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DISCRIMINATING BETWEEN GAMMA AND GENERALIZED EXPONENTIAL DISTRIBUTIONS

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Recently the two-parameter generalized exponential distribution was introduced by the authors. It is observed that a generalized exponential distribution has several properties which are quite similar to a gamma distribution. It is also observed that a generalized exponential distribution can be used quite effectively in many situations where a skewed distribution is needed. In this paper, we use the ratio of the maximized likelihoods in choosing between a generalized exponential distribution and a gamma distribution. We obtain asymptotic distributions of the logarithm of the ratio of the maximized likelihoods under null hypotheses and use them to determine the sample size needed to discriminate between two overlapping families of distributions for a user specified probability of correct selection and a tolerance limit.

Keywords: Asymptotic distributions; Gamma distribution; Generalized exponential distribution; Kolmogorov–Smirnov distances; Likelihood ratio statistic

1 INTRODUCTION

Recently the two-parameter generalized exponential distribution or the exponentiated exponential distribution has been introduced and studied quite extensively by Gupta and Kundu [12–15], Raqab and Ahsanullah [22] and Zheng [26]. The two-parameter generalized exponential distribution has the distribution function

$$F_{\mathrm{GE}}(x; \alpha, \lambda) = (1 - e^{-\lambda x})^{\alpha}; \quad \alpha, \lambda > 0, \tag{1}$$

density function

$$f_{\mathrm{GE}}(x; \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}; \quad \alpha, \lambda > 0, \tag{2}$$

survival function

$$S_{\mathrm{GE}}(x; \alpha, \lambda) = 1 - (1 - e^{-\lambda x})^{\alpha}; \quad \alpha, \lambda > 0, \tag{3}$$

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hazard function

$$h_{\mathrm{GE}}(x;\alpha,\lambda)=\frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}};\quad \alpha,\lambda>0 \tag{4}$$

and the reverse hazard function

$$r_{\mathrm{GE}}(x;\alpha,\lambda)=\frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{(1-e^{-\lambda x})^{\alpha}}=\frac{\alpha\lambda e^{-\lambda x}}{(1-e^{-\lambda x})};\quad \alpha,\lambda>0. \tag{5}$$

Here α is the shape parameter and λ is the scale parameter. For different values of α, density functions and hazard functions are plotted in Gupta and Kundu [13]. It is quite clear that the density function and the hazard function of a generalized exponential distribution are quite similar to the density function and the hazard function of a gamma distribution. When $\alpha = 1$, the generalized exponential distribution coincides with the exponential distribution. It is clear that the hazard function of a generalized exponential distribution can be increasing, decreasing and constant depending on the shape parameter similarly as a gamma distribution. Both generalized exponential distribution and gamma distribution can be considered as generalizations of exponential distribution in different ways. It is observed that in many situations generalized exponential distribution provides *better* fit than a gamma distribution. Therefore, an experimenter would like to choose one of the two models to analyze a skewed data set.

The problem of testing whether some given observations come from one of the two probability distributions, is quite old in the statistical literature. Atkinson [1, 2], Chen [6], Chambers and Cox [5], Cox [7, 8], Jackson [16], Dyer [10] considered this problem in general for discriminating between two models. Due to increasing applications of the lifetime distributions, special attention is given to the problem of discriminating between the log-normal and Weibull distributions [6, 9, 19, 21], between the log-normal and gamma [17, 21, 25] and between the gamma and Weibull distributions [3, 11]. Between two models, the effect of choosing a wrong model was originally discussed by Cox [7] in general and recently Wiens [25] demonstrated it nicely with a real data example. Therefore, for small or moderate sample sizes the gamma and GE models might provide similar data fit but it is still desirable to select the correct or more correct model. It is important to make the best possible decision based on whatever data are available even if large sample sizes are not available. We use the logarithm of the ratio of the maximized likelihoods (RML) in discriminating between the gamma and generalized exponential distributions. The idea was originally proposed by Cox [7, 8] in discriminating between two separate models and Bain and Engelhardt [3] used it in discriminating between Weibull and gamma distributions. We obtain asymptotic distributions of the logarithm of the RML statistics under null hypotheses. It is observed by a Monte Carlo simulation study that the asymptotic distributions work quite well even when the sample size is not too large. Using these asymptotic distributions and the distance between two distribution functions, we determine the minimum sample size needed to discriminate between two distribution functions at a user specified protection level. Two real life data sets are analyzed to verify how the proposed methods work in practice.

The rest of the paper is organized as follows. We briefly describe the likelihood ratio method in Section 2. Asymptotic distributions of the logarithm of the RML statistics under null hypotheses are obtained in Section 3. In Section 4, asymptotic distributions are used to determine the minimum sample size needed to discriminate between two distribution functions at a user specified protection level and a tolerance limit. Some numerical experiments are performed to observe how the asymptotic results behave for finite samples in Section 5. Two real data sets are analyzed in Section 6 and finally conclusions appear in Section 7. All the proofs have been presented in the Appendix.

