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Estimating the Fundamental Frequency of a Periodic Function

Estimating Fundamental Frequency

Swagata Nandi[1], **Debasis Kundu**[2]

[1] **Institut für Angewandte Mathematik**
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg
Germany
e-mail: nandi@statlab.uni-heidelberg.de

[2] **Department of Mathematics**
Indian Institute of Technology Kanpur
Kanpur, Pin 208016
India
e-mail: kundu@iitk.ac.in

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Abstract In this paper we consider the problem of estimation of the fundamental frequency of a periodic function, which has several applications in Speech Signal Processing. The problem was originally proposed by Hannan (1974) and later on Quinn and Thomson (1991) provided an estimation procedure of the unknown parameters. It is observed that the estimation procedure of Quinn and Thomson (1991) is quite involved numerically. In this paper we propose to use two simple estimators and it is observed that their performance are quite satisfactory. Asymptotic properties of the proposed estimators are obtained. The large sample properties of the estimators are compared theoretically. We present some simulation results to compare their small sample performance. One speech data is analyzed using this particular model.

Key words Fundamental frequency; least squares estimators; periodogram estimator; asymptotic distribution.

1 Introduction

In this paper we consider the estimation of parameters of the following model:

$$y(n) = \sum_{j=1}^{p} \rho_j^0 \cos(nj\lambda^0 - \phi_j^0) + X(n); \quad \text{for} \quad n = 1, \ldots, N. \tag{1}$$

Here ρ_j^0 (> 0), $j = 1, \ldots, p$ are unknown amplitudes, $0 < \lambda^0 < \frac{\pi}{p}$ is the fundamental frequency and $\phi_j^0 \in (-\pi, \pi)$, $j = 1, \ldots, p$ are unknown phase components. $X(n)$'s are zero mean additive errors which satisfy the following assumption 1.

Assumption 1 *$X(n)$ has the following representation;*

$$X(n) = \sum_{k=-\infty}^{\infty} a(k)e(n-k),$$

where $e(k)$'s are independent and identically distributed random variables with mean zero and finite variance σ^2. The constants $a(k)$'s satisfy

$$\sum_{k=-\infty}^{\infty} |a(k)| < \infty.$$

The problem is to estimate the unknown parameters assuming 'p' is known.

Estimating the fundamental frequency of a periodic function is an important problem in Speech Signal Processing. Note that an additional mean term μ^0 can always be added to the model (1) and an efficient estimator of μ^0 can be obtained as $\bar{y} = \frac{1}{N}\sum_{n=1}^{N} y(n)$. Therefore, if μ^0 is present one can work with the transformed data, namely $y(n) - \bar{y}$. Since the main concern of this paper is to estimate the non-linear parameter λ^0 efficiently, we consider the model (1) for brevity. Note that many authors considered the estimation and testing procedures for the related discrete time model

$$y(n) = \mu^0 + \sum_{j=1}^{p} \rho_j^0 \cos(n\lambda_j^0 - \phi_j^0) + X(n), \tag{2}$$

where λ_j^0's denote unknown frequencies $(0 < \lambda_1^0 < \ldots < \lambda_p^0 < \pi)$, ρ_j^0's and ϕ_j^0's are same as defined before. References may be made to the work of Fisher (1929), Whittle (1959), Walker (1971), Hannan (1973) and also see Stoica (1993) for an extensive list of references of this problem. Here we consider the particular case where unknown frequencies λ_j^0's are harmonics of a fundamental frequency λ^0. Interestingly, although the model (2) is a very well studied model, much work was not done on model (1). There are many signals, like speech, where the data indicate the presence of harmonics with respect to a fundamental frequency. In these situations, it is better to use model (1) than model (2), because model (1) has less number of non-linear parameters than model (2) for fixed $p > 1$. The problem was first proposed by Hannan (1974) and later on an estimation procedure was developed by Quinn and Thomson (1991).

