Analysis of Incomplete Data in Presence of Dependent Competing Risks

Debasis Kundu[1] and Sankarshan Basu[2]

[1] **Department of Mathematics, Indian Institute of Technology Kanpur Kanpur, Pin 208016, India**

[2] **Department of Actuarial Mathematics and Statistics, Heriot Watt University Riccarton, Edinburgh EH14 4AS, UK and Indian Institute of Management Bangalore, India**

Abstract: In reliability or life-testing experiments, the cause of failure of an individual or item may be due to one of several causes. The competing risks model assumes that the data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out under this assumption for the last several years. In many cases the investigation of the cause of failure is both expensive and time consuming. Hence sometimes the cause of failure is not observed, even if the failure time is observed. Miyakawa (1984) first considered this problem under the assumption that the lifetime of the different risks are independent. Kundu and Basu (2000) recently considered the same problem and studied several properties of the estimators under the same assumption as that of Miyakawa (1984). In this paper we consider the same problem as studied by Miyakawa (1984) and Kundu and Basu (2000), under the assumption that the lifetime distribution of the competing risks are dependent. We consider one real data set to illustrate the performances of the different methods.

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Corresponding Author: Debasis Kundu (e-mail: kundu@iitk.ac.in, Phone: 91-512-597141, FAX: 91-512-597500)

1. INTRODUCTION

The theory of competing risks has been of considerable interest to researchers who have been concerned with the assessment of a specific risk in the presence of other risk factors. The competing risks model assumes that the data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out for the competing risks model for both the parametric and non-parametric set up. In both the cases two major assumptions are made. First of all when the failure times are observed the causes of failure are also known. Secondly, the competing lifetimes are independently distributed. Both these assumptions are quite important in drawing any statistical inferences from a given data set.

In certain situations (Dinse; 1982, Miyakawa; 1982) it is observed that the determination of the cause of failure may be very expensive or may be very difficult to obtain. In those situations, one might observe the failure time but the corresponding cause of failure is not observed. Miyakawa (1984) first considered this problem from both parametric and non-parametric set up under the assumptions that the competing lifetimes are independently distributed. In the parametric set up Miyakawa (1984) considered the exponential lifetime and obtained the maximum likelihood estimators (MLE's) and uniformly minimum variance unbiased estimators (UMVUE's) of the unknown parameters. Kundu and Basu (2000) also considered the same problem and obtained different properties of the estimators under similar assumptions as that of Miyakawa (1984). Similar problems are considered by Mukhopadhyay and Basu (1997) and Gauss, Usher and Hodgson (1991). In all these cases it is assumed that the lifetime distributions are independent. In fact the independence of competing lifetimes is one of the major concerns in the general competing risks problem. Identifiability of the dependent competing risks is a well known problem in statistical literature; see for example Crowder (1991, 1994) and Kalbfleish and Prentice (1980) etc. In their seminal paper, Babu, Rao and Rao (1992) first considered the dependent competing risks problem using mixture model formulation in the non-parametric set up. The corresponding parametric formulation was considered by Kundu, Kannan and Mazumdar (1992). In both cases it is assumed that all causes of failure are known.

The aim of this paper is to consider the same problem of Miyakawa (1982) and Kundu and Basu (2000) under the assumption that the lifetimes may not be independent. In one sense it generalizes the usual competing risk methodology to include partially complete dependent outcomes. It is assumed that the lifetimes are dependent and they follow some parametric distribution. It is also assumed that individuals dying due to different causes have different lifetime distributions and that every member of a target population dies one of two specific causes, say, cancer, or due to other causes. We group all other causes under a different single category. A proportion π of the population die due to cancer (Cause 1) and the proportion $(1-\pi)$ die due to other causes (Cause 2). At the end of the study, we have three types of observations.

(a) Individuals who died of cancer and their lifetimes.

(b) Individuals who died of other causes and their lifetimes.

(c) Individuals whose lifetimes are observed but causes of death are unknown.

Types (a) and (b) will be referred to as complete observation and type (c) will be referred to as incomplete observations. For simplicity we assume that there is no censoring. The main aim of this paper is to formulate parametric model which can generate the above type of observations and analyze that model. We consider exponential and Weibull lifetime distributions. In both the cases, we obtain the maximum likelihood estimators (MLE's) of the different parameters and also compute the asymptotic distributions of the MLE's. The asymptotic distributions can be used to construct confidence intervals of the unknown parameters. We consider one real life data set and illustrate how the different methods work in this case.

2. PROBLEM FORMULATION AND DIFFERENT NOTATIONS

Consider a population, where every individual dies due to one of the two known causes; cause 1 and cause 2. An individual is selected at random from the population. Let

$$\begin{aligned} \Delta^* &= 1 \quad \text{if he dies of cause 1} \\ &= 2 \quad \text{if he dies of cause 2.} \end{aligned}$$

Also, let $P(\Delta^* = 1) = \pi$, $P(\Delta^* = 2) = 1 - \pi$, and T be the time until failure of the individual. We denote $S_1(t) = P(T > t|\Delta^* = 1)$ as the survival function of T given $\Delta^* = 1$ and similarly $S_2(t) = P(T > t|\Delta^* = 2)$ as the survival function of T given $\Delta^* = 2$. Let $S(t) = P(T > t)$ be the survival function of T, then

$$S(t) = \pi S_1(t) + (1 - \pi)S_2(t); \qquad 0 \le t < \infty.$$

Associated with each individual we have a pair (T, Δ^*), which is not observable in its entirety. In practice, we observe the pair (T, Δ), where

$$\begin{aligned} \Delta &= 1 \quad \text{if } \Delta^* = 1 \text{ and it is observed} \\ &= 2 \quad \text{if } \Delta^* = 2 \text{ and it is observed} \\ &= 0 \quad \text{if } \Delta^* \text{ is not observed.} \end{aligned}$$

