corresponding average confidence lengths (ALEN) are reported in the second row. Similarly the third row provides the coverage percentages of the ALSE's (ALEC) and the corresponding average confidence lengths are reported in the fourth row. In the fifth row we report the asymptotic theoretical confidence lengths (TLEN) obtained from Theorem 2. Similarly the results for different σ's are presented in different boxes.

Some of the points are very clear from the entries in Table 1. It is observed that as σ increases the MSE's and biases of both the methods increase. It verifies the consistency properties of both the estimators. Biases are quite small even

Table 1. The average estimates, their mean squared errors (MSE's) and the corresponding asymptotic variances (ASV's) for the least squares estimators (LSE's) and the approximate least squares estimators (ALSE's) of the different parameters when $M = N = 40$.

σ		A	B	λ	μ
	LSE	3.999662	4.000973	1.100004	1.885716
	MSE	3.7595609e-04	3.4174215e-04	2.1142712e-08	2.0661711e-08
0.25	ALSE	3.998600	4.001786	1.100009	1.885722
	MSE	4.3035491E-04	3.9372378E-04	2.3548409E-08	2.2889141E-08
	AVAR	3.5907235e-04	3.5907235e-04	2.1039394e-08	2.1039394e-08
	LSE	3.999281	4.001901	1.100008	1.885714
	MSE	1.5022277e-03	1.3646450e-03	8.4391715e-08	8.2922540e-08
0.50	ALSE	3.997061	4.003606	1.100015	1.885734
	MSE	1.6611648e-03	1.4794081e-03	9.0816705e-08	8.7569262e-08
	AVAR	1.4362894e-03	1.4362894e-03	8.4157577e-08	8.4157577e-08
	LSE	3.998799	4.002835	1.100012	1.885716
	MSE	3.3719174e-03	3.0715072e-03	1.8970439e-07	1.8607695e-07
0.75	ALSE	3.995591	4.005254	1.100022	1.885742
	MSE	3.7347965e-03	3.3114690e-03	1.9989317e-07	1.9594113e-07
	AVAR	3.2316511e-03	3.2316511e-03	1.8935457e-07	1.8935457e-07
	LSE	3.998390	4.003614	1.100016	1.885717
	MSE	5.9899781e-03	5.4539768e-03	3.3603939e-07	3.3173473e-07
1.00	ALSE	3.993906	4.007015	1.100030	1.885752
	MSE	6.5987734e-03	5.8453307e-03	3.5463844e-07	3.4791535e-07
	AVAR	5.7451576e-03	5.7451576e-03	3.3663031e-07	3.3663031e-07

when $\sigma = 1$. At $M = N = 0$, the LSE's and the ALSE's behave almost in an identical manner. MSE's of both methods are quite close to the corresponding asymptotic variances. It seems the asymptotic results can be used quite effectively for making finite sample inferences. From Table 2, it is observed that the LSE's and ALSE's behave in an almost identical manner in constructing confidence intervals also, *i.e.* their coverage percentages and the average confidence lengths are almost the same and they are quite close to the nominal level. The coverage percentages do not change much as σ changes. It is observed that the asymptotic confidence lengths are quite close to the average confidence lengths for different values of σ and for both the methods. It indicates that the estimate of $\sigma^2 c$ is also reasonable. Now let us compare the computational complexities of the LSE's and the ALSE's. Note that the LSE's involve a $2p$ dimensional search whereas the ALSE's involve p, 2 dimensional search. Therefore, for small p they are almost equal but for large p, LSE's are more computationally intensive than the ALSE's. Comparing all the points we recommend the use of ALSE's for all practical purposes.

Now we use the LSE's and the ALSE's to estimate the textures presented in Figure 1 from the noisy textures presented in Figure 2. Assuming that we do not have any idea about p, first we plot the periodogram function of the noisy data. It is presented in Figure 3. From Figure 3, it is clear that p should be 1. Assuming $p = 1$, we obtain the LSE's and the ALSE's of the different parameters. The estimates are given in

Table 2. The average 95% coverage percentages, the 95% average confidence lengths and the theoretical 95% confidence lengths for the least squares estimators and the approximate least squares estimators of the different parameters when $M = N = 40$.

