CLOSENESS OF GAMMA AND GENERALIZED EXPONENTIAL DISTRIBUTION

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ABSTRACT

Recently a new distribution, named as generalized exponential distribution or exponentiated exponential distribution was introduced and studied quite extensively by the authors. It is observed that the generalized exponential distribution can be used as an alternative to the gamma distribution in many situations. Different properties like monotonicity of the hazard functions and tail behaviors of the gamma distribution and the generalized exponential distribution are quite similar in nature, but the later one has a nice compact distribution function. It is observed that for a given gamma distribution there exists a generalized exponential distribution so that the two distribution functions are almost identical. Since the gamma distribution function does not have a compact form, efficiently generating gamma random numbers is known to be problematic. We observe that for all practical purposes it is possible to generate approximate gamma random numbers using generalized exponential distribution and the random samples thus obtained can not be differentiated using any statistical tests. Moreover, if there is a skewed data set where gamma distribution fits very well, the generalized exponential distribution also can be used. We use two real life data sets and

observe that the fitted distribution functions are 'almost identical' in many respects in both the cases.

1. INTRODUCTION

Recently a new distribution, named as generalized exponential (GE) distribution or exponentiated exponential distribution was introduced and studied quite extensively by the authors in a series of papers (Gupta and Kundu; 10, 11, 12, 13). The two-parameter GE distribution has the following distribution function;

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; \qquad \alpha,\lambda,x>0. \tag{1.1}$$

It has a density function

$$f(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}, \tag{1.2}$$

survival function

$$S(x;\alpha,\lambda) = 1-(1-e^{-\lambda x})^{\alpha}, \tag{1.3}$$

and a hazard function

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}. \tag{1.4}$$

Here α is the shape parameter and λ is the scale parameter. When the shape parameter α equals 1 it coincides with the 1-parameter exponential distribution. Therefore, GE distribution is a generalization of an 1-parameter exponential distribution having a shape parameter α. It is observed by Gupta and Kundu (10) that the two-parameter GE distribution can be used quite effectively in analyzing many skewed lifetime data, and the properties of the two-parameter GE distribution are quite close to the corresponding properties of the two-parameter gamma distribution. Both the gamma and the GE distributions have concave densities. The GE density is unimodal with mode at $\frac{1}{\lambda}ln(\alpha)$ (a function of α) for $\alpha > 1$ and it is reverse 'J' shaped for $\alpha < 1$ just like gamma density. The mean of both distributions diverges to ∞ as the shape parameter goes to ∞. The two-parameter GE distribution can also have increasing and decreasing hazard rates depending on the shape parameter. If the shape parameter is greater than one then for both gamma and GE hazard rates increase from 0 to λ and if the shape parameter is less than one they decrease from ∞ to λ. The tail behavior of the two distributions are also quite similar.

Some of the dissimilarities of the two distributions are as follows (see Gupta and Kundu (10)). Unlike gamma distribution, the distribution function, the density function, survival function and the hazard function of a GE distribution have convenient representations. The variance of the gamma distribution goes to ∞ as the shape parameter increases whereas the variance of the GE distribution goes to a constant $\frac{\pi^2}{6\lambda^2}$ as α tends to ∞. The gamma density almost becomes symmetric as the shape parameter increases and it can be seen that the skewness converges to zero as α increases unlike GE distribution. For GE, it is observed that the skewness does not go to zero as α increases, in fact the limiting value of the skewness is approximately 1.1395. It shows that even if α becomes very large the GE density always remains skewed.

Since the gamma distribution function does not have a simple form, it is well known (Johnson (15), Johnson, Kotz and Balakrishnan (16)) that generating pseudo random numbers from a gamma distribution is not very simple. There are several techniques available in the literature for generating pseudo gamma deviates. The well known ones are by Ahrens and Dieter (1), Atkinson (2), Cheng (4), Cheng and Feast (5, 6), Devroye (7), Fishman (8), Greenwood (9), Marsaglia (18), Schmeiser and Lal (19) and Tadikamalla (20, 21). For comparison and comments on the different methods see Tadikamalla and Johnson (22). Interestingly, most of the methods are mainly based on acceptance-rejection principle or some of its variants and all of them have their own drawbacks, see Johnson, Kotz and Balakrishnan (16). None of the methods work well for the entire range of the shape parameter. One needs different generators for different ranges of the shape parameter.