The variable Δ indicates the nature of the observation, *i.e.* death due to cause 1 ($\Delta = 1$), or death due to cause 2 ($\Delta = 2$), or death due to one of the causes 1 or 2 but it is unknown ($\Delta = 0$). We denote $f_1(t)$, $f_2(t)$ and $f(t)$ as the probability density functions corresponding to $S_1(t)$, $S_2(t)$ and $S(t)$ respectively. The joint probability density function of T and Δ, conditioning on whether the cause is observed or not, is given by

$$\begin{aligned} g(t, 1|\text{cause is observed}) &= \pi f_1(t) \\ g(t, 2|\text{cause is observed}) &= (1 - \pi) f_2(t) \\ g(t, 0|\text{cause is not observed}) &= \pi f_1(t) + (1 - \pi) f_2(t), \end{aligned}$$

for $0 \leq t < \infty$. It is clear from the above formulation that the net hazard function $h(t)$ is

$$h(t) = \frac{\pi f_1(t) + (1-\pi) f_2(t)}{\pi S_1(t) + (1-\pi) S_2(t)}.$$

The cause specific hazard functions are

$$h_1(t) = \frac{\pi f_1(t)}{\pi S_1(t) + (1-\pi) S_2(t)} \quad \text{and} \quad h_2(t) = \frac{(1-\pi) f_2(t)}{\pi S_1(t) + (1-\pi) S_2(t)}$$

due to cause 1 and cause 2 respectively. The failure time sub density functions for failure type 1 and 2 are

$$h_1(t)S(t) = \pi f_1(t) \quad \text{and} \quad h_2(t)S(t) = (1-\pi) f_2(t)$$

respectively.

Note that in the usual competing risks set up (Kalbfleish and Prentice; 1980) it is assumed that the two survival functions namely $S_1(t)$ and $S_2(t)$ are independent but under this mixture model formulation the lifetime survival functions of the two causes might be dependent (see for example Babu, Rao and Rao; 1992).

The main problem in this paper is to estimate π and also different other parameters that appear in $f_1(.)$ and $f_2(.)$. We are also interested in estimating the other parameters such as the expected lifetimes due to different causes, the expected lifetime of all the individuals, the cause specific mortality index etc. Without loss of generality, we assume that the first n_1 observations have failure type 1, the second n_2 observations have failure type 2, where $m = n_1 + n_2$, and the last $n - m$ observations have only failure times but not the cause of failure. It is also assumed throughout that m is fixed, strictly positive and not random. Note that n_1 and n_2 are random, here n_1 is Binomial (m, π) and n_2 is Binomial $(m, 1-\pi)$. With the above assumptions, let $(t_1, \delta_1), \ldots, (t_n, \delta_n)$ be n independent realization of (T, Δ), with $\delta_1 = \ldots = \delta_{n_1} = 1$, $\delta_{n_1+1} = \ldots = \delta_{n_1+n_2} = 2$ and $\delta_{n_1+n_2+1} = \ldots = \delta_{n_1+n_2+n_3} = 0$, here $n_1 + n_2 + n_3 = n$. From now on we denote $I_1 = \{1, \ldots, n_1\}$, $I_2 = \{n_1+1, \ldots, n_1+n_2\}$, $I_3 = \{n_1+n_2+1, \ldots, n\}$ and $I = I_1 \cup I_2 \cup I_3$. The log likelihood, L, of the observed data is

$$L = n_1 \; ln \; (\pi) + \sum_{i \in I_1} ln \; (f_1(t_i)) + n_2 \; ln \; (1-\pi) \sum_{i \in I_2} ln \; (f_2(t_i)) + \sum_{i \in I_3} ln \; (\pi f_1(t) + (1-\pi) f_2(t)). \tag{2.1}$$

In the subsequent sections we assume different structures on f_1 and f_2 and discuss different statistical inference procedures.

3. EXPONENTIAL FAILURE DISTRIBUTION: POINT ESTIMATION

In this section we assume that the lifetime distributions T_1 and T_2 due to cause 1 and cause 2 respectively are both exponential. The corresponding density functions $f_1(.)$ and $f_2(.)$ take the following form;

$$f_1(t) = \lambda_1 e^{-\lambda_1 t} \quad \text{for} \quad \lambda_1 > 0, t > 0,$$

$$f_2(t) = \lambda_2 e^{-\lambda_2 t} \quad \text{for} \quad \lambda_2 > 0, t > 0.$$

The log-likelihood function (2.1) takes the following form;

$$L = n_1 \; ln \; (\pi) + n_2 \; ln \; (1-\pi) + n_1 \; ln \; (\lambda_1) - \lambda_1 \sum_{i \in I_1} t_i + n_2 \; ln \; (\lambda_2) - \lambda_2 \sum_{i \in I_2} t_i +$$
$$\sum_{i \in I_3} ln \; \left(\pi \lambda_1 e^{-\lambda_1 t_i} + (1-\pi) \lambda_2 e^{-\lambda_2 t_i} \right). \tag{3.1}$$

We could maximize directly (3.1) using some standard maximization routine in three dimensions or we may use EM algorithm to obtain the maximum likelihood estimators of the unknown parameters.

We present the EM algorithm of Dempster, Laird and Rubin (1977) to obtain the maximum likelihood estimates of the unknown parameters. At the expectation step (E), the algorithm forms 'pseudo data' in which the complete observations are left intact and the unit mass associated with each incomplete observation T is fractioned and assigned to 2 partially complete 'pseudo observations' of the form $(T, \Delta^* = j)$ (see Louis; 1982). Specifically the fractional mass, $w_j(t, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \pi)$, assigned to this 'pseudo observation' is the conditional probability that the individual died from risk j given that the individual had died at time point t. Therefore,

$$w_j(t, \boldsymbol{\theta}) = P_{\boldsymbol{\theta}}[\Delta^* = j | t] = \frac{\pi \lambda_j e^{-\lambda_j t}}{\pi \lambda_1 e^{-\lambda_1 t} + (1-\pi) \lambda_2 e^{-\lambda_2 t}}, \qquad \text{for} \quad j = 1, 2. \tag{3.2}$$