σ		A	B	λ	μ
	LSEC	0.9600	0.9580	0.9580	0.9500
	ALEN	7.6987691e-02	7.6968186e-02	5.8918993e-04	5.8918993e-04
0.25	ALEC	0.9260	0.9500	0.9320	0.9460
	ALEN	7.6894447e-02	7.6848142e-02	5.8839534e-04	5.8839534e-04
	TLEN	7.4280880e-02	7.4280880e-02	5.6859455e-04	5.6859455e-04
	LSEC	0.9560	0.9580	0.9580	0.9480
	ALEN	0.1537375	0.1536600	1.1763248e-03	1.1763248e-03
0.50	ALEC	0.9440	0.9460	0.9440	0.9620
	ALEN	0.1535441	0.1533550	1.1745027e-03	1.1745027e-03
	TLEN	0.1485618	0.1485618	1.1371891e-03	1.1371891e-03
	LSEC	0.9560	0.9580	0.95800	0.9480
	ALEN	0.2305649	0.2303855	1.7638098e-03	1.7638098e-03
0.75	ALEC	0.9420	0.9520	0.9480	0.9560
	ALEN	0.2302901	0.2298715	1.7609874e-03	1.7609874e-03
	TLEN	0.2228426	0.2228426	1.7057837e-03	1.7057837e-03
	LSEC	0.9540	0.9580	0.9580	0.9480
	ALEN	0.3074149	0.3071044	2.3512922e-03	2.3512922e-03
1.00	ALEC	0.9380	0.9520	0.9440	0.9540
	ALEN	0.3070986	0.3063367	2.3474980e-03	2.3474980e-03
	TLEN	0.2971235	0.2971235	2.2743782e-03	2.2743782e-03

Table 3. The corresponding 95% confidence bands are also provided within brackets in each case. The estimated textures are presented in Figures 4 (using LSE's) and 5 (using ALSE's). Even in the presence of high noise both the estimators are quite good in extracting true textures.

6 Conclusions

In this paper we consider the general two dimensional frequency model which was originally discussed in Mandrekar and Zhang (1995). We consider the estimation of the unknown parameters under the assumptions of additive stationary errors. The assumptions are somewhat different from those of Mandrekar and Zhang (1995) and they are a natural generalization from the one dimensional model (see Hannan, 1971; Walker, 1971; Kundu, 1997) to the two dimensional one. Moreover the amplitude estimators provided by Mandrekar and Zhang (1995) are not consistent, although the estimators we have proposed are consistent and their asymptotic distributions are also obtained. It is observed that the asymptotic results can be used to draw small sample inference. Our results generalize many one and two dimensional results to their most general form. Note that we have not considered the estimation of p. In practice it is a very important problem. Two dimensional plot of the periodogram function gives some idea about the number of components present, as we observed while analyzing the simulated textures in section 5. Some Information Theoretic Criteria may be used to estimate p. More work is needed in this direction.

Now we would like to make some connections between our work and some of the work in the field of statistical signal processing. The two dimensional exponential model has the following form:

$$y(m,n) = \sum_{k=1}^{p} A_k e^{i(m\lambda_k + n\mu_k)} + Z(m,n). \tag{13}$$

Here $Z(m, n)$'s are complex valued mean zero random variables. The two dimensional exponential signal plays an important role in statistical signal processing, see for example the book of KAY (1988). RAO, ZHAO and ZHOU (1994) first obtained the asymptotic properties of the LSE's of a similar model under the assumptions that $Z(m, n)$'s are *i.i.d.* complex valued Gaussian random variables and it was generalized by KUNDU and MITRA (1996a) for any *i.i.d.* random variables with mean zero and finite variance. Our results can be used to generalize the results for the model (13), when $Z(m, n)$'s are from a stationary process. The work is in progress in that direction.

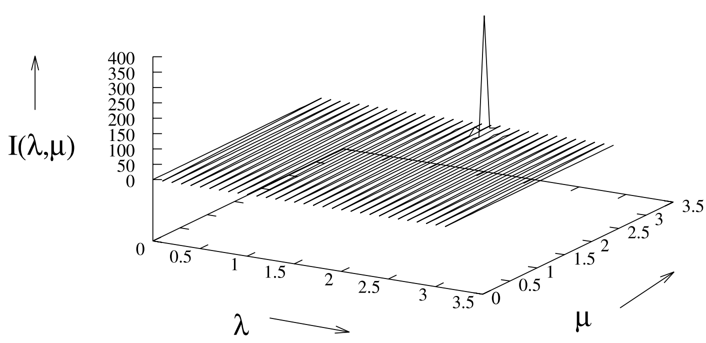

Fig. 3. The plot of the 2-D periodogram function of the data plotted in Figure 1.