Quinn and Thomson (1991) obtained an estimation procedure of the unknown parameters using a weighted least squares approach. From now on we abbreviate the estimators proposed by Quinn and Thomson (1991) as the QT estimators. Unfortunately to obtain the QT estimators, one needs to know the spectral density function of the error random variable. If it is unknown, it needs to be estimated, which is quite involved computationally. Quinn and Thomson (1991) proposed an estimation technique of the spectral density function by averaging the periodogram function, but it is observed that it may not be very easy to implement in practice. In particular, we observe that finding the 'length' of the window becomes quite subjective and it influences the performance of the QT estimators. It can also be pointed out that the asymptotic properties of the QT estimators are obtained assuming the spectral density function to be known and therefore the asymptotic properties do not work well for finite sample sizes when the spectral density function of the error distribution is not known, which is usually the case in practice. Moreover, the QT estimators require the spectral density function of the errors to be strictly positive, which is a stronger assumption than assumption 1. If the spectral density function of the error random variable is zero at any particular point then the QT estimators can not be calculated. For example, if

$$X(n) = e(n) + e(n-3),$$

then the QT estimators can not be obtained because the spectral density function of $X(n)$ is 0 at $\frac{\pi}{3}$.

In this paper we propose to use two simple methods to estimate unknown parameters of the model (1). The first one is the usual least squares estimators (LSE's) and the second one can be obtained by maximizing the periodogram function, we call them as the approximate least squares estimators (ALSE's). In both cases we do not need to have the exact knowledge of the error random variables (except satisfying the assumption 1) and computationally both the estimators are much less demanding than the QT estimators. It is observed that the performance of both the estimators are quite satisfactory. It is known that the QT estimators are consistent and asymptotically normal. In this paper we observe that both the LSE's and ALSE's are consistent and they are asymptotically normally distributed. Interestingly, it is observed that LSE's and ALSE's are asymptotically equivalent but they are different from the QT estimators even asymptotically. Some theoretical comparisons of their asymptotic distributions are also performed. We propose a test for detecting harmonics using the LSE's or ALSE's similarly as the test proposed by Quinn and Thomson (1991).

The rest of the paper is organized as follows. In section 2, we present two estimation procedures. Asymptotic properties of the different estimators are presented in section 3. A test for harmonics is presented in section 4. Small sample performance of different methods are compared using numerical simulations in section 5. One speech data is analyzed in section 6 and finally we conclude the paper in section 7.

2 Estimation Procedures

In this section we present how to obtain the LSE's and ALSE's. The LSE's of the unknown parameters can be obtained by minimizing the residual sum of squares;

$$Q(\boldsymbol{\theta}) = \sum_{n=1}^{N} \left[y(n) - \sum_{j=1}^{p} \rho_j \cos(nj\lambda - \phi_j) \right]^2, \qquad (3)$$

with respect to $\boldsymbol{\theta} = (\rho_1, \ldots, \rho_p, \phi_1, \ldots, \phi_p, \lambda)$. The quantity $\hat{\boldsymbol{\theta}} = (\hat{\rho}_1, \ldots, \hat{\rho}_p, \hat{\phi}_1, \ldots, \hat{\phi}_p, \hat{\lambda})$, which minimizes $Q(\boldsymbol{\theta})$ will be called the LSE of $\boldsymbol{\theta}^0 = (\rho_1^0, \ldots, \rho_p^0, \phi_1^0, \ldots, \phi_p^0, \lambda^0)$. Since $\rho_1^0, \ldots, \rho_p^0$ and $\phi_1^0, \ldots, \phi_p^0$ are all linear parameters, using separable regression technique Richards (1961), it is possible to write explicitly the LSE's of $\rho_1^0, \ldots, \rho_p^0$ and $\phi_1^0, \ldots, \phi_p^0$ as functions of λ^0. Therefore, obtaining the LSE's become a one dimensional minimization problem. Once $\hat{\lambda}$ is obtained, $\hat{\rho}_1, \ldots, \hat{\rho}_p$ and $\hat{\phi}_1, \ldots, \hat{\phi}_p$ can be obtained as functions of $\hat{\lambda}$.