The log-likelihood of the 'pseudo data' can be written as $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$, where

$$L_1(\boldsymbol{\theta}) = n_1 \; ln \; (\pi) + n_2 \; ln \; (1-\pi) + n_1 \; ln \; (\lambda_1) - \lambda_1 \sum_{i \in I_1} t_i + n_2 ln(\lambda_2) - \lambda_2 \sum_{i \in I_2} t_i$$

and

$$L_2(\boldsymbol{\theta}) = (ln \; (\pi) + ln \; (\lambda_1)) \sum_{i \in I_3} w_1(t_i, \boldsymbol{\theta}) + (ln \; (1-\pi) + ln \; (\lambda_2)) \sum_{i \in I_3} w_2(t_i, \boldsymbol{\theta}) -$$
$$\lambda_1 \sum_{i \in I_3} w_1(t_i, \boldsymbol{\theta}) \; t_i - \lambda_2 \sum_{i \in I_3} w_2(t_i, \boldsymbol{\theta}) \; t_i.$$

The maximization (M) step involves maximizing $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$ iteratively. It can be done very easily in the exponential case. Assuming $w_j(t_i, \boldsymbol{\theta})$'s to be known, maximize $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. If $\boldsymbol{\theta}^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \pi^{(i)})$ is the i^{th} iterate then, the $(i+1)^{th}$ iterate $\boldsymbol{\theta}^{(i+1)} = (\lambda_1^{(i+1)}, \lambda_2^{(i+1)}, \pi^{(i+1)})$ can be obtained as follows;

$$\lambda_1^{(i+1)} = \frac{n_1 + \sum_{i=n_1+n_2+1}^{n} w_1(t_i, \boldsymbol{\theta}^{(i)})}{\sum_{i=1}^{n_1} t_i + \sum_{i=n_1+n_2+1}^{n} w_1(t_i, \boldsymbol{\theta}^{(i)}) t_i},$$

$$\lambda_2^{(i+1)} = \frac{n_2 + \sum_{i=n_1+n_2+1}^{n} w_2(t_i, \boldsymbol{\theta}^{(i)})}{\sum_{i=n_1+1}^{n_1+n_2} t_i + \sum_{i=n_1+n_2+1}^{n} w_2(t_i, \boldsymbol{\theta}^{(i)}) t_i},$$

$$\pi^{(i+1)} = \frac{n_1 + \sum_{i=n_1+n_2+1}^{n} w_1(t_i, \boldsymbol{\theta}^{(i)})}{n}. \tag{3.3}$$

The iterative procedure is terminated when it satisfies some convergence criterion.

We are interested in the following parameters:

π: probability of death due to cause 1 (say cancer).

λ_1: hazard rate of T_1.

λ_2: hazard rate of T_2.

$\theta_1 = \lambda_1^{-1}$: expected lifetime of those dying due to cancer.

$\theta_2 = \lambda_2^{-1}$: expected lifetime of those dying due to other causes.

$\tau_1 = \pi\lambda_1^{-1} + (1-\pi)\lambda_2^{-1}$: the expected lifetime of all individuals.

$\tau_2 = \pi\tau_1^{-1}$: the cause specific mortality index.

Once we obtain the MLE's of $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \pi)$ the MLE's of all the above parameters can be obtained easily.

4. EXPONENTIAL FAILURE DISTRIBUTION: ASYMPTOTIC CONFIDENCE INTERVALS

In this section we obtain the confidence intervals of the different parameters. Since it is difficult to obtain the exact distribution of the MLE's we need to rely on asymptotic results only. Let us denote the Fisher information matrix by $\mathbf{I}$, $\mathbf{I} = ((I_{ij})), i, j = 1, 2, 3$, where

$$I_{ij} = E\left[-\frac{\partial^2 logL}{\partial\mu_i \partial\mu_j}\right], i, j = 1, 2, 3.$$

If $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\pi})$ is the MLE of $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \pi)$, then from the asymptotic theory of the MLE's, we have

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \rightarrow N_3(0, \mathbf{I}^{-1}). \tag{4.1}$$

We need to estimate $\mathbf{I}^{-1}(\boldsymbol{\theta})$ to obtain the confidence intervals of the unknown parameters. Using the idea of Louis (1982), the observed information matrix takes the form $\hat{\mathbf{I}} = \mathbf{B} - \mathbf{S}\mathbf{S}^T$. Here $\mathbf{B}$ is a 3×3 diagonal matrix and $\mathbf{S}$ is 3×1 vector. The diagonal elements of the matrix $\mathbf{B}$ and the elements of the vector $\mathbf{S}$ are as follows.

$$
\begin{aligned}
B(1,1) &= \frac{n_1}{\hat{\lambda}_1^2} + \frac{1}{\hat{\lambda}_1^2} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) \\
B(2,2) &= \frac{n_2}{\hat{\lambda}_2^2} + \frac{1}{\hat{\lambda}_2^2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) \\
B(3,3) &= \frac{n_1}{\hat{\pi}^2} + \frac{n_2}{(1-\hat{\pi})^2} + \frac{1}{\hat{\pi}^2} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) + \frac{1}{(1-\hat{\pi})^2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) \\
S(1) &= \frac{n_1}{\hat{\lambda}_1} - \sum_{i \in I_1} t_i + \frac{1}{\hat{\lambda}_1} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) - \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) t_i \\
S(2) &= \frac{n_2}{\hat{\lambda}_2} - \sum_{i \in I_2} t_i + \frac{1}{\hat{\lambda}_2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) - \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) t_i \\
S(3) &= \frac{n_1}{\hat{\pi}} - \frac{n_2}{(1-\hat{\pi})} + \frac{1}{\hat{\pi}} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) - \frac{1}{(1-\hat{\pi})} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}})
\end{aligned}
$$

Here $w_j(t_i, \hat{\boldsymbol{\theta}})$ can be obtained from (3.2) by putting $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. Using (4.1) and the observed information matrix it is possible to construct confidence intervals of the unknown parameters. The asymptotic results and the confidence intervals of the function of the parameters can be easily obtained using δ-method.