2 LIKELIHOOD RATIO TEST

Suppose $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables either from a generalized exponential distribution or from a gamma distribution. The density function of a gamma random variable with shape parameter β and scale parameter θ will be denoted as $f_{GA}(x; \beta, \theta)$, where

$$f_{GA}(x; \beta, \theta) = \frac{\theta^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\theta x}; \quad x, \beta, \theta > 0. \tag{6}$$

A generalized exponential distribution with shape parameter α and scale parameter λ will be denoted by GE(α, λ). A gamma distribution with shape parameter β and scale parameter θ will be denoted by GA(β, θ). We define the likelihood function of the data assuming that they are coming from a GE(α, λ) or GA(β, θ) as

$$L_{GE}(\alpha, \lambda) = \prod_{i=1}^{n} f_{GE}(x_i; \alpha, \lambda) \quad \text{and} \quad L_{GA}(\beta, \theta) = \prod_{i=1}^{n} f_{GA}(x_i, \beta, \theta) \tag{7}$$

respectively. The natural logarithm of the RML is defined as

$$T = \ln\left(\frac{L_{GE}(\hat{\alpha}, \hat{\lambda})}{L_{GA}(\hat{\beta}, \hat{\theta})}\right), \tag{8}$$

where $(\hat{\alpha}, \hat{\lambda})$ are maximum likelihood estimators of (α, λ) and similarly $(\hat{\beta}, \hat{\theta})$ are maximum likelihood estimators of (β, θ). The statistic T can be written as;

$$T = n\left[\ln(\hat{\alpha}\hat{\lambda}\tilde{X}) - \hat{\beta}\ln(\tilde{X}\hat{\theta}) - \frac{(\hat{\alpha}-1)}{\hat{\alpha}} + \ln(\Gamma(\hat{\beta})) - \bar{X}(\hat{\lambda} - \hat{\theta})\right], \tag{9}$$

here $\bar{X}$ and $\tilde{X}$ are arithmetic mean and geometric mean respectively of $X_1, \ldots, X_n$, *i.e.*

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \tilde{X} = \left(\prod_{i=1}^{n} X_i\right)^{1/n}.$$

It is observed by Gupta and Kundu [13] that if the data are assumed to come from a generalized exponential distribution, then

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \ln(1 - e^{-\hat{\lambda} X_i})}. \tag{10}$$

If the data are assumed to come from a gamma distribution then they satisfy the following relation

$$\hat{\theta} = \frac{\hat{\beta}}{\bar{X}}. \tag{11}$$

The likelihood ratio test as defined similarly in Bain and Englehardt [3] can be described as follows. Choose generalized exponential model if $T > 0$, otherwise choose gamma model. From Eq. (9), it is clear that if the data come from a generalized exponential population then the distribution of T is independent of λ and depends only of α. Similarly, if the data come from a gamma population, then also the distribution of T is independent of θ and depends only on β.

The probability of correct selection is estimated using extensive simulations for different shape parameters and for different sample sizes. First we generate a sample of size n from a GE(α, 1). From that sample we obtain maximum likelihood estimators of α and λ, similarly we obtain the maximum likelihood estimators of β and θ. We compute T as defined in (9) and observe whether T is positive or negative. We replicate the process 10,000 times and compute percentage of times it is positive. That provides estimates of probability of correct selections (PCS) when a generalized exponential distribution is the null distribution and the alternative is a gamma distribution. Similarly we estimate PCS when a gamma distribution is a null distribution and a generalized exponential distribution is the alternative distribution. The results are reported in Tables V and VI respectively.

Some of the points are quite clear from Tables V and VI. In both cases when the shape parameter is one, the estimate of PCS is close to 0.5. It is not very surprising because, when the shape parameter is 1, both the distributions coincide with the exponential distribution. Therefore, probability of selecting any one should be 0.5 and clearly it should be irrespective of sample sizes. It is also observed that as sample size increases the estimate of probability of correct selections increases for all values of the shape parameter ($\neq 1$). Moreover, as shape parameter moves away from one in either direction, the probability of correct selection increases. It is also not very surprising because as the shape parameters move away from one the distance between them increases and therefore it becomes easier to discriminate the two. The detailed discussions will be provided in Section 5.

3 ASYMPTOTIC PROPERTIES OF THE LOGARITHM OF RML UNDER NULL HYPOTHESES

In this section we obtain asymptotic distributions of the logarithm of RML statistics under null hypotheses in two different cases. We denote the almost sure convergence by *a.s.*

Case 1 The data come from a gamma distribution and the alternative is a generalized exponential distribution.

We assume that n data points are from a gamma distribution with the shape parameter β and scale parameter θ as given in (6). $\hat{\beta}$, $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\lambda}$ are same as defined earlier. We use following notations. For any Borel measurable function $h(\cdot)$, $E_{GA}(h(U))$ and $V_{GA}(h(U))$ denote mean and variance of $h(U)$ under the assumption that U follows GA$(\cdot, \cdot)$. Similarly we define $E_{GE}(h(U))$ and $V_{GE}(h(U))$ as mean and variance of $h(U)$ under the assumption that U follows GE$(\cdot, \cdot)$. If $g(\cdot)$ and $h(\cdot)$ are two Borel measurable functions, we define similarly $\text{cov}_{GA}(g(U), h(U)) = E_{GA}(g(U)h(U)) - E_{GA}(g(U))E_{GA}(h(U))$ and

$\text{cov}_{\text{GE}}(g(U), h(U)) = E_{\text{GE}}(g(U)h(U)) - E_{\text{GE}}(g(U))E_{\text{GE}}(h(U))$, where U follows GA$(\cdot, \cdot)$ and GE$(\cdot, \cdot)$ respectively. We have the following results.

THEOREM 1 *Under the assumption that the data come from a gamma distribution, the distribution of the statistic T as defined in* (9) *is approximately normally distributed with mean $E_{\text{GA}}(T)$ and variance $V_{\text{GA}}(T)$.*

The proof is provided in the Appendix I. Observe that $\lim_{n\to\infty}(E_{\text{GA}}(T)/n)$ and $\lim_{n\to\infty}(V_{\text{GA}}(T)/n)$ exist. Let us denote $\lim_{n\to\infty}(E_{\text{GA}}(T)/n) = \text{AM}_{\text{GA}}(\beta)$ and $\lim_{n\to\infty}(V_{\text{GA}}(T)/n) = \text{AV}_{\text{GA}}(\beta)$. The exact expressions of $\text{AM}_{\text{GA}}(\beta)$, $\text{AV}_{\text{GA}}(\beta)$ are provided in the Appendix II.