Now we describe the ALSE's. The ALSE of λ, say $\tilde{\lambda}$, can be obtained by maximizing $I_S(\lambda)$, the sum of the periodogram functions at $j\lambda$, $j = 1, \ldots, p$, defined as follows:

$$I_S(\lambda) = \frac{1}{N} \sum_{j=1}^{p} \left| \sum_{n=1}^{N} y(n) e^{inj\lambda} \right|^2. \qquad (4)$$

The ALSE's of the other parameters are as follows;

$$\hat{\rho}_j = \frac{2}{N} \left| \sum_{n=1}^{N} y(n) e^{inj\tilde{\lambda}} \right|, \qquad \tilde{\phi}_j = arg \left\{ \frac{1}{N} \sum_{n=1}^{N} y(n) e^{inj\tilde{\lambda}} \right\} \qquad (5)$$

for $j = 1, \ldots, p$. Therefore, obtaining the ALSE of λ^0 also involves one dimensional maximization of (4) and once we obtain $\tilde{\lambda}$ the other linear parameter estimators can be obtained simply as functions of $\tilde{\lambda}$.

Now for comparison purposes, we briefly mention how to obtain the QT estimators also. The QT estimator of λ^0 can be obtained by maximizing

$$R(\lambda) = \frac{1}{N} \sum_{j=1}^{p} \frac{1}{f(j\lambda)} \left| \sum_{n=1}^{N} y(n) e^{inj\lambda} \right|^2, \qquad (6)$$

where $f(.)$ is the spectral density function of $X(n)$ and it is assumed that $f(\omega)$ is known and strictly positive on $[0, \pi]$. If it is unknown, $f(j\lambda)$ in (6) is replaced by its estimate. The QT estimators of the other parameters are same as the ALSE's given in (5). Comparing (4) and (6) it is clear that the QT estimators and ALSE's are same when the error random variable is independent and identically distributed.

Now comparing the computational aspects of the different estimators, it is observed that the ALSE's are the easiest to obtain. It only involves one dimensional maximization of a function. The LSE's involve one dimensional minimization of a function and a $2p \times 2p$ matrix inversion to use the separable regression technique Richards (1961). If the spectral density function is known, the QT estimators also involve mainly maximization of a one dimensional function, but if it is unknown then at each iteration step it needs to be estimated. Therefore, if the spectral density function is not known, the QT estimator becomes the most expensive one.

3 Asymptotic Properties

In this section, we provide the asymptotic properties of the different estimators. Here $\hat{\boldsymbol{\theta}}$, $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$ are as defined before.
Main Results:

Theorem 1 *Under assumption 1, $\hat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}^0$ of the model (1), is a strongly consistent estimator of $\boldsymbol{\theta}^0$.*

Theorem 2 *Under assumption 1,*

$$\sqrt{N}\left[(\hat{\rho}_1 - \rho_1^0), \ldots, (\hat{\rho}_p - \rho_p^0), (\hat{\phi}_1 - \phi_1^0), \ldots, (\hat{\phi}_p - \phi_p^0), N(\hat{\lambda} - \lambda^0)\right] \\ \rightarrow N_{2p+1}(\mathbf{0}, 2\sigma^2 \mathbf{V})$$

as N tends to infinity. The dispersion matrix $\mathbf{V}$ is as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\mathbf{D}_{\rho^0}^{-1} + \frac{3\delta_G \mathbf{L}\mathbf{L}^T}{(\sum_{j=1}^p j^2 \rho_j^{0^2})^2} & \frac{6\delta_G \mathbf{L}}{(\sum_{j=1}^p j^2 \rho_j^{0^2})^2} \\ \mathbf{0} & \frac{6\delta_G \mathbf{L}^T}{(\sum_{j=1}^p j^2 \rho_j^{0^2})^2} & \frac{12\delta_G}{(\sum_{j=1}^p j^2 \rho_j^{0^2})^2} \end{bmatrix}$$