Since the two distributions have so many common properties, we felt that for a given a gamma distribution, at least within a certain range of the shape parameter where the gamma density function is not symmetric, there might exists a GE distribution which is very close to that gamma distribution. If there exists such a GE distribution, it can be useful in two different ways. First of all since the generation of gamma random deviates are always problematic, we can use the GE random deviate generator to generate approximate gamma random deviates at least if the shape parameter is within a range. Secondly for a skewed data if gamma can be fitted well to a given data set then it is expected that GE also should fit well to the same data set. Since the estimation of the GE parameters are easier compared to the gamma parameter estimation (see Gupta and Kundu (11, 12)), we can use GE distribution in place of the gamma distribution in data analysis also.

The main aim of this paper is to examine whether for a given gamma distribution, there exists a GE distribution which for all practical purposes is identical to the gamma distribution. The natural questions are how do we approximate a gamma distribution by a GE distribution and also how good is the approximation? Also if it is possible to approximate a gamma distribution by a GE distribution for a certain range of the shape parameter, what should be that range? We answer all these questions in the present paper. In approximating the gamma distribution by a GE distribution, we equate their two moments and their two L-moments (See Hosking (14) for details). The L-moments of any distribution are analogous to the conventional moments but they are based on quantiles and they can be estimated by the linear combinations of order statistics, i.e. by L-statistics. Let Z be any random variable having finite first moment and let $Z_{1:n} \leq \ldots \leq Z_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of Z. Then the L-moments of Z are defined as follows;

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(Z_{r-k:r}), \qquad r = 1, 2, \ldots,$$

see Hosking (14). L-moments have the theoretical advantages over the conventional moments of being able to characterize a wider range of distributions and, when estimated from a sample, of being more robust to the presence of outliers in the data. Unlike usual moment estimates, the parameter estimates obtained from L-moments are sometimes more accurate in small samples than even the maximum likelihood estimates. The moments and the L-moments of gamma and GE distributions will be provided in the next section.

The rest of the paper is organized as follows. In section 2, we give different notations used in this paper and some of the preliminaries. In section 3, we give two methods to approximate a gamma distribution by a GE distribution and discuss the goodness of these approximations. In section 4, we consider the generation of approximate gamma random deviates using GE generator. The data analysis are reported in section 5. Finally the conclusion appears in section 6.

2. NOTATIONS AND PRELIMINARIES

In this section we provide all the notations used in this paper and also the preliminaries we need for further development. Generalized exponential distribution as defined in section 1, having shape parameter α and the scale parameter λ will be denoted by $GE(\alpha, \lambda)$. If X follows $GE(\alpha, \lambda)$, then from Gupta and Kundu (10)

$$\mu = E(X) = \frac{1}{\lambda}\left(\psi(\alpha+1) - \psi(1)\right),$$
$$\sigma^2 = V(X) = \frac{1}{\lambda^2}\left(\psi'(1) - \psi'(\alpha+1)\right), \tag{2.1}$$

here $\psi(.)$ and $\psi'(.)$ are the digamma function and the polygamma function respectively, see Johnson, Kotz and Balakrishnan (16). The two L-moments (Gupta and Kundu (11)) of X are

$$\lambda_1 = \mu = \frac{1}{\lambda}\left(\psi(\alpha+1) - \psi(1)\right),$$
$$\lambda_2 = \frac{1}{\lambda}\left(\psi(2\alpha+1) - \psi(\alpha+1)\right). \tag{2.2}$$

Gamma(α, λ) denotes the gamma random variable with the density function

$$g(x, \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}. \tag{2.3}$$

Here α is the shape parameter and λ is the scale parameter. It is well known that if Y follows Gamma(α,λ), then

$$\mu = E(Y) = \frac{\alpha}{\lambda} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{\alpha}{\lambda^2}.$$

The two L-moments of Y are

$$\lambda_1 = E(Y) = \frac{\alpha}{\lambda}, \qquad \lambda_2 = \frac{1}{\lambda}\left[\frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\right] = \frac{1}{\lambda} B(\frac{1}{2}, \alpha),$$

see Hosking (14). We also provide the first two sample L-moments for convenience. Let $X_1, \ldots, X_n$ be a random sample of size n from any distribution and if $X_{(1)} < \ldots < X_{(n)}$ denote the ordered observations, then the first two sample L-moments say l_1 and l_2 are defined as;

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}, \qquad l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)X_{(i)} - \bar{X}.$$

In the next section we study how we can approximate a gamma distribution by a GE distribution. The closeness of the two distributions will also be studied.