Table 3. The LSEs and the ALSEs of the simulated data plotted in Figure 1.

Parameters	LSE	ALSE
$A = 3.0$	3.000933 (2.950178, 3.051687)	2.998732 (2.944025, 3.053439)
$B = 3.0$	2.998315 (2.947527, 3.049103)	3.005226 (2.950608, 3.059845)
$\lambda = 1.885$	1.884993 (1.884786, 1.885200)	1.885004 (1.884781, 1.885227)
$\mu = 2.827$	2.827001 (2.826794, 2.827208)	2.827011 (2.826788, 2.827234)

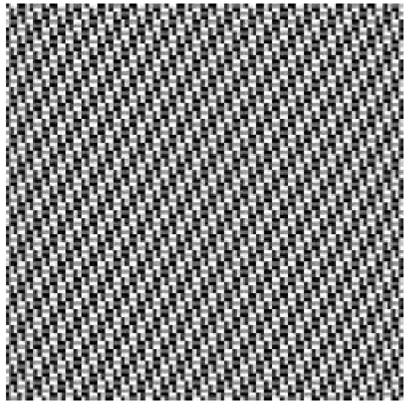

Fig. 4. Image plot of estimated data using LSE's of the parameters.

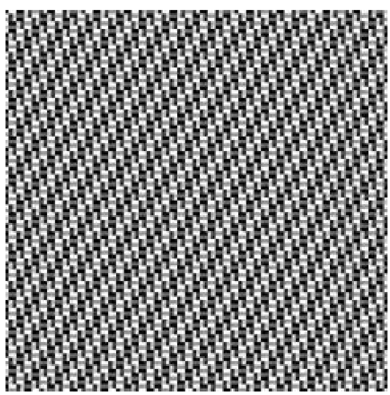

Fig. 5. Image plot of estimated data using ALSE's of the parameters.

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Appendix

PROOF OF LEMMA 1: Let us denote $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\lambda}, \hat{\mu})$ as $\hat{\boldsymbol{\theta}}_{MN} = (\hat{A}_{MN}, \hat{B}_{MN}, \hat{\lambda}_{MN}, \hat{\mu}_{MN})$, in this particular case, because $\hat{\boldsymbol{\theta}}$ depends on M and N. Suppose $\hat{\boldsymbol{\theta}}_{MN}$ does not converge to $\boldsymbol{\theta}^0$ almost surely (*a.s.*) as $min\{M, N\}$ tends to ∞, then either

Case I: For all subsequences $\{M_k, N_k\}$ of $\{M, N\}$, $|\hat{A}_{M_kN_k} + \hat{B}_{M_kN_k}| \rightarrow \infty$, or

Case II: There exists a $C > 0$, a $K < \infty$ and a subsequence $\{M_k, N_k\}$ of $\{M, N\}$ such that $\hat{\boldsymbol{\theta}}_{M_k N_k} \in S_{C,K}$ for all $k = 1, 2, \ldots$.

As $\hat{\boldsymbol{\theta}}_{M_k N_k}$ is the LSE of $\boldsymbol{\theta}^0$ when $M = M_k$ and $N = N_k$, so

$$Q_{M_k N_k}(\hat{\boldsymbol{\theta}}_{M_k N_k}) - Q_{M_k N_k}(\boldsymbol{\theta}^0) \leqslant 0. \tag{14}$$

Now observe that as $k \to \infty$, for Case I

$$\frac{1}{M_k N_k} Q_{M_k N_k}(\hat{\boldsymbol{\theta}}_{M_k N_k}) \to \infty \text{ and} \tag{15}$$

$$\frac{1}{M_k N_k} Q_{M_k N_k}(\boldsymbol{\theta}^0) \to \text{a finite positive constant.} \tag{16}$$