where

$$\mathbf{D}_{\rho^0} = diag\{\rho_1^{0^2}, \ldots, \rho_p^{0^2}\}, \quad \mathbf{L} = (1, 2, \ldots, p)^T,$$

$$\delta_G = \mathbf{L}^T \mathbf{D}_{\rho^0} \mathbf{C} \mathbf{L} = \sum_{j=1}^p j^2 \rho_j^{0^2} c(j), \quad \mathbf{C} = diag\{c(1), \ldots, c(p)\},$$

$$c(j) = \left\{ \sum_{k=-\infty}^{\infty} a(k) \cos(jk\lambda^0) \right\}^2 + \left\{ \sum_{k=-\infty}^{\infty} a(k) \sin(jk\lambda^0) \right\}^2$$

Theorem 3 *Under assumption 1, $\tilde{\boldsymbol{\theta}}$, the ALSE of $\boldsymbol{\theta}^0$ of the model (1), is a strongly consistent estimator of $\boldsymbol{\theta}^0$.*

Theorem 4 *Under assumption 1*

$$\sqrt{N}\left[(\tilde{\rho}_1-\rho_1^0),\ldots,(\tilde{\rho}_p-\rho_p^0),(\tilde{\phi}_1-\phi_1^0),\ldots,(\tilde{\phi}_p-\phi_p^0),N(\tilde{\lambda}-\lambda^0)\right] \rightarrow N_{2p+1}(\mathbf{0},2\sigma^2\mathbf{V})$$

as N tends to infinity. Here the matrix $\mathbf{V}$ is same as in theorem 2.

Comments: It is interesting to note that even if the errors are correlated the LSE's and ALSE's provide consistent estimators of unknown parameters. Now we compare asymptotic variances of the QT estimators, LSE's and ALSE's. Using our notation, asymptotic distributions of the QT estimators are as follows. Under the assumption of Quinn and Thomson (1991), the asymptotic distributions of the QT estimators are

$$\sqrt{N}\left[(\hat{\rho}_1-\rho_1^0),\ldots,(\hat{\rho}_p-\rho_p^0),(\hat{\phi}_1-\phi_1^0),\ldots,(\hat{\phi}_p-\phi_p^0),N(\hat{\lambda}-\lambda^0)\right] \rightarrow N_{2p+1}(\mathbf{0},2\sigma^2\mathbf{W})$$

as N tends to infinity. The dispersion matrix $\mathbf{W}$ is as follows:

$$\mathbf{W}=\begin{bmatrix}\mathbf{C} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\mathbf{D}_{\rho^0}^{-1}+3\beta^*\mathbf{L}\mathbf{L}^T & 6\beta^*\mathbf{L}\\ \mathbf{0} & 6\beta^*\mathbf{L}^T & 12\beta^*\end{bmatrix}$$

where

$$\beta^*=\left(\sum_{j=1}^{p}\frac{j^2\rho_j^{0^2}}{c(j)}\right)^{-1}.$$

If we define

$$x_{jk}=c(1)\ldots c(j-1)c(j+1)\ldots c(k-1)c(k+1)\ldots c(p)=\prod_{\substack{l=1\\ l\neq j,k}}^{p}c(l)$$

and

$$z_j=c(1)\ldots c(j-1)c(j+1)\ldots c(p)=\prod_{\substack{l=1\\ l\neq j}}^{p}c(l),$$

then it can be shown by straight forward calculations that

$$\frac{\delta_G}{(\sum_{j=1}^{p}j^2\rho_j^{0^2})^2}-\beta^*=\frac{\sum\sum_{1\leq j<k\leq p}j^2k^2\rho_j^{0^2}\rho_k^{0^2}(c(j)-c(k))^2x_{jk}}{(\sum_{j=1}^{p}j^2\rho_j^{0^2})^2(\sum_{j=1}^{1}j^2\rho_j^{0^2}z_j)}\geq 0,$$