From the Lemma 1 (Appendix I) we obtain that $\hat{\alpha}$ and $\hat{\lambda}$ converge to $\tilde{\alpha}$ and $\tilde{\lambda}$ respectively. Now we discuss how to compute $\tilde{\alpha}$, $\tilde{\lambda}$, $E_{\text{GA}}(T)$ and $V_{\text{GA}}(T)$. Let us define

$$\begin{aligned} g(\alpha, \lambda) &= E_{\text{GA}}[\ln(f_{\text{GE}}(X; \alpha, \lambda))] \\ &= E_{\text{GA}}[\ln(\alpha) + \ln(\lambda) - \lambda X + (\alpha - 1)\ln(1 - e^{-\lambda X})] \\ &= \ln(\alpha) + \ln(\lambda) - \beta\frac{\lambda}{\theta} + (\alpha - 1)u\left(\beta, \frac{\lambda}{\theta}\right), \end{aligned}$$

where

$$u\left(\beta, \frac{\lambda}{\theta}\right) = \frac{1}{\Gamma(\beta)}\int_0^\infty \ln(1 - e^{-(\lambda/\theta)y})y^{\beta-1}e^{-y}\,dy.$$

Therefore, $\tilde{\alpha}$ and $\tilde{\lambda}$ can be obtained as solutions of

$$\frac{\partial g}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial g}{\partial \lambda} = 0.$$

i.e.

$$\frac{1}{\tilde{\alpha}} + u\left(\beta, \frac{\tilde{\lambda}}{\theta}\right) = 0, \tag{12}$$

$$\frac{1}{\tilde{\lambda}} - \frac{\beta}{\theta} + \frac{\tilde{\alpha} - 1}{\theta}u_2\left(\beta, \frac{\tilde{\lambda}}{\theta}\right) = 0. \tag{13}$$

Here $u_2(\cdot, \cdot)$ is the derivative of $u(\cdot, \cdot)$ with respect to the second argument. From (12), it is clear that $\tilde{\lambda}/\theta$ is a function of $\tilde{\alpha}$ and β. From (12) and (13) it is immediate that $\tilde{\alpha}$ is a function of β only. Now let us consider the Case 2.

Case 2 The data come from a generalized exponential distribution and the alternative is a gamma distribution.

It is assumed that a random sample $X_1, \ldots, X_n$, of size n comes from a GE(α, λ) and the alternative is a GA(β, θ). Here we denote $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\theta}$ as the MLEs of α, λ, β and θ respectively. In this case we have the following theorem.

THEOREM 2 *Under the assumption that the data come from the GE distribution, T is approximately normally distributed with mean $E_{\text{GE}}(T)$ and variance $V_{\text{GE}}(T)$.*

The proof is presented in Appendix I. Similarly as before, we observe that $\lim_{n\to\infty} (E_{GE}(T)/n)$ and $\lim_{n\to\infty} (V_{GE}(T)/n)$ exist. Suppose, we denote $\lim_{n\to\infty} (E_{GE}(T)/n) = AM_{GE}(\alpha)$ and $\lim_{n\to\infty} (V_{GE}(T)/n) = AV_{GE}(\alpha)$. The exact expressions are provided in the Appendix II.

Now we discuss how to obtain $\tilde{\beta}$ and $\tilde{\theta}$ (see Appendix I), the almost sure limits of $\hat{\beta}$ and $\hat{\theta}$ respectively. First let us define

$$\begin{aligned} h(\beta, \theta) &= E_{GE}[\ln(f_{GA}(X; \beta, \theta))] = E_{GE}[\beta \ln(\theta) - \ln(\Gamma(\beta)) + (\beta - 1)\ln(X) - X\theta] \\ &= \beta \ln(\theta) - \ln(\Gamma(\beta)) + (\beta - 1)E_{GE}[\ln(X)] - \frac{\theta}{\lambda} E_{GE}(Z), \\ &= \beta \ln(\theta) - \ln(\Gamma(\beta)) + (\beta - 1)E_{GE}[\ln(Z) - \ln(\lambda)] - \frac{\theta}{\lambda}(\psi(\alpha + 1) - \psi(1)) \end{aligned} \tag{14}$$

here Z follows GE(α, 1). Therefore, $\tilde{\beta}$ and $\tilde{\theta}$ can be obtained as solutions of

$$\tilde{\beta} = \frac{\tilde{\theta}}{\lambda}[\psi(\alpha + 1) - \psi(1)] \tag{15}$$

and

$$\ln\left(\frac{\tilde{\theta}}{\lambda}\right) = \psi(\tilde{\beta}) - E[\ln(Z)]. \tag{16}$$

From (15) it is clear that $(\tilde{\theta}/\lambda)$ is a function of $\tilde{\beta}$ and α only. From (15) and (16), we obtain that $\tilde{\beta}$ is a function of α only.

It may be noted that $\tilde{\alpha}$, $\tilde{\lambda}$, $\tilde{\beta}$, $\tilde{\theta}$, $AM_{GA}(\beta)$, $AV_{GA}(\beta)$, $AM_{GE}(\alpha)$ and $AV_{GE}(\alpha)$ may not be easy to obtain. We present some of these values in Tables I and II and they may be helpful during data analysis.

4 DETERMINATION OF SAMPLE SIZE

In this section, we propose a method to determine the minimum sample size required to discriminate between the gamma and GE distributions. It is expected that user specifies the probability of correct selection before hand. The distance or equivalently the measure of closeness between two distribution functions can be defined several ways. Among the popular ones we can mention are the Kolmogorov–Smirnov (K–S) distance measure or the Hellinger distance measure. It is very clear that if the distance measure between two distribution

TABLE I Different Values of $AM_{GE}(\alpha)$, $AV_{GE}(\alpha)$, $\tilde{\beta}$ and $\tilde{\theta}$ for Different α, When $\lambda = 1$.