(15) and (16) contradict (14) for large M_k and N_k. For Case II, since (5) is true, it contradicts (14), therefore,

$$\hat{\boldsymbol{\theta}}_{MN} \to \boldsymbol{\theta}^0 \quad a.s.$$

PROOF OF LEMMA 2: Observe that

$$\begin{aligned}
&\sup_{\alpha,\beta} \frac{1}{MN} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} X(m,n) e^{i(m\alpha + n\beta)} \right| \\
&= \sup_{\alpha,\beta} \frac{1}{MN} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a(j,k) e(m-j, n-k) e^{i(m\alpha + n\beta)} \right| \\
&\leqslant \frac{1}{MN} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a(j,k)| \sup_{\alpha,\beta} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} e(m-j, n-k) e^{i(m\alpha + n\beta)} \right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E\left(\sup_{\alpha,\beta} \frac{1}{MN} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} X(m,n) e^{i(m\alpha + n\beta)} \right| \right) \\
&\leqslant E\left(\frac{1}{MN} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a(j,k)| \sup_{\alpha,\beta} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} e(m-j, n-k) e^{i(m\alpha + n\beta)} \right| \right) \\
&\leqslant \frac{1}{MN} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a(j,k)| E\left(\sup_{\alpha,\beta} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} e(m-j, n-k) e^{i(m\alpha + n\beta)} \right| \right) \\
&\leqslant \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a(j,k)| \frac{1}{MN} \left\{ E\left(\sup_{\alpha,\beta} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} e(m-j, n-k) e^{i(m\alpha + n\beta)} \right|^2 \right) \right\}^{\frac{1}{2}}.
\end{aligned} \tag{17}$$

Now consider

$$\frac{1}{MN}\left\{E\left(\sup_{\alpha,\beta}\left|\sum_{m=1}^{M}\sum_{n=1}^{N}e(m-j,n-k)e^{i(m\alpha+n\beta)}\right|^{2}\right)\right\}^{\frac{1}{2}}$$

$$=\frac{1}{MN}\left[E\sup_{\alpha,\beta}\left\{\left(\sum_{m=1}^{M}\sum_{n=1}^{N}e(m-j,n-k)\cos(m\alpha+n\beta)\right)^{2}\right.\right.$$

$$\left.\left.+\left(\sum_{m=1}^{M}\sum_{n=1}^{N}e(m-j,n-k)\sin(m\alpha+n\beta)\right)^{2}\right\}\right]^{\frac{1}{2}}$$

$$\leqslant\frac{1}{MN}\left(E\sup_{\alpha,\beta}\left(\sum_{m=1}^{M}\sum_{n=1}^{N}e(m-j,n-k)\cos(m\alpha+n\beta)\right)^{2}\right.$$

$$\left.+E\sup_{\alpha,\beta}\left(\sum_{m=1}^{M}\sum_{n=1}^{N}e(m-j,n-k)\sin(m\alpha+n\beta)\right)^{2}\right)^{\frac{1}{2}}. \tag{18}$$

Using the one dimensional result of KUNDU (1997, page 225) it follows that the right hand side of (18) is of the order

$$\frac{1}{MN}\left(MN+(MN)^{\frac{3}{2}}\right)^{\frac{1}{2}}=(MN)^{-\frac{1}{4}}.$$

Therefore, the right hand side of (17) is also $O(MN)^{-1/4}$. Now observe that if we take any subsequence $\{(MN)\}^{\delta}$ of $\{MN\}$, where $\delta > 4$, then the right hand side of (17) is summable over that subsequence. Let us take in particular $\delta = 5$. Therefore, by using Chebyshev's inequality we can say that

$$\sup_{\alpha,\beta}\frac{1}{(MN)^{5}}\left|\sum_{m=1}^{M^5}\sum_{n=1}^{N^5}X(m,n)e^{i(m\alpha+n\beta)}\right|\to 0 \quad \text{a.s.}$$

as $min\{M,\ N\}\to\infty$. Now consider

$$\sup_{\alpha,\beta}\sup_{\substack{M^5<R\leqslant(M+1)^5\\N^5<S\leqslant(N+1)^5}}\left|\frac{1}{(MN)^{5}}\sum_{m=1}^{R}\sum_{n=1}^{S}X(m,n)e^{i(m\alpha+n\beta)}-\frac{1}{(MN)^{5}}\sum_{m=1}^{M^5}\sum_{n=1}^{N^5}X(m,n)e^{i(m\alpha+n\beta)}\right|$$