as $c(j)$'s are non negative. Note that asymptotic distributions of the estimators of ρ_j^0's are same in all three cases, but asymptotic distributions of the estimators of ϕ_j^0's and λ^0 are different for $p \geq 2$. For $p > 1$, asymptotic variances of the QT estimators are smaller than the corresponding asymptotic variances of the LSE's or ALSE's of ϕ_j^0' and λ^0, if $c(j)$'s are distinct. Therefore, theoretically the QT estimators have certain advantages over the LSE's or ALSE's but in practice we observe (see Section 5) that the QT estimators may not behave that well. It may be mentioned that if in the model (1) μ^0 is present then the asymptotic distribution of $\bar{y}$ is same as the corresponding asymptotic distribution of the QT estimator and that is asymptotically independent of the other parameters.

Proof of the Main Results:

We need the following lemmas to prove the main results.

Lemma 1 *If $X(n)$ satisfies assumption 1, then*

$$\lim_{N\to\infty} \sup_{\beta} \left| \frac{1}{N^{L+1}} \sum_{n=1}^{N} n^L X(n) \cos(n\beta) \right| = 0 \quad a.s.,$$

$$\lim_{N\to\infty} \sup_{\beta} \left| \frac{1}{N^{L+1}} \sum_{n=1}^{N} n^L X(n) \sin(n\beta) \right| = 0 \quad a.s.,$$

for $L = 0, 1, \ldots$.

Proof For $L = 0$, the result is available in Kundu (1997). For general L, the result follows using the fact that $\frac{n}{N} < 1$. □

Lemma 2 *Let $\boldsymbol{\theta}$ be same as before and define*

$$S_{\delta,M} = \{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta \text{ or } |\rho_j - \rho_j^0| > \delta \text{ or } |\phi_j - \phi_j^0| > \delta, \\ \text{for any } j = 1, \ldots, p \text{ and } \rho_k \leq M, \text{ for all } k = 1, \ldots, p\}.$$

If for any $\delta > 0$ and for some $M < \infty$

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in S_{\delta,M}} \frac{1}{N} \left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0) \right] > 0 \quad a.s., \tag{7}$$

then $\hat{\boldsymbol{\theta}}$ which minimizes $Q(\boldsymbol{\theta})$ is a strongly consistent estimator of $\boldsymbol{\theta}^0$.

Proof In this proof only we denote $Q(\boldsymbol{\theta})$ by $Q_N(\boldsymbol{\theta})$ and $\hat{\boldsymbol{\theta}}$ by $\hat{\boldsymbol{\theta}}_N = (\hat{\rho}_{1N}, \ldots, \hat{\rho}_{pN}, \hat{\phi}_{1N}, \ldots, \hat{\phi}_{pN}, \hat{\lambda}_N)$ just to emphasize that they depend on N. Suppose $\hat{\boldsymbol{\theta}}_N$ is not consistent, then we can have one of the following two cases.
Case 1: For all subsequences $\{N_k\}$ of $\{N\}$, at least one $|\hat{\rho}_{jN_k}|$ tends to ∞.
Case 2: There exists a $\delta > 0$ and an $0 < M < \infty$ and a subsequence $\{N_k\}$ of $\{N\}$ such that $\hat{\boldsymbol{\theta}}_{N_k} \in S_{\delta,M}$ for all $k = 1, 2 \ldots$.

Now for both cases under the definition of $Q_N(\boldsymbol{\theta})$ (see (3)) and because of (7), there exits a K_0, such that for all $k > K_0$,

$$Q_{N_k}(\hat{\boldsymbol{\theta}}_{N_k}) - Q_{N_k}(\boldsymbol{\theta}^0) > 0 \quad a.s.$$

This contradicts the fact that $\hat{\boldsymbol{\theta}}_{N_k}$ minimizes $Q_{N_k}(\boldsymbol{\theta})$. $\square$