α	AM_{GE}	AV_{GE} (β)	β	θ
0.50	0.0003	0.0004	0.517	0.845
0.75	0.0001	0.0001	0.769	0.938
2.00	0.0005	0.0015	1.906	1.275
3.00	0.0017	0.0042	2.638	1.458
4.00	0.0029	0.0071	3.318	1.608
5.00	0.0041	0.0097	3.901	1.728
6.00	0.0052	0.0124	4.435	1.830

TABLE II Different Values of $\mathrm{AM_{GA}}(\beta)$, $\mathrm{AV_{GA}}(\beta)$, $\tilde{\alpha}$ and $\tilde{\lambda}$ for Different β When $\theta = 1$.

β	AM_{GA}	AV_{GA}	$\tilde{\alpha}$	$\tilde{\lambda}$
0.50	−0.00006	0.0003	0.498	1.241
0.75	−0.00003	0.0001	0.750	1.107
2.00	−0.0006	0.0016	2.215	0.790
3.00	−0.0024	0.0041	3.589	0.657
4.00	−0.0041	0.0069	5.211	0.574
5.00	−0.0054	0.0117	7.245	0.519
6.00	−0.0085	0.0149	9.299	0.471

functions are *small*, one naturally needs very large sample size to discriminate between the two distribution functions if the probability of correct selection is kept fixed. On the other hand, if the corresponding distance is *large* then relatively small samples should be enough to discriminate between the two distribution functions. Also if the two distribution functions are very close to each other, then naturally, one may not need to differentiate the two distributions for any practical point of view. We assume that the user will specify before hand the probability of correct selection and also the tolerance limit in terms of the distance function between the two distribution functions. The user specified tolerance limit indicates that the two distribution functions are not considered to be significantly different if their distance is less than the tolerance limit. The tolerance limit and the probability of correct selection are equivalent to the type-I error and power in the corresponding testing of hypothesis problem. Based on the probability of correct selection and the tolerance limit, the required minimum sample size can be determined.

In this paper, we develop the methodology based on the K–S distance but similar methodology can be developed using the Hellinger distance also, which is not pursued here. We observed in Section 3 that the logarithm of RML statistics follow normal distribution for large n. Now it will be used with the help of the K–S distance to determine the required sample size n such that the probability of correct selection achieves a certain protection level p^* for a given tolerance level D^*. We explain the procedure assuming Case 1 and Case 2 follows exactly along the same line.

Since T is asymptotically normally distributed with mean $E_{\mathrm{GA}}(T)$ and variance $V_{\mathrm{GA}}(T)$, therefore the probability of correct selection (PCS) is

$$\mathrm{PCS}(\beta) = P[T < 0|\beta] \approx \Phi\left(\frac{-E_{\mathrm{GA}}(T)}{\sqrt{V_{\mathrm{GA}}(T)}}\right) = \Phi\left(\frac{-n \times \mathrm{AM_{GA}}(\beta)}{\sqrt{n \times \mathrm{AV_{GA}}(\beta)}}\right). \tag{17}$$

Here Φ is the distribution function of the standard normal random variable. $\mathrm{AM_{GA}}(\beta)$ and $\mathrm{AV_{GA}}(\beta)$ are same as defined in (22) and (23) respectively. Now to determine the sample size needed to achieve at least a p^* protection level we equate

$$\Phi\left(\frac{-n \times \mathrm{AM_{GA}}(\beta)}{\sqrt{n \times \mathrm{AV_{GA}}(\beta)}}\right) = p^* \tag{18}$$

and solve for n. We obtain

$$n = \frac{z_{p^*}^2 \mathrm{AV_{GA}}(\beta)}{(\mathrm{AM_{GA}}(\beta))^2}. \tag{19}$$

Here z_{p^*} is the $100p^*$ percentile point of a standard normal distribution. For $p^* = 0.7$ and for different β, the values of n are reported in Table III. Similarly for Case 2, we need

$$n = \frac{z_{p^*}^2 \mathrm{AV_{GE}}(\alpha)}{(\mathrm{AM_{GE}}(\alpha))^2}. \tag{20}$$

Here $\mathrm{AM_{GE}}(\alpha)$ and $\mathrm{AV_{GE}}(\alpha)$ are as defined in (24) and (25) respectively. We report n, with the help of Table II for different values of α when $p^* = 0.7$ in Table IV. From Tables III and IV it is immediate that as α and β move away from 1, for a given probability of correct selection, the required sample size decreases as expected. From (19) and (20) it is clear that if one knows the range of the shape parameter of the null distribution then the minimum sample size can be obtained using (19) or (20) and using the fact that n decreases as the shape parameter moves away from 1.

Unfortunately, in practice the shape parameters may be completely unknown. To determine the minimum sample size needed to discriminate the two distributions the following assumptions have been made. It is assumed that the experimenter would like to choose the minimum sample size for a given protection level when the distance between the two distribution functions is greater than a pre-specified tolerance level. The distance between the two distribution functions is defined by the K–S distance. The K–S distance between two distribution functions, say $F(x)$ and $G(x)$ is defined as

$$\sup_x |F(x) - G(x)|. \tag{21}$$

We report the K–S distance between $\mathrm{GA}(\beta, 1)$ and $\mathrm{GE}(\tilde{\alpha}, \tilde{\lambda})$ for different values of β in Table III. Here $\tilde{\alpha}$ and $\tilde{\beta}$ are same as defined before and they have reported in Table I. Similarly, the K–S distance between $\mathrm{GE}(\alpha, 1)$ and the $\mathrm{GA}(\tilde{\beta}, \tilde{\theta})$ for different values of α is reported in Table IV. Here $\tilde{\beta}$ and $\tilde{\theta}$ are same as defined before and they were reported in Table II. From Tables III and IV it is clear that in both the cases the K–S distance between the two distribution functions increases as the shape parameter moves away from 1.