3. APPROXIMATION OF A GAMMA DISTRIBUTION BY A GE DISTRIBUTION

In this section we try to approximate a Gamma(α, λ) distribution by a GE distribution. Therefore for a given α and λ, our problem is to find α^* and λ^* such that $GE(\alpha^*, \lambda^*)$ is *closest* to Gamma(α, λ). Since both the distributions have two parameters each, we want the two distributions should have at least the same measure of locations and the measure of spreads. It can be achieved in two different ways.

Method 1: For the given α and λ of the gamma distribution, we equate the mean and variance of the gamma with GE and obtain λ^* and α^*. They are as follows;

$$\alpha = \frac{(\psi(\alpha^*+1) - \psi(1))^2}{(\psi'(1) - \psi'(\alpha^*+1))}, \tag{3.1}$$

$$\lambda^* = \frac{\lambda}{\alpha}\left(\psi(\alpha^*+1) - \psi(1)\right). \tag{3.2}$$

Therefore, for given α and λ, α^* can be obtained by solving (3.1) and then λ^* can be obtained from (3.2). We denote this pair as $(\alpha_1^*, \lambda_1^*)$.

Method 2: In this method, instead of equating the two moments of Gamma(α, λ) and $GE(\alpha^*, \lambda^*)$, we equate the two L- moments to solve for α^* and λ^*. For a given α and λ, α^* can be obtained by solving

$$\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} = \frac{\psi(\alpha^*+1) - \psi(1)}{\psi(2\alpha^*+1) - \psi(\alpha^*+1)}. \tag{3.3}$$

Once α^* is obtained, λ^* can be obtained from equation (3.2). We denote this pair as $(\alpha_2^*, \lambda_2^*)$.

The next question is how good are these approximations? The goodness of these approximations can be tested in different ways. Two important measure of discrepancies are (i)

Kolmogorov discrepancy and (ii) Kullback-Leibler discrepancy (Kullback and Leibler (17)). Kolmogorov discrepancy measure (KDM) between the two distribution functions, say between $F_1(.)$ and $F_2(.)$ can be defined as

$$KDM(F_1, F_2) = \sup_x |F_1(x) - F_2(x)|.$$

Let $f_i(x)$ be the probability density function corresponding to $F_i(x)$, $i = 1, 2$. Then logarithm of the likelihood ratio, $log\left(\frac{f_1(x)}{f_2(x)}\right)$, is considered as the information in x for discrimination in favor of $F_1(x)$ against $F_2(x)$. The mean information for discrimination in favor of $F_1(x)$ against $F_2(x)$ is called Kullback-Leibler discrepancy measure ($KLDM$) of $F_2(x)$ from $F_1(x)$ and is defined as

$$KLDM(F_1, F_2) = E_{F_1} log\left(\frac{f_1(x)}{f_2(x)}\right).$$

Note that although KDM is symmetric with respect to F_1 and F_2 but $KLDM$ is not. However, we could consider a measure of divergence between $F_1(.)$ and $F_2(.)$, which is symmetric and is defined as

$$SKLDM(F_1, F_2) = KLDM(F_1, F_2) + KLDM(F_2, F_1).$$

Now we compute the Kullbuck-Leibler discrepancy measures between different distributions. Let's denote $KLDM$ between Gamma (α_1, λ_1) and $GE(\alpha_1^*, \lambda_1^*)$ as $KLDM(G_1, GE_1)$, Gamma (α_1, λ_1) and Gamma(α_2, λ_2) as $KLDM$ (G_1, G_2), similarly we define the $KLDM$ between $GE(\alpha_1^*, \lambda_1^*)$ and $GE(\alpha_2^*, \lambda_2^*)$ as $KLDM(GE_1, GE_2)$ etc. We denote $f_{GE_1}(x) = f(x; \alpha_1^*, \lambda_1^*)$ (see (1.2)), similarly $f_{GE_2}(x) = f(x; \alpha_2^*, \lambda_2^*)$, moreover, $f_{G_1}(x)$ and $f_{G_2}(x)$ as the density functions of Gamma(α_1, λ_1) and Gamma(α_2, λ_2) respectively (see (2.3)). Therefore,