Proof of Theorem 1: Note that the set $S_{\delta,M}$, defined in lemma 2, can be written as

$$S_{\delta,M} = P_1 \cup P_2 \cup \ldots \cup P_p \cup \Phi_1 \cup \Phi_2 \cup \ldots \Phi_p \cup \Lambda,$$

where for $j = 1, \ldots, p$,

$$\begin{aligned}
P_j &= \left\{\boldsymbol{\theta} : |\rho_j - \rho_j^0| > \delta,\ \rho_k \le M,\ \forall\ \ k = 1, \ldots, p\right\}, \\
\Phi_j &= \left\{\boldsymbol{\theta} : |\phi_j - \phi_j^0| > \delta,\ \rho_k \le M,\ \forall\ \ k = 1, \ldots, p\right\} \quad \text{and} \\
\Lambda &= \left\{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta,\ \rho_k \le M,\ \forall\ \ k = 1, \ldots, p\right\}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] \\
&= \frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{j=1}^{p}\rho_j^0\cos(nj\lambda^0 - \phi_j^0) - \sum_{j=1}^{p}\rho_j\cos(nj\lambda - \phi_j)\right\}^2 \\
&+ \frac{2}{N}\sum_{n=1}^{N}X(n)\left\{\sum_{j=1}^{p}\rho_j^0\cos(nj\lambda^0 - \phi_j^0) - \sum_{j=1}^{p}\rho_j\cos(nj\lambda - \phi_j)\right\} \\
&= f_N(\boldsymbol{\theta}) + g_N(\boldsymbol{\theta}) \quad \text{(say)}.
\end{aligned}$$

For any $\delta > 0$ and a fixed $0 < M < \infty$,

$$\begin{aligned}
&\liminf_{N\to\infty}\inf_{\boldsymbol{\theta}\in P_j} f_N(\boldsymbol{\theta}) \\
&= \liminf_{N\to\infty}\inf_{\boldsymbol{\theta}\in P_j}\frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{k=1}^{p}\rho_k^0\cos(nk\lambda^0 - \phi_k^0) - \sum_{k=1}^{p}\rho_k\cos(nk\lambda - \phi_k)\right\}^2 \\
&= \liminf_{N\to\infty}\inf_{|\rho_j - \rho_j^0| > \delta}\frac{1}{N}\sum_{n=1}^{N}\left[(\rho_j^0 - \rho_j)\cos(nj\lambda^0 - \phi_j^0)\right]^2 \\
&= \inf_{|\rho_j - \rho_j^0| > \delta}\ \frac{1}{2}(\rho_j - \rho_j^0)^2 \ > \ \frac{1}{2}\delta^2 \ > \ 0 \ \ a.s. \quad j = 1, \ldots, p.
\end{aligned}$$

Similarly it can be proved that for $j = 1, \ldots, p$,

$$\liminf_{N\to\infty}\inf_{\boldsymbol{\theta}\in\Phi_j} f_N(\boldsymbol{\theta}) \ > \ 0 \ \ a.s. \qquad \text{and} \qquad \liminf_{N\to\infty}\inf_{\boldsymbol{\theta}\in\Lambda} f_N(\boldsymbol{\theta}) \ > \ 0 \ \ a.s.$$

which imply that

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in S_{\delta,M}} f_N(\boldsymbol{\theta}) \quad > \quad 0 \quad a.s. \tag{8}$$

Using lemma 1 it can be easily shown that

$$\lim_{N\to\infty} \sup_{\boldsymbol{\theta}\in S_{\delta,M}} g_N(\boldsymbol{\theta}) = 0 \quad a.s. \tag{9}$$