Now we explain how we can determine the minimum sample size required to discriminate between Gamma and GE distribution functions for a user specified protection level and for a given tolerance level between the two distribution functions. Suppose the protection level is $p^* = 0.7$ and the tolerance level is given in terms of the K–S distance as $D^* = 0.01$. Here tolerance level $D^* = 0.01$ means that the practitioner wants to discriminate between gamma and GE distribution functions only when their K–S distance is more than 0.01. From the Table III, it is clear that for Case 1, the K–S distance will be more than 0.01 if $\beta \geq 4$. Similarly from Table IV, it is clear that for Case 2, the K–S distance will be more than 0.01 if $\alpha \geq 6.0$. Therefore, if the null distribution is gamma, then for the tolerance level $D^* = 0.01$, one needs $n \geq 312$, to meet the PCS, $p^* = 0.7$. Similarly if the null distribution is GE then one needs at least $n = 348$ to meet the above protection level $p^* = 0.7$ and when the tolerance level, $D^* = 0.01$. Therefore, for the given tolerance level 0.01 one needs $\max(312, 348) = 348$ to meet the protection level $p^* = 0.7$ simultaneously for both the

TABLE III The Minimum Sample Size $n = z_{0.70}^2 \mathrm{AV_{GA}}(\beta)/(\mathrm{AM_{GA}}(\beta))^2$, Using (19), for $p^* = 0.7$ and When the Null Distribution is Gamma is Presented. The K–S Distance Between $\mathrm{GA}(\beta, 1)$ and $\mathrm{GE}(\tilde{\alpha}, \tilde{\lambda})$ for Different Values of β is Reported.

β	0.50	0.75	2.00	3.00	4.00	5.00	6.00
n	63,167	84,222	3369	540	312	304	157
K–S	0.0029	0.0019	0.0065	0.0076	0.0093	0.0110	0.0139

TABLE IV The Minimum Sample Size $n = z^2_{0.70}\text{AV}_{\text{GE}}(\alpha)/(\text{AM}_{\text{GE}}(\alpha))^2$, Using (20), for $p^* = 0.7$ and When the Null Distribution is GE is Presented. The K–S Distance Between GE(α, 1) and GA($\tilde{\beta}$, $\tilde{\theta}$) for Different Values of α is Reported.

α	0.50	0.75	2.00	3.00	4.00	5.00	6.00
n	3368	7580	4548	1102	640	438	348
K–S	0.0048	0.0032	0.0062	0.0071	0.0083	0.0096	0.0100

cases. It is clear that if $\alpha < 6.0$ or $\beta < 4.0$ then one needs larger sample size to discriminate between the two distribution functions to meet the same PCS.

5 NUMERICAL EXPERIMENTS

In this section we present some experimental results to examine how the asymptotic results derived in Section 3 behave for finite sample sizes. All the computations are performed at the Indian Institute of Technology, Kanpur, using Pentium-II processor. We use the random deviate generator of Press *et al.* [20] and all the programs are written in FORTRAN-77. They can be obtained from the authors on request. We compare PCSs obtained using simulations and based on the asymptotic results derived in Section 3. We consider different sample sizes and different shape parameters of the null distributions. The details are explained below.

First we consider the case when the null distribution is gamma and the alternative is GE. In this case we consider $n = 20, 40, 60, 80, 100$ and $\beta = 0.5, 0.75, 2.0, 3.0, 4.0, 5.0$ and 6.0. For a fixed β and n we generate a random sample of size n from the GA(β, 1), we finally compute T as defined in (8) and check whether T is positive or negative. We replicate the process 10,000 times and obtain an estimate of PCS. We also compute the PCS's by using the asymptotic results as given in (17). The results are reported in Table V. Similarly, we obtain the results when the null distribution is GE and the alternative is gamma. In this case we consider the same set of n and $\alpha = 0.5, 0.75, 2.0, 3.0, 4.0, 5.0$ and 6.0. The results are reported in Table VI. In each box

TABLE V The Probability of Correct Selection Based on Monte Carlo Simulations and also Based on Asymptotic Results When the Null Distribution is Gamma.

	n				
β	*20*	*40*	*60*	*80*	*100*
0.50	0.51	0.52	0.52	0.53	0.54
	(0.51)	(0.51)	(0.51)	(0.51)	(0.52)
0.75	0.47	0.48	0.49	0.51	0.52
	(0.50)	(0.51)	(0.51)	(0.51)	(0.52)
2.00	0.54	0.55	0.56	0.57	0.57
	(0.53)	(0.54)	(0.55)	(0.56)	(0.56)
3.00	0.59	0.62	0.64	0.65	0.67
	(0.57)	(0.60)	(0.62)	(0.63)	(0.65)
4.00	0.60	0.63	0.67	0.69	0.71
	(0.59)	(0.63)	(0.65)	(0.67)	(0.69)
5.00	0.61	0.65	0.68	0.70	0.72
	(0.60)	(0.64)	(0.66)	(0.68)	(0.71)
6.00	0.62	0.67	0.73	0.74	0.75
	(0.62)	(0.67)	(0.72)	(0.73)	(0.75)

Note: The element in the first row in each box represents the results based on Monte Carlo Simulations (10,000 replications) and the number in bracket immediately below represents the result obtained by using asymptotic results.