$$\begin{aligned}
KLDM(G_1, GE_1) &= \int_0^\infty f_{G_1}(x) ln\left(\frac{f_{G_1}(x)}{f_{GE_1}(x)}\right) dx \\
&= \int_0^\infty f_{G_1}(x) ln(f_{G_1}(x)) dx - \int_0^\infty f_{G_1}(x) ln(f_{GE_1}(x)) dx \\
&= \alpha_1 ln(\lambda_1) - (\alpha_1 + ln(\Gamma(\alpha_1)) + (\alpha_1 - 1)(\psi(\alpha_1) - ln(\lambda_1)) - ln(\alpha_1^* \lambda_1^*) \\
&\quad + \frac{\lambda_1^* \alpha_1}{\lambda_1} - (\alpha_1^* - 1)\lambda_1^{\alpha_1} \int_0^\infty ln(1 - e^{-\lambda_1^* x}) \frac{x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} e^{-\lambda_1 x} dx. \qquad (3.5)
\end{aligned}$$

$$
\begin{aligned}
KLDM(GE_1, G_1) &= \int_0^\infty f_{GE_1}(x) ln\left(\frac{f_{GE_1}(x)}{f_{G_1}(x)}\right) dx \\
&= \int_0^\infty f_{GE_1}(x) ln(f_{GE_1}(x)) dx - \int_0^\infty f_{GE_1}(x) ln(f_{G_1}(x)) dx \\
&= -1 + ln(\alpha_1^* \lambda_1^*) + \psi(1) - \psi(\alpha_1^*) + ln(\Gamma(\alpha_1)) + \frac{\lambda_1}{\lambda_1^*}(\psi(\alpha_1^* + 1) - \psi(1)) \\
&\quad -\alpha_1 ln(\lambda_1) - (\alpha_1 - 1) \int_0^\infty \lambda_1^* \alpha_1^* ln(x)(1 - e^{-\lambda_1^* x})^{\alpha_1^* - 1} e^{-\lambda_1^* x} dx.
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
KLDM(G_1, G_2) &= \int_0^\infty f_{G_1}(x) ln\left(\frac{f_{G_1}(x)}{f_{G_2}(x)}\right) dx \\
&= \int_0^\infty f_{G_1}(x) ln(f_{G_1}(x)) dx - \int_0^\infty f_{G_1}(x) ln(f_{G_2}(x)) dx \\
&= \alpha_1 ln(\lambda_1) - \alpha_2 ln(\lambda_2) + ln\left(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\right) \\
&\quad +(\alpha_1 - \alpha_2)\left(\psi(\alpha_1) - ln(\lambda_1)\right) + \alpha_1\left(\frac{\lambda_2}{\lambda_1} - 1\right).
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
KLDM(GE_1, GE_2) &= \int_0^\infty f_{GE_1}(x) ln\left(\frac{f_{GE_1}(x)}{f_{GE_2}(x)}\right) dx \\
&= \int_0^\infty f_{GE_1}(x) ln(f_{GE_1}(x)) dx - \int_0^\infty f_{GE_1}(x) ln(f_{GE_2}(x)) dx \\
&= ln\left(\frac{\alpha_1^* \lambda_1^*}{\alpha_2^* \lambda_2^*}\right) - \frac{(\alpha_1^* - 1)}{\alpha_1^*} + \left(\frac{\lambda_2^*}{\lambda_1^*} - 1\right)(\psi(\alpha_1^* + 1) - \psi(1)) \\
&\quad -(\alpha_2^* - 1)\alpha_1^* \lambda_1^* \int_0^\infty ln(1 - e^{-\lambda_2^* x})(1 - e^{-\lambda_1^* x})^{\alpha_1^* - 1} e^{-\lambda_1^* x} dx.
\end{aligned}
\tag{3.8}
$$

Similarly, we can get the other $KLDM$'s also. Using (3.5)-(3.8) we can compute easily the different $SKLDM$'s also. We compute numerically the KDM and $KLDM$ between gamma and the corresponding GE approximate distributions for different values of the shape parameter when the scale parameter of gamma is fixed at one. The results are reported in Tables 1 and 2.