Now using (8) and (9) in lemma 2, theorem 1 follows immediately. □

Proof of Theorem 2: Let $Q'(\boldsymbol{\theta})$, the derivative vector of $Q(\boldsymbol{\theta})$, be defined as follows:

$$Q'(\boldsymbol{\theta}) = \left(\frac{\partial Q(\boldsymbol{\theta})}{\partial \rho_1}, \ldots, \frac{\partial Q(\boldsymbol{\theta})}{\partial \rho_p}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \phi_1}, \ldots, \frac{\partial Q(\boldsymbol{\theta})}{\partial \phi_p}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \lambda} \right).$$

where at $\boldsymbol{\theta}^0$,

$$\begin{aligned}
\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \rho_j} &= -2\sum_{n=1}^{N} X(n)\cos(nj\lambda^0 - \phi_j^0), \quad j = 1, \ldots, p,\\
\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \phi_j} &= -2\sum_{n=1}^{N} X(n)\rho_j^0 \sin(nj\lambda^0 - \phi_j^0), \quad j = 1, \ldots, p,\\
\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \lambda} &= 2\sum_{j=1}^{p}\sum_{n=1}^{N} X(n)nj\rho_j^0 \sin(nj\lambda^0 - \phi_j^0).
\end{aligned}$$

Similarly the $(2p+1)\times(2p+1)$ second derivative matrix $Q''(\boldsymbol{\theta})$ is also defined. Expanding $Q'(\boldsymbol{\theta})$ at $\hat{\boldsymbol{\theta}}$ around the true parameter $\boldsymbol{\theta}^0$, we obtain

$$Q'(\hat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^T Q''(\bar{\boldsymbol{\theta}}), \tag{10}$$

where $\bar{\boldsymbol{\theta}}$ is a point between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Consider the $(2p+1)\times(2p+1)$ diagonal matrix $\mathbf{D}$ as follows. The diagonal matrix $\mathbf{D}$ has first $2p$ diagonal entries as $N^{-\frac{1}{2}}$ and the last diagonal entry as $N^{-\frac{3}{2}}$. Therefore, (10) can be re-written as

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^T\mathbf{D}^{-1} = -\left[Q'(\boldsymbol{\theta}^0)\mathbf{D}\right]\left[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}.$$

It can be shown that as $\hat{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\theta}}$ tend to $\boldsymbol{\theta}^0$,

$$\lim_{N\to\infty}\left[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}\right] = \lim_{N\to\infty}\left[\mathbf{D}Q''(\boldsymbol{\theta}^0)\mathbf{D}\right] = \boldsymbol{\Sigma},$$

where $\boldsymbol{\Sigma}$ can be written as follows

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\rho^0} & -\frac{1}{2}\mathbf{D}_{\rho^0}\mathbf{L} \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}^T\mathbf{D}_{\rho^0} & \frac{1}{3}\mathbf{L}^T\mathbf{D}_{\rho^0}\mathbf{L} \end{bmatrix}.$$

Here $\mathbf{D}_{\rho^0}$ and $\mathbf{L}$ are same as defined earlier. Now using the Central Limit theorem of a linear process (Fuller; 1976, page 251-256), $Q'(\boldsymbol{\theta}^0)\mathbf{D}$ converges to a (2p+1)-variate normal distribution with mean vector $\mathbf{0}$ and variance-covariance matrix

$$2\sigma^2 \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\rho^0}\mathbf{C} & -\frac{1}{2}\mathbf{C}\mathbf{D}_{\rho^0}\mathbf{L} \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}^T\mathbf{D}_{\rho^0}\mathbf{C} & \frac{1}{3}\mathbf{L}^T\mathbf{D}_{\rho^0}\mathbf{C}\mathbf{L} \end{bmatrix} = 2\sigma^2\mathbf{G} \quad (say).$$

Here $\mathbf{C}$ is same as defined before. Now theorem 2 follows by observing the fact that $\mathbf{V} = \boldsymbol{\Sigma}^{-1}\mathbf{G}\boldsymbol{\Sigma}^{-1}$. $\square$

Now to prove theorem 3, we need the following lemmas.

Lemma 3 *Let us denote* $S_c = \{\lambda : |\lambda - \lambda^0| > c\}$. *If for any* $c > 0$

$$\limsup_{N\to\infty} \sup_{S_c} \frac{1}{N}\left[I_S(\lambda) - I_S(\lambda^0)\right] < 0 \quad a.s.$$

then $\tilde{\lambda}$ *which maximizes* $I_S(\lambda)$ *(see (4)) is a strongly consistent estimator of* λ^0.