$$\leqslant\frac{1}{(MN)^{5}}\sup_{\alpha,\beta}\sup_{\substack{M^5<R\leqslant(M+1)^5\\N^5<S\leqslant(N+1)^5}}\left|\sum_{m=1}^{M^5}\sum_{n=N^5+1}^{S}X(m,n)e^{i(m\alpha+n\beta)}\right|$$

$$+\frac{1}{(MN)^{5}}\sup_{\alpha,\beta}\sup_{\substack{M^5<R\leqslant(M+1)^5\\N^5<S\leqslant(N+1)^5}}\left|\sum_{m=M^5+1}^{R}\sum_{n=1}^{N^5}X(m,n)e^{i(m\alpha+n\beta)}\right|$$

$$+ \frac{1}{(MN)^5} \sup_{\alpha,\beta} \sup_{\substack{M^5<R\leqslant(M+1)^5 \\ N^5<S\leqslant(N+1)^5}} \left| \sum_{m=M^5+1}^{R} \sum_{n=N^5+1}^{S} X(m,n)e^{i(m\alpha+n\beta)} \right|$$

$$\leqslant \sup_{N^5<S\leqslant(N+1)^5} \frac{1}{N^5} \sum_{n=N^5+1}^{S} \sup_{\alpha} \left| \frac{1}{M^5} \sum_{m=1}^{M^5} X(m,n)e^{im\alpha} \right|$$

$$+ \sup_{M^5<S\leqslant(M+1)^5} \frac{1}{M^5} \sum_{m=M^5+1}^{R} \sup_{\beta} \left| \frac{1}{N^5} \sum_{n=1}^{N^5} X(m,n)e^{in\beta} \right|$$

$$+ \frac{1}{(MN)^5} \sup_{\alpha,\beta} \sup_{\substack{M^5<R\leqslant(M+1)^5 \\ N^5<S\leqslant(N+1)^5}} \left| \sum_{m=M^5+1}^{R} \sum_{n=N^5+1}^{S} X(m,n)e^{i(m\alpha+n\beta)} \right|.$$

Now consider the first term

$$\sup_{N^5<S\leqslant(N+1)^5} \frac{1}{N^5} \sum_{n=N^5+1}^{S} |Y(m,n)| \leqslant \frac{1}{N^5} \sum_{n=N^5+1}^{(N+1)^5} |Y(m,n)|$$

where $|Y(m,\, n)| = \sup_{\alpha} \left| \frac{1}{M^5} \sum_{m=1}^{M^5} X(m,n)e^{im\alpha} \right|$. It is shown in KUNDU and GUPTA (2000) that

$$E(|Y(m,n)|) = O(M^{-\frac{5}{4}}).$$

Therefore,

$$E\left(\frac{1}{N^5} \sum_{n=N^5+1}^{(N+1)^5} |Y(m,n)| \right) \leqslant O(N^{-2}M^{-\frac{5}{4}}).$$

Therefore, it converges to 0 almost surely. Similarly the second term also goes to 0. Consider the third term

$$\frac{1}{(MN)^5} \sup_{\alpha,\beta} \sup_{\substack{M^5<R\leqslant(M+1)^5 \\ N^5<S\leqslant(N+1)^5}} \left| \sum_{m=M^5+1}^{R} \sum_{n=N^5+1}^{S} X(m,n)e^{i(m\alpha+n\beta)} \right|$$

$$\leqslant \frac{1}{(MN)^5} \sum_{m=M^5+1}^{(M+1)^5} \sum_{n=N^5+1}^{(N+1)^5} |X(m,n)|.$$

Therefore, the mean squared error of the third term is $O((MN)^{-2})$. So the third term converges to zero almost surely. That proves the lemma.

PROOF OF THEOREM 1: Let us write

$$\frac{1}{MN}\left[Q_{MN}(\boldsymbol{\theta}) - Q_{MN}(\boldsymbol{\theta}^0)\right] = f_{MN}(\boldsymbol{\theta}) + g_{MN}(\boldsymbol{\theta}) \quad \text{(say)},$$

where $f_{MN}(\boldsymbol{\theta}) = \frac{1}{MN} \times$

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\left[A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)\right.$$
$$\left.-A\cos(m\lambda+n\mu)-B\sin(m\lambda+n\mu)\right]^2$$

and $g_{MN}(\boldsymbol{\theta}) = \frac{2}{MN} \times$

$$\sum_{m=1}^{M}\sum_{n=1}^{N}X(m,n)\left[A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)\right.$$
$$\left.-A\cos(m\lambda+n\mu)-B\sin(m\lambda+n\mu)\right].$$