From the tables it is clear that the discrepancy between Gamma and GE is minimum when the shape parameter of gamma is one. In that case both the distributions become exponential distribution. As the shape parameter goes away from one (in both directions) the discrepancy between the two increases. From the tables it is clear that $GE(\alpha_2^*, \lambda_2^*)$ is closer to the gamma compared to $GE(\alpha_1^*, \lambda_1^*)$. At this point we make the conjecture that for any given α between 0 and 2.5, there exists a GE distribution which can approximate the gamma distribution for all practical purposes. Now we try to verify this conjecture in the next section using numerical simulations.

4. GENERATING GAMMA DISTRIBUTION USING GE GENERATOR

In the previous section we have observed that we can approximate a gamma distribution reasonably well by a GE distribution if the shape parameter of the gamma distribution is close to one. It is also observed that the GE distribution having the same two L-moments with the gamma distribution, is closer than the GE distribution having the same two raw moments. In this section, we discuss how we can generate Gamma (α,1) using GE random deviate generator. Note that once we can generate pseudo Gamma (α,1) random number, then the generation of pseudo Gamma (α, λ) random number is immediate. The main idea of generator of pseudo Gamma (α,1) random number using GE is as follows. First, we identify a GE distribution which is 'closest' to the Gamma (α,1) distribution by using methods 1 or 2 as described in the previous section. Once we identify the GE distribution, we generate a random sample from the GE distribution in a cost effective and efficient way. We simply generate pseudo random numbers $u_1, u_2, \ldots, u_n$ from uniform (0,1) and set $x_i = -\frac{1}{\lambda^*} ln(1 - u_i^{\frac{1}{\alpha^*}})$ for $i = 1, \ldots, n$. Then $x_1, \ldots, x_n$ is a pseudo random sample from $GE(\alpha^*, \lambda^*)$. For all practical purposes for a certain range of α, generating pseudo random number from Gamma(α,1) is same as generating pseudo random number from $GE(\alpha_1^*, \lambda_1^*)$ or from $GE(\alpha_2^*, \lambda_2^*)$. To verify our assertions, we test the following hypotheses.

$$H_0 : \text{The sample is from Gamma}(\alpha, 1)$$

$$H_a : \text{The sample is not from Gamma}(\alpha, 1) \tag{4.1}$$

We perform three different tests.

Test-I Generate a random sample of size n from a Gamma(α,1) distribution and test the hypothesis (4.1).

Test-II Generate a random sample of size n from a $GE(\alpha_1^*, \lambda_1^*)$ distribution and test the hypothesis (4.1).

Test-III Generate a random sample of size n from $GE(\alpha_2^*, \lambda_2^*)$ distribution and test the hypothesis (4.1).

We perform the one sample Kolmogorov-Smirnov (K-S) test to test the above hypotheses. Note that the K-S statistic is same as the Kolmogorov discrepancy measure between the empirical distribution and the distribution function under H_0. We compute the K-S statistics and also compute the corresponding p-values. We take different n = 15, 25, 40, 50 and different α = .4, 1.2, 2.5. We replicate the process 1000 time and obtain the average p-values and the standard deviation of the p-values over 1000 replications. We also report the average K-S statistics and the standard deviation of the K-S statistics over 1000 replications. The results are reported in the tables 3-5.

From tables 3-5, it is quite clear that all the three tests behave very similarly. In all the cases the average p-values, the average K-S statistics and the standard deviation of the K-S statistics are quite close to each other. The average high p-values indicate that we accept the null hypothesis in all the cases. The results from the tests II and III indicate that if we generate a random sample from a GE distribution, then statistically we can not reject the hypothesis that it is coming from a gamma distribution. Comparing the results between test I with tests II, III, it is clear that the sample drawn from a GE distribution will behave as if it has been drawn from a gamma distribution. Therefore, if the shape parameter α is between 0.0 to 2.5, we can generate good quality pseudo gamma random numbers, using GE random deviate generator. For larger values of α, we can use the convolution property of the gamma to generate gamma random sample. However, some more research needs to be done and see if the quality will be better using convolution than using GE approximation.