5. WEIBULL FAILURE DISTRIBUTION

5.1 Estimation of the Parameters

In this section we assume that the lifetime distribution of the different causes T_1 and T_2 follow Weibull distributions with different scale parameters but the same shape parameter. The density functions $f_1(.)$ and $f_2(.)$ take the following form in this case:

$$
f_1(t) = \alpha\lambda_1 t^{\alpha-1} e^{-\lambda_1 t^{\alpha}} \qquad \text{for} \quad \alpha, \lambda_1, t > 0,
$$

$$
f_2(t) = \alpha\lambda_2 t^{\alpha-1} e^{-\lambda_2 t^{\alpha}} \qquad \text{for} \quad \alpha, \lambda_2, t > 0.
$$

In this case also the relative risk rate due to cancer is π. The hazard rate or the instantaneous death rate of T_1 at the time point t is given by $\alpha\lambda_1 t^{\alpha}$ and that due to T_2 is given by $\alpha\lambda_2 t^{\alpha}$.

The log-likelihood function (2.1) takes the following form;

$$
\begin{aligned}
L &= n_1\, ln\,(\pi) + n_1\, ln\,(\alpha) + n_1\, ln\,(\lambda_1) + (\alpha-1) \sum_{i \in I_1} ln(t_i) - \lambda_1 \sum_{i \in I_1} t_i^{\alpha} \\
&+ n_2\, ln\,(1-\pi) + n_2\, ln\,(\alpha) + n_2\, ln\,(\lambda_2) + (\alpha-1) \sum_{i \in I_2} ln\,(t_i) - \lambda_2 \sum_{i \in I_2} t_i^{\alpha} \\
&+ \sum_{i \in I_3} ln\left[\pi\alpha\lambda_1 t_i^{\alpha} e^{-\lambda_1 t_i^{\alpha}} + (1-\pi)\alpha\lambda_2 t_i^{\alpha} e^{-\lambda_2 t_i^{\alpha}}\right]. \qquad (5.1)
\end{aligned}
$$

Again in this case it is possible to obtain the MLE's of the unknown parameters by maximizing (5.1) directly with respect to the unknown parameters and it involves four dimensional optimization. On the other hand it is possible to use the EM algorithm as before. In this case the weight vector $w_j(x, \boldsymbol{\theta})$ takes the following form

$$w_j(t, \boldsymbol{\theta}) = P_{\boldsymbol{\theta}}[\Delta^* = j|t] = \frac{\pi\lambda_j e^{-\lambda_j t^\alpha}}{\pi\lambda_1 e^{-\lambda_1 t^\alpha} + (1-\pi)\lambda_2 e^{-\lambda_2 t^\alpha}}, \qquad \text{for} \quad j = 1, 2, \tag{5.2}$$

where $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \pi, \alpha)$. The log-likelihood function of the 'pseudo data' can be written as $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$, where

$$\begin{aligned}
L_1(\boldsymbol{\theta}) &= n_1\ ln\ (\pi) + n\ ln\ (\alpha) + n_1\ ln\ (\lambda_1) + n_2\ ln\ (1-\pi) + n_2\ ln\ (\lambda_2) \\
&\quad +(\alpha-1)\sum_{i\in I} ln\ (t_i) - \lambda_1 \sum_{i\in I_1} t_i^\alpha - \lambda_2 \sum_{i\in I_2} t_i^\alpha, \\
L_2(\boldsymbol{\theta}) &= (ln\ (\pi) + ln\ (\lambda_1)) \sum_{i\in I_3} w_1(t_i, \boldsymbol{\theta}) + (ln\ (1-\pi) + ln\ (\lambda_2)) \sum_{i\in I_3} w_2(t_i, \boldsymbol{\theta}) \\
&\quad -\lambda_1 \sum_{i\in I_3} w_1(t_i, \boldsymbol{\theta}) t_i^\alpha - \lambda_2 \sum_{i\in I_3} w_2(t_i, \boldsymbol{\theta}) t_i^\alpha.
\end{aligned}$$

The maximization of the (M) step involves maximizing $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$ iteratively. It can be done similarly as before. Assuming $w_j(t_i, \boldsymbol{\theta})$ to be known we need to maximize $L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$, with respect to $\boldsymbol{\theta}$. If $\boldsymbol{\theta}^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \pi^{(i)}, \alpha^{(i)})$ is the i^{th} iterate, then $(i+1)^{th}$ iterate $\boldsymbol{\theta}^{(i+1)} = (\lambda_1^{(i+1)}, \lambda_2^{(i+1)}, \pi^{(i+1)}, \alpha^{(i)})$ can be obtained as follows :

$$\begin{aligned}
\lambda_1^{(i+1)} &= \frac{n_1 + \sum_{i\in I_3} w_1(t_i, \boldsymbol{\theta}^{(i)})}{\sum_{i\in I_1} t_i^{\alpha^{(i)}} + \sum_{i\in I_3} t_1(x_i, \boldsymbol{\theta}^{(i)}) t_i^{\alpha^{(i)}}}, \\
\lambda_2^{(i+1)} &= \frac{n_2 + \sum_{i\in I_2} w_2(t_i, \boldsymbol{\theta}^{(i)})}{\sum_{i\in I_2} t_i^{\alpha^{(i)}} + \sum_{i\in I_3} w_2(t_i, \boldsymbol{\theta}^{(i)}) t_i^{\alpha^{(i)}}}, \\
\pi^{(i+1)} &= \frac{n_1 + \sum_{i\in I_3} w_1(t_i, \boldsymbol{\theta}^{(i)})}{n}.
\end{aligned}$$

Finally $\alpha^{(i+1)}$ can be obtained as follows :

$$\alpha^{(i+1)} = \arg\max\ g(\alpha),$$

where

$$\begin{aligned}
g(\alpha) = n\ ln\ (\alpha) + (\alpha-1)\sum_{i\in I} ln\ (t_i) - \lambda_1^{(i)} \sum_{i\in I_1} t_i^\alpha - \lambda_2^{(i)} \sum_{i\in I_2} t_i^\alpha \\
-\lambda_1^{(i)} \sum_{i\in I_3} w_1(t_i, \boldsymbol{\theta}^{(i)}) t_i^\alpha - \lambda_2^{(i)} \sum_{i\in I_3} w_2(t_i, \boldsymbol{\theta}^{(i)}) t_i^\alpha.
\end{aligned}$$

5.2 Confidence Intervals

In this subsection we discuss how to obtain the asymptotic confidence intervals of the different parameters. Similarly as in the exponential case, if we define $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\pi}, \hat{\alpha})$ to be the MLE 's of $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \pi, \alpha)$, then from the asymptotic theory of the MLE's we also have