Re-write the set $S_{C,\,K}$ as

$$S_{C,K} = A_C \cup B_C \cup \Lambda_C \cup M_C,$$

where

$$A_C = \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \lambda, \mu), |A - A^0| \geqslant C, |A| \leqslant K, |B| \leqslant K\right\},$$
$$B_C = \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \lambda, \mu), |B - B^0| \geqslant C, |A| \leqslant K, |B| \leqslant K\right\},$$
$$\Lambda_C = \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \lambda, \mu), |\lambda - \lambda^0| \geqslant C, |A| \leqslant K, |B| \leqslant K\right\},$$
$$M_C = \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \lambda, \mu), |\mu - \mu^0| \geqslant C, |A| \leqslant K, |B| \leqslant K\right\}.$$

So

$$\underline{\lim}\inf_{\boldsymbol{\theta}\in A_C} f_{MN}(\boldsymbol{\theta}) = \underline{\lim}\inf_{\boldsymbol{\theta}\in A_C}\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left[A^0\cos(m\lambda^0+n\mu^0)+B^0\sin(m\lambda^0+n\mu^0)\right.$$
$$\left.-A\cos(m\lambda+n\mu)-B\sin(m\lambda+n\mu)\right]^2$$
$$=\underline{\lim}\inf_{\boldsymbol{\theta}\in A_C}\frac{1}{MN}\left[\sum_{m=1}^{M}\sum_{n=1}^{N}\left(A^0\cos(m\lambda^0+n\mu^0)-A\cos(m\lambda+n\mu)\right)^2\right.$$
$$+\sum_{m=1}^{M}\sum_{n=1}^{N}\left(B^0\sin(m\lambda^0+n\mu^0)-B\sin(m\lambda+n\mu)\right)^2$$
$$+2\sum_{m=1}^{M}\sum_{n=1}^{N}\left(A^0\cos(m\lambda^0+n\mu^0)-A\cos(m\lambda+n\mu)\right)$$
$$\left.\times\left(B^0\sin(m\lambda^0+n\mu^0)-B\sin(m\lambda+n\mu)\right)\right]$$
$$=\underline{\lim}\inf_{|A-A^0|\geqslant C}\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left[A^0\cos(m\lambda^0+n\mu^0)-A\cos(m\lambda^0+n\mu^0)\right]^2$$
$$\geqslant\frac{1}{2}C^2>0\ \ a.s.$$

Similarly it can be proved for B_C, Λ_C and M_C. Thus

$$\varliminf \inf_{\boldsymbol{\theta} \in S_{C,K}} f_{MN}(\boldsymbol{\theta}) > 0 \quad a.s.$$

Using Lemma 2, it follows that

$$\lim_{M,N \to \infty} \sup_{\boldsymbol{\theta}} g_{MN}(\boldsymbol{\theta}) \to 0 \quad a.s.$$

Since

$$\varliminf \inf_{\boldsymbol{\theta} \in S_{C,K}} \frac{1}{MN} \left[Q_{MN}(\boldsymbol{\theta}) - Q_{MN}(\boldsymbol{\theta}^0) \right] > 0 \quad a.s.,$$

using Lemma 1, Theorem 1 follows.

Proof of Lemma 5: Let us denote $\mathbf{I}'(\boldsymbol{\omega})$ as the 2×1 first derivative matrix and $\mathbf{I}''(\boldsymbol{\omega})$ as the 2×2 second derivative matrix of $\mathbf{I}(\boldsymbol{\omega})$. Using the multivariate Taylor series expansion of $\mathbf{I}'(\tilde{\boldsymbol{\omega}})$ around $\boldsymbol{\omega}^0$ yields;

$$\mathbf{I}'(\tilde{\boldsymbol{\omega}}) - \mathbf{I}'(\boldsymbol{\omega}^0) = (\tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}^0)\mathbf{I}''(\bar{\boldsymbol{\omega}}),$$

where $\bar{\boldsymbol{\omega}}$ is a point between $\tilde{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}^0$. Since, $\mathbf{I}'(\tilde{\boldsymbol{\omega}}) = 0$, we have

$$(\tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}^0)\mathbf{D_1}^{-1} = -\left[\frac{1}{MN}\mathbf{I}'(\boldsymbol{\omega}^0)\mathbf{D_1}\right]\left[\frac{1}{MN}\mathbf{D}_1\mathbf{I}''(\bar{\boldsymbol{\omega}})\mathbf{D}_1\right]^{-1}.$$