TABLE VI The Probability of Correct Selection Based on Monte Carlo Simulations and also Based on Asymptotic Results When the Null Distribution is GE.

	n				
β	*20*	*40*	*60*	*80*	*100*
0.50	0.54	0.55	0.56	0.57	0.58
	(0.53)	(0.54)	(0.55)	(0.55)	(0.56)
0.75	0.53	0.54	0.55	0.55	0.56
	(0.52)	(0.52)	(0.53)	(0.54)	(0.54)
2.00	0.49	0.51	0.51	0.52	0.53
	(0.52)	(0.53)	(0.54)	(0.55)	(0.55)
3.00	0.51	0.54	0.55	0.58	0.59
	(0.54)	(0.57)	(0.58)	(0.59)	(0.60)
4.00	0.51	0.56	0.57	0.61	0.62
	(0.56)	(0.59)	(0.61)	(0.62)	(0.63)
5.00	0.52	0.58	0.61	0.63	0.66
	(0.57)	(0.60)	(0.63)	(0.64)	(0.66)
6.00	0.54	0.59	0.62	0.65	0.68
	(0.58)	(0.61)	(0.64)	(0.66)	(0.68)

Note: The element in the first row in each box represents the results based on Monte Carlo Simulations (10,000 replications) and the number in bracket immediately below represents the result obtained by using asymptotic results.

the first row represents the results obtained by using Monte Carlo simulations and the second row represents the results obtained by using the asymptotic theory.

It is quite clear from the Tables V and VI that as the sample size increases the PCS increases as expected. It is also clear that as the shape parameter moves away from 1, the PCS increases. Even when the sample size is 20, the asymptotic results work reasonable well for both the distributions and for all possible ranges of the parameters. From the simulation study it is recommended that the asymptotic results can be used quite effectively even when the sample size is as small as 20 for all possible choices of the shape parameters.

6 DATA ANALYSIS

In this section two real life data sets will be analyzed.

6.1 Data Set 1

The first data set is given by Birnbaum and Saunders [4] on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data are presented below after subtracting 65.

5, 25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43, 43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55, 55, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64, 64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68, 69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76, 76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83, 84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97, 98, 98, 99, 101, 101, 103, 105, 109, 136, 147.

Under the assumption that the data come from a GE distribution, the MLEs of the different parameters are $\hat{\alpha} = 9.8776$ and $\hat{\lambda} = 0.0409$, also $\ln(L_{GE}(\hat{\alpha}, \hat{\lambda})) = -1216.151$, Similarly under the assumption that the data come from a gamma distribution, the MLEs of the different gamma parameters are $\hat{\beta} = 7.7302$ and $\hat{\theta} = 0.1124$. In this case $\ln(L_{GA}(\hat{\beta}, \hat{\theta})) = 1212.792$ and that provides $T = -1216.151 + 1212.792 = -3.359 < 0$. Therefore, we choose the gamma model in this case. Under the assumptions that the data come from the GA(7.7302, 0.1124) distribution we obtain the PCS = 0.7828 based on simulation results (10,000 replications) and also under the assumptions that the data come from GE(9.8776, 0.0409), we obtain the PCS = 0.7134 based on simulations. We obtain $\text{AM}_{\text{GA}}(7.7302) = -0.0112$, $\text{AV}_{\text{GA}}(7.7302) = 0.0209$, $\text{AM}_{\text{GE}}(9.8776) = 0.0084$, $\text{AV}_{\text{GE}}(9.8776) = 0.0213$ and we have $E_{\text{GA}}(T) \approx -1.1312$, $V_{\text{GA}}(T) \approx 2.1109$, $E_{\text{GE}}(T) \approx 0.8484$ and $V_{\text{GE}} \approx 2.1513$. Therefore using large sample approximation, under the assumption that the data come from a gamma distribution, the PCS is $\Phi(0.7786) \approx 0.78$ and using simulations we obtain the PCS = 0.7828. Under the assumption that the data come from a gamma distribution, the approximate p value of the observed $T = -3.359$ is clearly more than 0.5. Similarly, under the assumption that the data come from GE, using the large sample approximation we obtain the PCS is $\Phi(0.5784) \approx 0.72$ and using simulations the PCS = 0.7183. In this case the corresponding approximate p value of the observed T is 0.002. The K–S distances between the empirical distribution functions (EDF) and the fitted gamma and GE distribution functions are 0.1060 and 0.1114 respectively. Therefore, the p value and the K–S distance also suggest to choose the gamma model for this data set and in this case the PCS is at least $\min\{0.7183, 0.7828\} \approx 0.72$.

6.2 Data Set 2

The second data set is from Lawless [18, p. 28]. The data given arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

If we use the GE model the MLEs of the different parameters are $\hat{\alpha} = 5.2589$, $\hat{\lambda} = 0.0314$ and $\ln(L_{GE}(\hat{\alpha}, \hat{\lambda})) = -112.9763$. Similarly, if we use the gamma model the MLEs of the different parameters are $\hat{\beta} = 4.0196$ and $\hat{\theta} = 0.0556$ and $\ln(L_{WE}(\hat{\beta}, \hat{\theta})) = -113.0274$. Therefore, $T = -112.9763 + 113.0274 = 0.0511 > 0$, which indicates to choose the GE model. Assuming that the population distribution is GE with $\alpha = 5.2589 = \hat{\alpha}$ and $\lambda = 0.0314 = \hat{\lambda}$, we compute the PCS by computer simulations (based on 10,000 replications) similarly as in Section 5 and we obtain the PCS = 0.5450. On the other hand assuming that the original distribution is gamma with the shape parameter $\beta = 4.0196 = \hat{\beta}$ and the scale parameter $\theta = 0.0556 = \hat{\theta}$, then similarly as before based on 10,000 replications we obtain the PCS = 0.5881, yielding an estimated risk approximately 41% to choose the wrong model. Now we compute the PCSs based on large sample approximations. Assuming that the data come from GE, we obtain $\text{AM}_{\text{GE}}(5.2589) = 0.0030$ and $\text{AV}_{\text{GE}}(5.2589) = 0.0110$, it implies that $E_{\text{GE}}(T) \approx 0.0690$ and $V_{\text{GE}}(T) \approx 0.2523$. Therefore, assuming that the data come from GE, T is approximately normally distributed with mean 0.0690 and variance 0.2523 and the PCS is $1 - \Phi(-0.1373) = \Phi(0.1373) \approx 0.55$, which is almost equal to the above simulation result. Moreover under the same assumption that the data come from the GE, we obtain the approximate p value of the observed $T = 0.0511$ is greater than 0.50. Similarly, assuming that the data come from gamma, we compute $\text{AM}_{\text{GA}}(4.0196) = -0.0040$ and $\text{AV}_{\text{GA}}(2.1050) = 0.0081$. Using, (3.6) and (3.7) we have $E_{\text{GA}}(T) \approx -0.0917$ and $V_{\text{GA}}(T) \approx 0.1863$. Therefore, assuming that the data come from gamma distribution the probability of miss classification (1 − PCS) is $1 - \Phi(0.2124) \approx 0.42$, which is also very close to the simulated results. In this case the approximate p value of the observed $T = 0.0511$ is approximately 0.38. We also compute K–S distances between the EDF and fitted gamma and GE distribution functions and they are 0.1237 and 0.1261 respectively. Interestingly in this case K–S distance suggests to choose the gamma model and based on the p value we can not reject the hypothesis that the data come from gamma. Therefore in this case although our criterion suggest the GE model but definitely the choice is not very clear because the two fitted distributions are quite close to each other.