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \rightarrow N_4(\mathbf{0}, \mathbf{I}^{-1}), \tag{5.3}$$

where $\mathbf{I}$ is the information matrix. The observed information matrix can be obtained using the EM algorithm (Louis; 1982) and in this case it takes the following form $\mathbf{I} = \mathbf{B} - \mathbf{S}\mathbf{S}^T$, where $\mathbf{B}$ is a 4×4 symmetric matrix and $\mathbf{S}$ is a 4×1 vector. The elements of the matrix $\mathbf{B}$ are as follows :

$$\begin{aligned}
B(1,1) &= \frac{n_1}{\hat{\lambda}_1^2} + \frac{1}{\hat{\lambda}_1^2} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) \\
B(2,2) &= \frac{n_2}{\hat{\lambda}_2^2} + \frac{1}{\hat{\lambda}_2^2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) \\
B(3,3) &= \frac{n_1}{\hat{\pi}^2} + \frac{n_2}{(1-\hat{\pi})^2} + \frac{1}{\hat{\pi}^2} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) + \frac{1}{(1-\hat{\pi})^2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) \\
B(4,4) &= \frac{n}{\hat{\alpha}^2} + \hat{\lambda}_1 \left[\sum_{i \in I_1} t_i^{\hat{\alpha}} (ln(t_i))^2 + \sum_{i \in I_3} t_i^{\hat{\alpha}} (ln(t_i))^2 w_1(t_i, \hat{\boldsymbol{\theta}}) \right] \\
&+ \hat{\lambda}_2 \left[\sum_{i \in I_2} t_i^{\hat{\alpha}} (ln(t_i))^2 + \sum_{i \in I_3} t_i^{\hat{\alpha}} (ln(t_i))^2 w_2(t_i, \hat{\boldsymbol{\theta}}) \right] \\
B(1,4) &= B(4,1) = \sum_{i \in I_1} t_i^{\hat{\alpha}}\ ln\ (t_i) + \sum_{i \in I_3} t_i^{\hat{\alpha}}\ ln\ (t_i) w_1(t_i, \hat{\boldsymbol{\theta}}) \\
B(2,4) &= B(4,2) = \sum_{i \in I_2} t_i^{\hat{\alpha}}\ ln\ (t_i) + \sum_{i \in I_3} t_i^{\hat{\alpha}}\ ln\ (t_i) w_2(t_i, \hat{\boldsymbol{\theta}}).
\end{aligned}$$

The rest of the elements of the matrix $\mathbf{B}$ are zero. The elements of the vector $\mathbf{S}$ are as follows:

$$\begin{aligned}
S(1) &= \frac{n_1}{\hat{\lambda}_1} - \sum_{i \in I_1} t_i^{\hat{\alpha}} + \frac{1}{\hat{\lambda}_1} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) - \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) t_i^{\hat{\alpha}} \\
S(2) &= \frac{n_2}{\hat{\lambda}_2} - \sum_{i \in I_2} t_i^{\hat{\alpha}} + \frac{1}{\hat{\lambda}_2} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) - \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) t_i^{\hat{\alpha}} \\
S(3) &= \frac{n_1}{\hat{\pi}} - \frac{n_2}{(1-\hat{\pi})} + \frac{1}{\hat{\pi}} \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) - \frac{1}{(1-\hat{\pi})} \sum_{i \in I_3} w_2(t_i, \hat{\boldsymbol{\theta}}) \\
S(4) &= \frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} ln(t_i) - \hat{\lambda}_1 \left[\sum_{i \in I_1} x_i^{\hat{\alpha}} \ln(t_i) + \sum_{i \in I_3} w_1(t_i, \hat{\boldsymbol{\theta}}) t_i^{\hat{\alpha}} \ln(t_i) \right] \\
&- \hat{\lambda}_2 \left[\sum_{i \in I_1} t_i^{\hat{\alpha}} \ln(t_i) + \sum_{i \in I_3} w_2(t_i, \boldsymbol{\theta}) t_i^{\hat{\alpha}} \ln(t_i) \right].
\end{aligned}$$

Therefore, using (5.3) and the observed information matrix it is possible to construct confidence intervals of the unknown parameters. The asymptotic results and the confidence intervals of the other parameters can be obtained using the δ-method similarly as before.

6. NUMERICAL EXPERIMENTS:

In this section we present some numerical results to illustrate how the different methods behave for small sample sizes and also for different parameters. We consider both the cases, namely when the lifetimes follow exponential and Weibull distributions. We observe the bias and the variance of the MLE's of the different parameters. We also compute the confidence intervals of the unknown parameters for different sample sizes based on asymptotic results.

First we present the results when the lifetimes are assumed to be exponentially distributed. From the asymptotic distribution of the MLE's it is clear that the asymptotic distributions of $\frac{\hat{\lambda}_1}{\lambda_1}$ and $\frac{\hat{\lambda}_2}{\lambda_2}$ are independent of λ_1 and λ_2 respectively. Therefore, without loss of generality we keep $\lambda_1 = 1$ fixed and consider different values of λ_2 namely 0.75 and 0.90. We take sample sizes n = 20, 30, 40 and 50 and π = .40 and .60. In all cases we assume that 10% of the failure times are not known. For a given λ_1, λ_2 and π, we draw a random sample of size n and we compute the MLE's of λ_1, λ_2, and the corresponding 95% asymptotic confidence intervals (Asymp) for different parameters. We replicate the process one thousand times and compute the average of the MLE's and the average mean squared errors (mse's). For different confidence intervals we compute the coverage percentages and the average confidence lengths. The results are reported in Table 1 and Table 2. In Table 1, we report the average estimates with the corresponding mean squared errors within parentheses. In Table 2 we present the average confidence lengths and the corresponding coverage percentages are presented within parentheses for different parameters.