Let us consider the elements of $\frac{1}{MN}\mathbf{I}'(\boldsymbol{\omega}^0)\mathbf{D_1}$.

$$\begin{aligned}
&\frac{\partial I(\boldsymbol{\omega}^0)}{\partial \lambda}\frac{1}{M^2N} \\
&= \frac{2}{M^3N^2}\left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\cos(m\lambda^0 + n\mu^0)\right\}\left\{-\sum_{m=1}^{M}\sum_{n=1}^{N} my(m,n)\sin(m\lambda^0 + n\mu^0)\right\} \\
&\quad + \frac{2}{M^3N^2}\left\{\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\sin(m\lambda^0 + n\mu^0)\right\}\left\{\sum_{m=1}^{M}\sum_{n=1}^{N} my(m,n)\cos(m\lambda^0 + n\mu^0)\right\} \\
&= 2\left[\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\{A^0\cos(m\lambda^0 + n\mu^0) + B^0\sin(m\lambda^0 + n\mu^0) + X(m,n)\}\cos(m\lambda^0 + n\mu^0)\right] \\
&\quad \times \left[-\frac{1}{M^2N}\sum_{m=1}^{M}\sum_{n=1}^{N} m\{A^0\cos(m\lambda^0 + n\mu^0) + B^0\sin(m\lambda^0 + n\mu^0) + X(m,n)\}\sin(m\lambda^0 + n\mu^0)\right] \\
&\quad + 2\left[\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\{A^0\cos(m\lambda^0 + n\mu^0) + B^0\sin(m\lambda^0 + n\mu^0) + X(m,n)\}\sin(m\lambda^0 + n\mu^0)\right] \\
&\quad \times \left[\frac{1}{M^2N}\sum_{m=1}^{M}\sum_{n=1}^{N} m\{A^0\cos(m\lambda^0 + n\mu^0) + B^0\sin(m\lambda^0 + n\mu^0) + X(m,n)\}\cos(m\lambda^0 + n\mu^0)\right]
\end{aligned}$$

$$\to -2.A^0.\frac{1}{2}.B^0.\frac{1}{4} + 2.B^0.\frac{1}{2}.A^0.\frac{1}{4} = 0.$$

(Using Lemma 2, corollary of Lemma 2 and the trigonometric identities stated in the proof of Theorem 2).

Similarly it can be shown that $\frac{1}{MN^2}\frac{\partial I(\omega^0)}{\partial \mu} \to 0$ *a.s.* Thus $\frac{1}{MN}\mathbf{I}'(\boldsymbol{\omega}^0)\mathbf{D_1} \to \mathbf{0}$ almost surely. Using similar method we can show that $\frac{1}{MN}\mathbf{D}_1\mathbf{I}''(\boldsymbol{\omega}^0)\mathbf{D}_1 \to \frac{1}{24}(A^{0^2} + B^{0^2})\mathbf{I}_2$, which is a positive definite matrix. Therefore, using Lemma 4 the result follows.

Proof of Theorem 4: Observe that

$$\begin{aligned}
\frac{1}{MN}Q_{MN}(\theta) &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)^2 - \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\{A\cos(m\lambda+n\mu) \\
&\quad + B\sin(m\lambda+n\mu)\} + \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\{A\cos(m\lambda+n\mu)+B\sin(m\lambda+n\mu)\}^2 \\
&= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)^2 - \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\{A\cos(m\lambda+n\mu) \\
&\quad + B\sin(m\lambda+n\mu)\} + \frac{1}{2}(A^2+B^2) + O\left(\frac{1}{MN}\right) \quad \text{(using (8))} \\
&= C - \frac{1}{MN}J_{MN}(\boldsymbol{\theta}) + O\left(\frac{1}{MN}\right) \quad \text{(say)},
\end{aligned}$$

where

$$C = \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)^2$$

and

$$\frac{1}{MN}J_{MN}(\boldsymbol{\theta}) = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\{A\cos(m\lambda+n\mu)+B\sin(m\lambda+n\mu)\} - \frac{1}{2}(A^2+B^2).$$