From the two examples it is clear that not only the sample size but the model parameters also play very important roles in choosing between the two overlapping distributions. For comparison purposes we compute the K–S distances in both the cases. It is observed that for data set 1, the K–S distance between the two fitted distributions is 0.0436 and for data set 2, the corresponding K–S distance is 0.0123. We plot the two fitted distributions for both the data sets in Figures 1 and 2 respectively. It also shows as expected that the distance between the two fitted distributions is very important in discriminating the two distribution functions.

7 CONCLUSIONS

In this paper we consider the problem of discriminating the two overlapping families of distribution functions, namely the gamma family and the GE family. We consider the statistic based on the logarithm of the ratio of the maximized likelihoods and obtain the asymptotic distributions of the test statistics under null hypotheses. We compare the probability of correct selection using Monte Carlo simulations with the asymptotic results and it is observed that even when the sample size is very small the asymptotic results work quite well for a wide range of the parameter space. Therefore, the asymptotic results can be used to estimate the probability of correct selection. We use these asymptotic results to calculate the minimum sample size for a user specified probability of correct selection. We use the concept of tolerance level based on the distance between the two distribution functions. For a particular D^* tolerance level the minimum sample size is obtained for a given user specified protection level. Two small tables are provided for the protection level 0.70 but for the other protection level the tables can be easily used as follows. For example if we need the protection level $p^* = 0.8$, then all the entries corresponding to the row of n, will be simply multiplied by $z_{0.8}^2/z_{0.7}^2$. Therefore, Tables III and IV can be used for any given protection level. We have just presented two small tables for illustration purposes, the extensive tables for different values of β and α can be easily obtained. It is observed that sample size and the model parameters play very important roles in discriminating between two distribution functions. For certain ranges of the parameter values the distance (K–S) between the two distribution functions is very small and therefore it may not be possible to discriminate them unless the sample size is very large.

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APPENDIX I

The following lemma is needed to prove the main results.

LEMMA 1 *Under the assumption that the data come from* GA(β, θ) *as* $n \to \infty$, *we have*

(i) $\hat{\beta} \to \beta$ *a.s.*, $\hat{\theta} \to \theta$ *a.s.*, *where*

$$E_{\mathrm{GA}}[\ln(f_{\mathrm{GA}}(X;\beta,\theta))] = \max_{\bar{\beta},\bar{\theta}} E_{\mathrm{GA}}[\ln(f_{\mathrm{GA}}(X;\bar{\beta},\bar{\theta}))].$$

(ii) $\hat{\alpha} \to \tilde{\alpha}$ *a.s.*, $\hat{\lambda} \to \tilde{\lambda}$ *a.s.*, *where*

$$E_{\mathrm{GA}}[\ln(f_{\mathrm{GE}}(X;\tilde{\alpha},\tilde{\lambda}))] = \max_{\alpha,\lambda} E_{\mathrm{GA}}[\ln(f_{\mathrm{GE}}(X;\alpha,\lambda))].$$

Note that $\tilde{\alpha}$ and $\tilde{\lambda}$ may depend on β and θ but we do not make it explicit for brevity. Let us denote

$$T^* = \ln\left(\frac{L_{\mathrm{GE}}(\tilde{\alpha}, \tilde{\lambda})}{L_{\mathrm{GA}}(\beta, \theta)}\right).$$

(iii) *$n^{-1/2}[T - E_{\mathrm{GA}}(T)]$ is asymptotically equivalent to $n^{-1/2}[T^* - E_{\mathrm{GA}}(T^*)]$*

Proof of Lemma 1 The proof follows using the similar argument of White [24, Theorem 1] and therefore it is omitted.

Proof of Theorem 1 Using the Central limit theorem and from part (ii) of Lemma 1, it follows that $n^{-1/2}[T^* - E_{\mathrm{GA}}(T^*)]$ is asymptotically normally distributed with mean zero and variance $V_{\mathrm{GA}}(T^*) = V_{\mathrm{GA}}(T)$. Therefore using part (iii) of Lemma 1, the result immediately follows.

To prove Theorem 2, we need the following lemma.

LEMMA 2 *Under the assumption that the data come from a GE distribution and as $n \to \infty$, we have*

(i) *$\hat{\alpha} \to \alpha$ a.s., $\hat{\lambda} \to \lambda$ a.s., where*

$$E_{\mathrm{GE}}[\ln(f_{\mathrm{GE}}(X; \alpha, \lambda))] = \max_{\bar{\alpha}, \bar{\lambda}} E_{\mathrm{GE}}[\ln(f_{\mathrm{GE}}(X; \bar{\alpha}, \bar{\lambda}))].$$

(ii) *$\hat{\beta} \to \tilde{\beta}$ a.s., $\hat{\theta} \to \tilde{\theta}$ a.s., where*

$$E_{\mathrm{GE}}[\ln(f_{\mathrm{GA}}(X; \tilde{\beta}, \tilde{\theta}))] = \max_{\beta, \theta} E_{\mathrm{GE}}[\ln(f_{\mathrm{GA}}(X; \beta, \theta))].$$

Note that here also $\tilde{\beta}$ and $\tilde{\theta}$ may depend on α and λ but we do not make it explicit for brevity. Let us denote $T_* = \ln\left(\frac{L_{\mathrm{GE}}(\alpha, \lambda)}{L_{\mathrm{GA}}(\tilde{\beta}, \tilde{\theta})}\right)$.