It is clear from Table 1 that as sample size increases the biases and the mean squared errors decrease for all the parameters. It verifies the consistency property of the maximum likelihood estimators. The present simulations indicate that the MLE's are asymptotically unbiased and consistent estimators of the corresponding parameters. For a fixed sample size the mse's of $\hat{\pi}$'s are fixed, that is independent of λ_1 and λ_2. The biases of $\hat{\pi}$'s are quite small and even when the sample size is only 20, the relative bias of $\hat{\pi}$'s is only 1%. For fixed sample sizes and for fixed π, the average estimates and the mse's of $\hat{\lambda}_1$ are fixed. If π increases the mse's of $\hat{\lambda}_1$ decrease. It is not very surprising because when π is high the sample has more information about λ_1. Note that for fixed π, the mse's of $\hat{\lambda}_2/\lambda_2^2$ are constant. It indicates that $\text{Var}(\hat{\lambda}_2/\lambda_2)$ is independent of λ_2. When π increases the mse's of $\hat{\lambda}_2$ increase and the reason has been explained before. Observing the behavior of the MLE's it can be said that if π is not too far away from 0.5, the MLE's behave quite well even when the sample size is reasonably small.

Some points are quite clear from Table 2 as well. It is observed that as sample size increases the average confidence lengths decrease and also the coverage percentages become closer to the nominal value (95%) for all the parameters. When estimating π, as expected the average confidence lengths and the coverage percentages remain same. For fixed π and n, the average confidence lengths and the coverage percentages of λ_1 are fixed for different values of λ_2 and when π increases the average confidence lengths decrease. Exactly the opposite behavior is observed in case of λ_2, that is when π increases the average confidence lengths increase. For λ_2, the average confidence length/λ_2 is fixed for different values of λ_2, when π is fixed. From the simulation study it can be said that the performance of the MLE's are quite good. For π the coverage percentages of the asymptotic confidence intervals are slightly lower than the nominal value but for λ_1 and λ_2 they achieve the nominal value even when the sample size is 20. But if the sample size is more than 40 the asymptotic results perform very well for all the parameters.

Now we present the results when the lifetimes follow Weibull distributions. In this case also we keep $\lambda_1 = 1$ and consider different values of λ_2 namely 0.75 and 0.90 as before. We take sample sizes as $n = 20, 30, 40, 50$, $\pi = 0.40, 0.60$ and $\alpha = 1$. Here also in all cases we assume that 10% of the data are incomplete. Exactly as before we estimate the parameters using EM algorithms and consider three different 95% confidence intervals. For different confidence intervals we compute the average confidence lengths and the coverage percentages. The results are reported in Tables 3 and 4. In Table 3, we report the average estimates with the corresponding mse's are reported within parentheses. In Table 4, we report the average confidence lengths and the corresponding coverage percentages are reported within parentheses, for different parameters.

Table 3 provides some clear insights into certain points. It is immediate that as sample size increases the biases and the mse's decrease for all the parameters and it confirms the consistency property of the MLE's. It verifies that the MLE's provide consistent and asymptotically unbiased estimators. The asymptotic behavior of $\hat{\pi}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are same as in the exponential case and for $\hat{\alpha}$ the biases and the mse's are independent of π, λ_1 and λ_2. The biases of $\hat{\alpha}$ are quite small ($\approx$ 10%) even when the sample size is 20. It is clear that MLE's are working quite well and EM algorithm can be used quite effectively in finding the MLE's even when the lifetimes have Weibull distributions.

Table 4 clarifies a few other points as well. As it is expected, it is observed that as sample size increases the average confidence lengths decrease and the also the coverage percentages increase to the nominal value (95%) for all methods and for all parameters. Interestingly in this case, the average confidence lengths and also the coverage percentages of $\hat{\pi}$ are very similar with the exponential case. It indicates that even if one parameter is increased it is not affecting the asymptotic property of $\hat{\pi}$. The average confidence lengths for λ_1 and λ_2 become larger compared to the corresponding exponential case. For the Weibull case also, it can be said that the performance of the MLE's are quite well. For π, the coverage percentages of the asymptotic confidence intervals are slightly lower than the nominal value but for λ_1, λ_2 and α they achieve the nominal value even when the sample size is 20. But if the sample size is more than 40 in this case also the asymptotic results perform very well for all the parameters.

7. DATA ANALYSIS

For illustration purposes, we consider one life data sets from Lawless (1982,page 491). It consists of failure or censoring times for 36 appliances subjected to an automatic life test. Failures were classified into 18 different modes, though among 33 observed failures only 7 modes are present and only modes 6 and 9 appear more than once. We are mainly interested in failure mode 9. The data consists of two causes of failure, δ = 1 (failure mode 9) and δ = 2 (all other failure modes). The data are given below, consisting of the failure times and the cause of failure ($\delta = 1$ or $\delta = 2$) if available.

Data Set: (1167,1), (1925,1), (1990,1), (2223,1), (2400,1), (2471,1), (2551,1), (2568,1), (2694,1), (3034,1),(3112,1), (3124,1), (3478,1), (3504,1), (4329,1), (6976,1), (7846,1), (11,2), (35,2), (49,2), (170,2), (329,2), (381,2), (708,2), (958,2), (1062,2), (1594,2), (2327,2), (2451,2), (2702,2), (2761,2), (2831,2), (3059,2) (2565,*), (6367,*), (13403,*).

Here '*' indicates that the corresponding cause of failure is unknown. We have $n = 36$, $n_1 = 17$, $n_2 = 16$. First we consider the case when the lifetimes are assumed to be exponentially distributed. The initial estimates of π, λ_1 and λ_2 are 0.5152, 0.000306 and 0.000747 respectively. After six iterations the EM algorithm converges and the final estimates are $\hat{\pi} = 0.5404$, $\hat{\lambda}_1 = 0.000256$ and $\hat{\lambda}_2 = 0.000709$. We stopped the iteration when the difference between the estimates at the i^{th} and $(i+1)^{th}$ iterations are less than 10^{-8} for all the three parameters. We obtain the Kolmogorov-Smirnov (K-S) distance between the empirical distribution function and the estimated distribution function. The K-S distance between them is 0.2469 and the corresponding p value is 0.0248. The 95% asymptotic confidence intervals of λ_1, λ_2 and π are (0.000142,0.000370), (0.000367, 0.001051) and (0.3775, 0.7031) respectively. The estimate of the expected lifetime due to cause 1, $\hat{\theta}_1 = 3906.25$ and the corresponding 95% asymptotic confidence band is (2170.08, 5642.42). Similarly the estimate of the expected lifetime due to other causes, $\hat{\theta}_2 = 1410.44$ and the corresponding asymptotic 95% confidence band is (720.92,2099.96). It clearly indicates that the expected lifetime due to cause 1 is significantly higher than cause 2. We also obtain the estimate of the expected lifetime of all individuals, $\hat{\tau}_1 = 2759.18$ and the corresponding 95% confidence band is (1690.08,3828.28) The estimate of the mortality index due to cause 1, $\hat{\tau}_2 = 0.000196$ and the corresponding 95% confidence interval is (0.000147,.000245).