5. DATA ANALYSIS

In this section we use two data sets and fit both the gamma and the GE distribution functions. We estimate the unknown parameters in all the cases by maximum likelihood method (Method 1) and by the method of L-moments (Method 2). We examine the closeness of the two fitted distributions by several ways. The following two data sets are used. The two data sets represent the failure times of the air conditioning system of two different air planes see Bain and Engelhardt (3, page 101).

Data 1: Plane 7912: 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90, 95, 120, 120, 225, 246, 261.

Data 2: Plane 7911: 33, 47, 55, 56, 104, 176, 182, 220, 239, 246, 320.

A simple exploratory data analysis indicates that the shape of the density is reverse 'J' shaped for Data set 1 and unimodal for Data set 2. We first fit the exponential distribution to the Data set 1. The K-S statistic between the empirical distribution function and the fitted exponential distribution function is 0.2419 and the corresponding 'p' value is 0.0497. This indicates that the exponential fit is not very good for Data set 1. Next we try to fit gamma distribution and GE distribution to these data sets. In both the cases we estimate the unknown parameters by Method 1 and Method 2. We compute the K-S statistic and the corresponding 'p' values in all the cases. The results are reported in Table A.

From Table A, it is clear that both the distributions fit quite well to these data sets. Interestingly if the MLE's are used then the K-S statistics and the 'p' values are quite close to each other and similarly when the LME's are used the corresponding K-S statistics and the 'p' values are also quite close to each other for both the data sets. As in the previous section, it is also observed here that the fitted distributions provide better fit when the parameters are estimated by the L-moments than by the MLE's. Surprisingly, in both cases the estimated shape parameters by two different methods are quite different. We can not find a proper reason for that. To observe how different are the corresponding distribution functions, we compute the KDM's and $SKLDM$'s between all fitted distributions in both the cases. The results are reported in Table B. In each box, the first figure represents the

KDM and the figure below represents the $SKLDM$ for Data Set 1 and the corresponding values of the Data Set 2 are reported within brackets.

Table A

The goodness of fit of gamma and *GE* distributions

Data Set	Distribution	Methods	α	λ	$K-S$	p
	Gamma	MLE	0.8134	0.0136	0.1706	0.3105
1		LME	0.6190	0.0104	0.1459	0.5002
	GE	MLE	0.8130	0.0145	0.1744	0.2926
		LME	0.6006	0.0118	0.1491	0.4857
	Gamma	MLE	2.1457	0.0141	0.2167	0.6064
2		LME	1.9394	0.0127	0.2165	0.6072
	GE	MLE	2.2427	0.0104	0.2194	0.5978
		LME	2.1303	0.0102	0.2227	0.5781

From Table B, it is clear that even though the shape parameters are significantly different between the MLE's and the LME's, the corresponding distribution functions are quite close to each other. Here also it is observed that the KDM's are slightly higher than the corresponding $SKLDM$. For Data Set 1, the maximum KDM is observed between the GE MLE and the GE LME and for Data Set 2, the maximum KDM is observed between Gamma MLE and Gamma LME. The minimum KDM is observed between the GE MLE and Gamma MLE for Data Set 1 and between the Gamma LME and GE LME for Data Set 2. Interestingly, it is also observed that the KDM between Gamma MLE and Gamma LME is more than the KDM between Gamma MLE and GE MLE or between Gamma LME and GE LME.

Based on these, we can say that for certain data set (if the shape parameter of the fitted gamma is less that 2.5) if a gamma distribution fits well, then we can use a GE

distribution also and it is very difficult to differentiate the two fitted distribution functions. Since numerically the estimation of the GE parameters is much simpler than the estimation of the gamma parameters, we recommend to use GE distribution in place of the gamma distribution if the data are skewed.