Now we compute the elements of $\frac{1}{MN}J'_{MN}(\boldsymbol{\theta}^0)$, where

$$\frac{1}{MN}J'_{MN}(\boldsymbol{\theta}) = \left(\frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta})}{\partial A}, \frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta})}{\partial B}, \frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta})}{\partial \lambda}, \frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta})}{\partial \mu}\right).$$

It can be seen after some straight forward calculations that

$$\frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta}^0)}{\partial A} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} X(m,n)\cos(m\lambda^0 + n\mu^0) + O\left(\frac{1}{MN}\right)$$

$$\frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta}^0)}{\partial B} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} X(m,n)\sin(m\lambda^0 + n\mu^0) + O\left(\frac{1}{MN}\right)$$

$$\frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta}^0)}{\partial\lambda} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} mX(m,n)\{-A^0\sin(m\lambda^0+n\mu^0) + B^0\cos(m\lambda^0+n\mu^0)\} + O\left(\frac{1}{N}\right)$$

$$\frac{1}{MN}\frac{\partial J_{MN}(\boldsymbol{\theta}^0)}{\partial\mu} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} nX(m,n)\{-A^0\sin(m\lambda^0+n\mu^0) + B^0\cos(m\lambda^0+n\mu^0)\} + O\left(\frac{1}{M}\right).$$

So comparing $\frac{1}{MN}Q'_{MN}(\boldsymbol{\theta}^0)$ and $\frac{1}{MN}J'_{MN}(\boldsymbol{\theta}^0)$, we have

$$\frac{1}{MN}Q'_{MN}(\boldsymbol{\theta}^0) = -\frac{1}{MN}J'_{MN}(\boldsymbol{\theta}^0) + \begin{bmatrix} O(\frac{1}{MN}) \\ O(\frac{1}{MN}) \\ O(\frac{1}{N}) \\ O(\frac{1}{M}) \end{bmatrix} \text{ or } Q'_{MN}(\boldsymbol{\theta}^0) = -J'_{MN}(\boldsymbol{\theta}^0) + \begin{bmatrix} O(1) \\ O(1) \\ O(M) \\ O(N) \end{bmatrix}. \tag{19}$$

Also observe that $\boldsymbol{\theta}$ which maximizes $J_{MN}(\boldsymbol{\theta})$ is same as the ALSE's of $\boldsymbol{\theta}^0$. Now similarly as (7), we obtain the following

$$(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} = -\left[J'_{MN}(\boldsymbol{\theta}^0)\mathbf{D}\right]\left[\mathbf{D}J''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}. \tag{20}$$

Therefore, using (19) and (7), (20) can be written as

$$(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} = \left[Q'_{MN}(\boldsymbol{\theta}^0)\mathbf{D} + \begin{bmatrix} O(1) \\ O(1) \\ O(M) \\ O(N) \end{bmatrix}\mathbf{D}\right]\left[\mathbf{D}J''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}$$
$$= \left[-(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1}\left[\mathbf{D}Q''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right] + \begin{bmatrix} O(1) \\ O(1) \\ O(M) \\ O(N) \end{bmatrix}\mathbf{D}\right]\left[\mathbf{D}J''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}.$$

It can be checked that

$$\lim_{M,N\to\infty}\left[\mathbf{D}J''_{MN}(\bar{\boldsymbol{\theta}})\mathbf{D}\right] = \lim_{M,N\to\infty}\left[\mathbf{D}J''_{MN}(\boldsymbol{\theta}^0)\mathbf{D}\right] = -\lim_{M,N\to\infty}\left[\mathbf{D}Q''_{MN}(\boldsymbol{\theta}^0)\mathbf{D}\right] = -\Sigma.$$

Since

$$\lim_{M,N\to\infty}\begin{bmatrix} O(1) \\ O(1) \\ O(M) \\ O(N) \end{bmatrix}\mathbf{D} = \mathbf{0},$$

it follows that the LSE and ALSE of $\boldsymbol{\theta}^0$ of the model (4) are asymptotically equivalent in distribution, therefore the asymptotic distribution of the ALSE's is same as that of the LSE's.

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