Now we do the analysis under the assumption that the lifetimes follow Weibull distributions with the same shape parameter. Using the initial estimates of π, λ_1, λ_2 and α as 0.5152, 0.000306, 0.000747 and 1 respectively, the final estimates are $\hat{\pi} = 0.5419$, $\hat{\lambda}_1 = 0.000102$, $\hat{\lambda}_2 = 0.000314$ and $\hat{\alpha} = 1.1092$. In this case the EM algorithm takes 57 iterations to reach the final estimates. Here also we used the same stopping criterion as in the exponential case. In this case the K-S distance between the empirical distribution function and the estimated distribution function is 0.2331 and the corresponding p value is 0.0398. The 95% asymptotic confidence intervals of the λ_1, λ_2, π and α are (-0.000040, 0.000285), (-0.000039, 0.000847), (0.3792, 0.7046), (0.8713,1.5058). The lower bound of the confidence intervals of λ_1 and λ_2 are slightly lower than zero and for practical purposes then can be replaced by zero. Using the Weibull mixture model the estimate of the expected lifetime due to cause 1, $\hat{\theta}_1 = 3817.68$ and the corresponding 95% confidence interval is (2074.75, 5560.61). The estimate of the expected lifetime due to other causes is $\hat{\theta}_2 = 1385.31$ and the corresponding 95% confidence interval is (674.65, 2095.97). Similarly we obtain the estimate of the expected lifetime of all individuals, $\hat{\tau}_1 = 2703.41$ and the estimate of the mortality index due to cause 1, $\hat{\tau}_2 = 0.000200$. The corresponding 95% confidence intervals are (1605.30,3801.52) and (0.000149, 0.000251) respectively.

Note that the confidence intervals of α includes one and therefore, it may not be unreasonable to assume that the lifetimes are exponential. Also if we perform the Chi-square test between the exponential mixture model and the Weibull mixture model then also it indicates that the exponential mixture model is the preferred one in this case.

8. CONCLUSIONS:

In this paper we consider estimation of the parameters of the competing risks model when the data may not be complete and when the competing risks are not independent. In this paper the main difference from Babu, Rao and Rao (1992) or Kundu, Kannan and Mazumdar (1992) is that here the data may be partially incomplete in the sense that for some individual lifetime is known but the corresponding cause of failure may be unknown. We consider two different lifetime distributions of the competing causes, namely exponential and Weibull distributions. We use the MLE's to estimate the unknown parameters. EM algorithm is used quite effectively to compute the MLE's and also to obtain the asymptotic dispersion matrix. The asymptotic dispersion matrix is used to compute asymptotic confidence bounds for different parameters. By extensive simulations it is observed that the MLE's work very well in estimating unknown parameters and the MLE's are asymptotically unbiased. Even when the sample size is quite small the asymptotic results work quite well for all the parameters. It is observed that the exponential mixture model or the Weibull mixture model can be used quite effectively for analyzing competing risks data when some of the causes are unknown and with out the assumptions that the competing risks are independent.

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Table 1

Average values of $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\pi}$ and the corresponding mean squared errors. The mean squared errors of the estimates are reported within parentheses against each average estimates. Here the lifetimes are exponential.

n		$\lambda_2 = 0.75$ $\pi = .40$	$\lambda_2 = 0.90$ $\pi = .40$	$\lambda_2 = 0.75$ $\pi = .60$	$\lambda_2 = 0.90$ $\pi = .60$
	$\hat{\pi}$	0.396(0.013)	0.396(0.013)	0.603(0.013)	0.602(0.013)
20	$\hat{\lambda}_1$	1.172(0.386)	1.174(0.391)	1.097(0.153)	1.098(0.152)
	$\hat{\lambda}_2$	0.839(0.091)	1.006(0.132)	0.913(0.338)	1.093(0.482)
	$\hat{\pi}$	0.398(0.009)	0.398(0.009)	0.598(0.009)	0.598(0.009)
30	$\hat{\lambda}_1$	1.098(0.160)	1.099(0.160)	1.061(0.081)	1.063(0.081)
	$\hat{\lambda}_2$	0.809(0.053)	0.969(0.077)	0.836(0.087)	1.003(0.126)
	$\hat{\pi}$	0.400(0.007)	0.400(0.006)	0.601(0.007)	0.601(0.007)
40	$\hat{\lambda}_1$	1.062(0.091)	1.064(0.091)	1.042(0.056)	1.043(0.055)
	$\hat{\lambda}_2$	0.793(0.034)	0.951(0.049)	0.819(0.058)	0.983(0.084)
	$\hat{\pi}$	0.398(0.005)	0.398(0.005)	0.600(0.005)	0.600(0.005)
50	$\hat{\lambda}_1$	1.047(0.072)	1.048(0.072)	1.033(0.043)	1.033(0.043)
	$\hat{\lambda}_2$	0.786(0.026)	0.943(0.038)	0.805(0.043)	0.966(0.062)

Table 2

Average confidence lengths of λ_1, λ_2, π and their coverage percentages. Here the lifetimes are exponential.