Table B

The KDM's and the $SKLDM$'s between the fitted distributions

Estimated Dist.	Gamma MLE	Gamma LME	GE MLE	GE LME
Gamma MLE	0 (0) 0 (0)	0.0659(0.0186) 0.0516(0.0059)	0.0042(0.0076) 0.0001(0.0026)	0.0688(0.0173) 0.0579(0.0051)
Gamma LME	0.0659(0.0186) 0.0516(0.0059)	0 (0) 0 (0)	0.0687(0.0147) 0.0546(0.0045)	0.0042(0.0063) 0.0005(0.0018)
GE MLE	0.0042(0.0076) 0.0001(0.0026)	0.0687(0.0147) 0.0546(0.0045)	0 (0) 0 (0)	0.0714(0.0106) 0.0607(0.0026)
GE LME	0.0688(0.0173) 0.0579(0.0051)	0.0042(0.0063) 0.0005(0.0018)	0.0714(0.0106) 0.0607(0.0026)	0 (0) 0 (0)

6. CONCLUSIONS

In this paper, we study the closeness between the gamma distribution and the generalized exponential distribution by using different methods. It is observed that if the shape parameter of the gamma distribution is not very high then a gamma distribution can be approximated very well by a GE distribution. This closeness between the two distributions can be used to generate gamma random numbers using GE generator. It is observed by extensive simulations that it is not possible to differentiate between a random sample from a gamma distribution and a random sample from a GE distribution. Further, by closely studying the two distributions by Kolmogorov discrepancy measure and by Kullback-Leibler discrepancy measure, it is observed that for α close to one the two distributions are almost indistinguishable. In this paper, we come to the following two final conclusions. It is always possible to generate gamma random numbers using generalized exponential distribution for $\alpha < 2.5$. For a given data set, if gamma fits well and $\alpha < 2.5$, then the two-parameter GE also fits equally well to the same data set.

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Table 1

Kolmogorov and Kullback-Leibler discrepancy measures between Gamma and GE Distributions

α	α_1^*	λ_1^*	KDM(G, GE_1)	KLDM(G, GE_1)
0.1	0.089759	1.386839	0.034177	0.004808
0.2	0.181456	1.322081	0.026809	0.003412
0.3	0.275249	1.265259	0.020899	0.002363
0.4	0.371272	1.214898	0.016116	0.001582
0.5	0.469641	1.169876	0.012193	0.001008
0.6	0.570461	1.129324	0.008929	0.000596
0.7	0.673824	1.092558	0.006176	0.000311
0.8	0.779816	1.059031	0.003821	0.000129
0.9	0.888517	1.028299	0.001783	0.000030
1.0	1.000000	1.000000	0.000000	0.000000
1.2	1.231593	0.949543	0.002979	0.000101
1.4	1.475128	0.905787	0.005380	0.000364
1.6	1.731104	0.867384	0.007368	0.000743
1.8	2.000000	0.833333	0.009052	0.001208
2.0	2.282285	0.802878	0.010503	0.001737
2.2	2.578420	0.775432	0.011774	0.002315
2.4	2.888866	0.750535	0.012901	0.002929
2.5	3.049600	0.738925	0.013419	0.003247
2.6	3.214084	0.727819	0.013911	0.003572
3.0	3.910696	0.687790	0.015656	0.004915
3.5	4.872939	0.645721	0.017459	0.006659
4.0	5.943969	0.610339	0.018963	0.008436
4.5	7.131775	0.580055	0.020249	0.010219

Table 2

Kolmogorov and Kullback-Leibler discrepancy measures between Gamma and GE Distributions

α	α_2^*	λ_2^*	KDM(G, GE_2)	KLDM(G, GE_2)
0.1	0.094851	1.460559	0.015591	0.001221
0.2	0.190045	1.377430	0.012501	0.001022
0.3	0.286006	1.306810	0.009958	0.000808
0.4	0.383060	1.247960	0.007837	0.000593
0.5	0.481465	1.192470	0.006043	0.000406
0.6	0.581430	1.145333	0.004504	0.000254
0.7	0.683130	1.103279	0.003165	0.000140
0.8	0.786710	1.065458	0.001988	0.000060
0.9	0.892296	1.031207	0.000940	0.000015
1.0	1.000000	1.000000	0.000000	0.000000
1.2	1.222138	0.945095	0.001628	0.000053
1.4	1.453799	0.898186	0.003000	0.000199
1.6	1.695558	0.857506	0.004183	0.000419
1.8	1.947920	0.821788	0.005220	0.000699
2.0	2.211349	0.790101	0.006646	0.001029
2.2	2.486277	0.761740	0.007830	0.001397
2.4	2.773119	0.736161	0.008958	0.001796
2.5	2.921133	0.724278	0.009502	0.002006
2.6	3.072276	0.712938	0.010034	0.002221
3.0	3.709108	0.672265	0.012039	0.003129
3.5	4.581387	0.629829	0.014300	0.004335
4.0	5.544099	0.594358	0.016329	0.005586
4.5	6.603520	0.564139	0.018161	0.006859