(iii) *$n^{-1/2}[T - E_{\mathrm{GE}}(T)]$ is asymptotically equivalent to $n^{-1/2}[T_* - E_{\mathrm{GE}}(T_*)]$*

Proof of Lemma 2 Follows similarly as Lemma 1.

Proof of Theorem 2 Using Lemma 2, it follows along the same line as the proof of Theorem 1.

APPENDIX II

For large n,

$$\begin{aligned}\frac{E_{\mathrm{GA}}(T)}{n} &\approx \mathrm{AM}_{\mathrm{GA}}(\beta) = E_{\mathrm{GA}}[\ln(f_{\mathrm{GE}}(\tilde{\alpha}, \tilde{\lambda})) - \ln(f_{\mathrm{GA}}(\beta, \theta))] \\ &= \ln(\tilde{\alpha}) + \ln\left(\frac{\tilde{\lambda}}{\theta}\right) - \left(\frac{\tilde{\lambda}}{\theta}\right)\beta + (\tilde{\alpha} - 1)E\left[\ln(1 - e^{-(\tilde{\lambda}/\theta)Y})\right] \\ &\quad + \ln(\Gamma(\beta)) - (\beta - 1)\psi(\beta) + \beta. \end{aligned} \tag{22}$$

Here Y is a gamma random variable with shape parameter β and scale parameter 1, $\psi(\cdot)$ is a digamma function. Since $(\tilde{\lambda}/\theta)$ is a function of $\tilde{\alpha}$ and β only, therefore (22) is a function of β only. Similarly for large n

$$\begin{aligned}\frac{V_{\text{GA}}(T)}{n} &\approx \text{AV}_{\text{GA}}(\beta) = V_{\text{GA}}[\ln(f_{\text{GE}}(\tilde{\alpha}, \tilde{\lambda})) - \ln(f_{\text{GA}}(\beta, \theta))] \\ &= \beta\left(1 - \frac{\tilde{\lambda}}{\theta}\right)^2 + (\tilde{\alpha} - 1)^2 V\left[\ln(1 - e^{-(\tilde{\lambda}/\theta)Y})\right] + (\beta - 1)^2\psi'(\beta) \\ &\quad - 2\beta(\beta - 1)\left(1 - \frac{\tilde{\lambda}}{\theta}\right)\psi(\beta + 1) + 2\left(1 - \frac{\tilde{\lambda}}{\theta}\right)(\tilde{\alpha} - 1)\text{cov}(Y,\ \ln(1 - e^{-(\tilde{\lambda}/\theta)Y})) \\ &\quad - 2(\tilde{\alpha} - 1)(\beta - 1)\text{cov}(\ln(Y),\ \ln(1 - e^{-(\tilde{\lambda}/\theta)Y})). \end{aligned} \tag{23}$$

Here Y is same as defined above. Clearly (23) is also a function of β only.

For large n,

$$\begin{aligned}\frac{E_{\text{GE}}(T)}{n} &\approx \text{AM}_{\text{GE}}(\alpha) = E_{\text{GE}}[\ln(f_{\text{GE}}(\alpha, \lambda)) - \ln(f_{\text{GA}}(\tilde{\beta}, \tilde{\theta}))] \\ &= \ln(\alpha) - [\psi(\alpha + 1) - \psi(1)] - \frac{(\alpha - 1)}{\alpha} - \tilde{\beta}\psi(\tilde{\beta}) + \ln(\Gamma(\tilde{\beta})) \\ &\quad + E(\ln(Z)) + \tilde{\beta}. \end{aligned} \tag{24}$$

Also

$$\begin{aligned}\frac{V_{\text{GE}}(T)}{n} &\approx \text{AV}_{\text{GE}}(\alpha) = V_{\text{GE}}[\ln(f_{\text{GE}}(\alpha, \lambda)) - \ln(f_{\text{GA}}(\tilde{\beta}, \tilde{\theta}))] \\ &= (\alpha - 1)^2 V[\ln(1 - e^{-Z})] + \left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]^2 V(Z) + (\tilde{\beta} - 1)^2 V(\ln(Z)) \\ &\quad - 2(\alpha - 1)\left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]\text{cov}[\ln(1 - e^{-Z}), Z] \\ &\quad - 2(\alpha - 1)(\tilde{\beta} - 1)\text{cov}[\ln(1 - e^{-Z}),\ \ln(Z)] \\ &\quad + 2(\tilde{\beta} - 1)\left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]\text{cov}(Z,\ \ln(Z)) \\ &= \left(\frac{\alpha - 1}{\alpha}\right)^2 + \left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]^2 (\psi'(1) - \psi'(\alpha + 1)) + (\tilde{\beta} - 1)^2 V(\ln(Z)) \\ &\quad - 2(\alpha - 1)\left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]\text{cov}[\ln(1 - e^{-Z}), Z] \\ &\quad - 2(\alpha - 1)(\tilde{\beta} - 1)\text{cov}[\ln(1 - e^{-Z}),\ \ln(Z)] \\ &\quad + 2(\tilde{\beta} - 1)\left[1 - \frac{\tilde{\beta}}{\psi(\alpha + 1) - \psi(1)}\right]\text{cov}(Z,\ \ln(Z)) \end{aligned} \tag{25}$$