n		$\lambda_2 = 0.75$ $\pi = .40$	$\lambda_2 = 0.90$ $\pi = .40$	$\lambda_2 = 0.75$ $\pi = .60$	$\lambda_2 = 0.90$ $\pi = .60$
	$\hat{\pi}$	0.416(0.903)	0.416(0.920)	0.416(0.917)	0.417(0.925)
20	$\hat{\lambda}_1$	1.685(0.952)	1.686(0.953)	1.259(0.952)	1.258(0.958)
	$\hat{\lambda}_2$	0.935(0.952)	1.122(0.951)	1.344(0.952)	1.612(0.953)
	$\hat{\pi}$	0.343(0.921)	0.343(0.921)	0.344(0.932)	0.344(0.934)
30	$\hat{\lambda}_1$	1.290(0.946)	1.290(0.947)	0.995(0.929)	0.996(0.933)
	$\hat{\lambda}_2$	0.748(0.935)	0.898(0.932)	0.961(0.949)	1.153(0.947)
	$\hat{\pi}$	0.299(0.921)	0.299(0.920)	0.299(0.916)	0.299(0.912)
40	$\hat{\lambda}_1$	1.072(0.939)	1.071(0.939)	0.840(0.939)	0.840(0.940)
	$\hat{\lambda}_2$	0.629(0.956)	0.755(0.959)	0.795(0.955)	0.955(0.952)
	$\hat{\pi}$	0.268(0.922)	0.268(0.922)	0.268(0.938)	0.269(0.936)
50	$\hat{\lambda}_1$	0.932(0.940)	0.933(0.944)	0.747(0.945)	0.744(0.948)
	$\hat{\lambda}_2$	0.562(0.959)	0.676(0.957)	0.702(0.946)	0.843(0.945)

Table 3

Average values of $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\pi}$, $\hat{\alpha}$ and the corresponding mean squared errors. The mean squared errors of the estimates are reported within parentheses against each average estimates. Here the lifetimes are Weibull.

n		$\lambda_2 = 0.75$ $\pi = .40$	$\lambda_2 = 0.90$ $\pi = .40$	$\lambda_2 = 0.75$ $\pi = .60$	$\lambda_2 = 0.90$ $\pi = .60$
20	$\hat{\pi}$	0.396(0.013)	0.397(0.013)	0.602(0.013)	0.602(0.013)
	$\hat{\lambda}_1$	1.186(0.577)	1.187(0.591)	1.086(0.206)	1.087(0.206)
	$\hat{\lambda}_2$	0.802(0.112)	0.981(0.168)	0.895(0.509)	1.096(0.766)
	$\hat{\alpha}$	1.122 (0.065)	1.122(0.065)	1.121(0.064)	1.122(0.064)
30	$\hat{\pi}$	0.398(0.009)	0.398(0.009)	0.599(0.009)	0.598(0.009)
	$\hat{\lambda}_1$	1.094(0.192)	1.094(0.190)	1.048(0.092)	1.048(0.091)
	$\hat{\lambda}_2$	0.782(0.062)	0.950(0.088)	0.814(0.098)	0.989(0.146)
	$\hat{\alpha}$	1.075 (0.034)	1.076(0.033)	1.074(0.033)	1.074(0.033)
40	$\hat{\pi}$	0.400(0.007)	0.400(0.007)	0.601(0.007)	0.601(0.007)
	$\hat{\lambda}_1$	1.054(0.101)	1.054(0.101)	1.031(0.062)	1.031(0.062)
	$\hat{\lambda}_2$	0.774(0.039)	0.936(0.053)	0.800(0.064)	0.969(0.092)
	$\hat{\alpha}$	1.053 (0.021)	1.053(0.022)	1.054(0.022)	1.054(0.022)
50	$\hat{\pi}$	0.399(0.005)	0.399(0.005)	0.600(0.005)	0.600(0.005)
	$\hat{\lambda}_1$	1.040(0.079)	1.040(0.079)	1.023(0.047)	1.023(0.046)
	$\hat{\lambda}_2$	0.769(0.030)	0.929(0.040)	0.789(0.048)	0.953(0.069)
	$\hat{\alpha}$	1.044 (0.017)	1.044(0.016)	1.044(0.017)	1.045(0.017)

Table 4

Average confidence lengths of λ_1, λ_2, π, $\hat{\alpha}$ and their coverage percentages. The coverage percentages are reported within parentheses. Here the lifetimes are Weibull.

n		$\lambda_2 = 0.75$ $\pi = .40$	$\lambda_2 = 0.90$ $\pi = .40$	$\lambda_2 = 0.75$ $\pi = .60$	$\lambda_2 = 0.90$ $\pi = .60$
20	$\hat{\pi}$	0.416(0.902)	0.416(0.903)	0.417(0.907)	0.417(0.904)
	$\hat{\lambda}_1$	1.796(0.918)	1.798(0.917)	1.299(0.920)	1.299(0.927)
	$\hat{\lambda}_2$	0.999(0.919)	1.187(0.921)	1.422(0.918)	1.722(0.929)
	$\hat{\alpha}$	0.786(0.942)	0.787(0.943)	0.785(0.941)	0.786(0.942)
30	$\hat{\pi}$	0.344(0.928)	0.344(0.928)	0.344(0.926)	0.344(0.930)
	$\hat{\lambda}_1$	1.309(0.937)	1.309(0.939)	1.015(0.923)	1.015(0.926)
	$\hat{\lambda}_2$	0.791(0.913)	0.930(0.915)	0.992(0.925)	1.182(0.929)
	$\hat{\alpha}$	0.610(0.946)	0.610(0.946)	0.609(0.943)	0.610(0.945)
40	$\hat{\pi}$	0.299(0.919)	0.299(0.920)	0.299(0.919)	0.299(0.919)
	$\hat{\lambda}_1$	1.076(0.943)	1.076(0.944)	0.860(0.936)	0.860(0.935)
	$\hat{\lambda}_2$	0.677(0.929)	0.792(0.933)	0.845(0.930)	1.001(0.933)
	$\hat{\alpha}$	0.516(0.945)	0.516(0.947)	0.516(0.941)	0.516(0.942)
50	$\hat{\pi}$	0.268(0.927)	0.268(0.927)	0.269(0.939)	0.269(0.936)
	$\hat{\lambda}_1$	0.947(0.934)	0.947(0.933)	0.761(0.922)	0.761(0.925)
	$\hat{\lambda}_2$	0.600(0.927)	0.701(0.935)	0.740(0.937)	0.874(0.936)
	$\hat{\alpha}$	0.456(0.952)	0.456(0.947)	0.456(0.949)	0.456(0.952)