Table 3

K-S Goodness of Fit and P-values of the Different Tests when $\alpha = .4$

n	Statistics	Test-I	Test-II	Test-III
15	Average p-value	0.6708	0.6791	0.6825
	S.D. of p-value	0.1987	0.1969	0.1952
	Average K-S Stat.	0.7089	0.7045	0.7027
	S.D. of K-S Stat.	0.1060	0.1050	0.1042
25	Average p-value	0.6471	0.6587	0.6635
	S.D. of p-value	0.1704	0.1664	0.1663
	Average K-S Stat.	0.7035	0.6975	0.6950
	S.D. of K-S Stat.	0.0886	0.0865	0.0865
40	Average p-value	0.6330	0.6386	0.6438
	S.D. of p-value	0.1423	0.1452	0.1448
	Average K-S Stat.	0.7005	0.6977	0.6950
	S.D. of K-S Stat.	0.0731	0.0745	0.0742
50	Average p-value	0.6280	0.6323	0.6377
	S.D. of p-value	0.1257	0.1293	0.1292
	Average K-S Stat.	0.6997	0.6975	0.6947
	S.D. of K-S Stat.	0.0641	0.0660	0.0660

Table 4

K-S Goodness of Fit and P-values of the Different Tests when $\alpha = 1.2$

n	Statistics	Test-I	Test-II	Test-III
15	Average p-value	0.8023	0.7958	0.7954
	S.D. of p-value	0.1699	0.1754	0.1754
	Average K-S Stat.	0.6388	0.6423	0.6425
	S.D. of K-S Stat.	0.0906	0.0936	0.0936
25	Average p-value	0.7844	0.7910	0.7904
	S.D. of p-value	0.1555	0.1555	0.1558
	Average K-S Stat.	0.6321	0.6287	0.6290
	S.D. of K-S Stat.	0.0809	0.0808	0.0810
40	Average p-value	0.7806	0.7855	0.7849
	S.D. of p-value	0.1373	0.1395	0.1398
	Average K-S Stat.	0.6245	0.6219	0.6223
	S.D. of K-S Stat.	0.0712	0.0723	0.0725
50	Average p-value	0.7737	0.7849	0.7841
	S.D. of p-value	0.1284	0.1251	0.1255
	Average K-S Stat.	0.6252	0.6195	0.6199
	S.D. of K-S Stat.	0.0659	0.0642	0.0644

Table 5

K-S Goodness of Fit and P-values of the Different Tests when $\alpha = 2.5$

n	Statistics	Test-I	Test-II	Test-III
15	Average p-value	0.8288	0.8234	0.8220
	S.D. of p-value	0.1650	0.1605	0.1633
	Average K-S Stat.	0.6247	0.6275	0.6283
	S.D. of K-S Stat.	0.0880	0.0856	0.0860
25	Average p-value	0.8288	0.8287	0.8274
	S.D. of p-value	0.1428	0.1431	0.1436
	Average K-S Stat.	0.6090	0.6091	0.6098
	S.D. of K-S Stat.	0.0743	0.0744	0.0747
40	Average p-value	0.8288	0.8287	0.8309
	S.D. of p-value	0.1428	0.1318	0.1286
	Average K-S Stat.	0.6090	0.6091	0.5983
	S.D. of K-S Stat.	0.0743	0.0744	0.0672
50	Average p-value	0.8385	0.8343	0.8325
	S.D. of p-value	0.1168	0.1167	0.1173
	Average K-S Stat.	0.5918	0.5940	0.5949
	S.D. of K-S Stat.	0.0604	0.0603	0.0606