Proof The proof is quite similar to the proof of lemma 2. For details see Lemma 2.3 (chapter 2) of Nandi (2001). $\square$

Lemma 4 *The ALSE* $\tilde{\lambda}$ *is a strongly consistent estimator of* λ^0.

Proof Using lemma 1, for a fixed $c > 0$, we have,

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{S_c} \frac{1}{N}[I_S(\lambda) - I_S(\lambda^0)] \\ &= -\lim_{N\to\infty} \frac{1}{N^2}\sum_{k=1}^{p}\left\{\sum_{n=1}^{N}\left(\sum_{j=1}^{p}\rho_j^0\cos(nj\lambda^0 - \phi_j^0) + X(n)\right)\cos(nk\lambda^0)\right\}^2 \\ &\quad -\lim_{N\to\infty} \frac{1}{N^2}\sum_{k=1}^{p}\left\{\sum_{n=1}^{N}\left(\sum_{j=1}^{p}\rho_j^0\cos(nj\lambda^0 - \phi_j^0) + X(n)\right)\sin(nk\lambda^0)\right\}^2 \\ &= -\sum_{k=1}^{p}\left[\left(\frac{\rho_k^0\cos(\phi_k^0)}{2}\right)^2 + \left(\frac{\rho_k^0\sin(\phi_k^0)}{2}\right)^2\right] \\ &= -\frac{1}{4}\sum_{k=1}^{p}{\rho_k^0}^2 < 0 \quad a.s. \end{aligned}$$

Therefore, by using lemma 3, the result follows. $\square$

Lemma 5 *If* $\tilde{\lambda}$ *is the ALSE of* λ^0 *then* $N(\tilde{\lambda} - \lambda^0) \longrightarrow 0 \quad a.s.$

Proof Expanding $I'_S(\lambda)$ at $\tilde{\lambda}$ around λ^0, by Taylor series expansion, we obtain

$$I'_S(\tilde{\lambda}) - I'_S(\lambda^0) = (\tilde{\lambda} - \lambda^0) I''_S(\bar{\lambda}).$$

Here $\bar{\lambda}$ is a point between $\tilde{\lambda}$ and λ^0. Since $I'_S(\tilde{\lambda}) = 0$,

$$N(\tilde{\lambda} - \lambda^0) = -\frac{\frac{1}{N^2} I'_S(\lambda^0)}{\frac{1}{N^3} I''_S(\bar{\lambda})}.$$

Now it can be easily shown that

$$\lim_{N\to\infty} \frac{1}{N^3} I''_S(\bar{\lambda}) = \lim_{N\to\infty} \frac{1}{N^3} I''_S(\lambda^0) = -\frac{1}{24} \sum_{k=1}^{p} {\rho_k^0}^2$$

$$\text{and} \quad \lim_{N\to\infty} \frac{1}{N^2} I'_S(\lambda^0) = 0 \quad a.s.$$

Therefore, the result follows. □

Lemma 6 *The ALSE's of ρ_j^0's and ϕ_j^0's given in (5) are strongly consistent estimators of ρ_j^0's and ϕ_j^0's for $j = 1, \ldots, p$.*

Proof Observe that

$$\begin{aligned}
\tilde{\rho_j}^2 &= \frac{4}{N^2} \left[\sum_{n=1}^{N} y(n) \cos(nj\tilde{\lambda}) \right]^2 + \frac{4}{N^2} \left[\sum_{n=1}^{N} y(n) \sin(nj\tilde{\lambda}) \right]^2 \\
&= \frac{4}{N^2} \left[\sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \rho_k^0 \cos(nk\lambda^0 - \phi_k^0) + X(n) \right\} \times \right. \\
&\qquad \left. \left\{ \cos(nj\lambda^0) - n(\tilde{\lambda} - \lambda^0) \sin(nj\tilde{\lambda}) \right\} \right]^2 \\
&+ \frac{4}{N^2} \left[\sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \rho_k^0 \cos(nk\lambda^0 - \phi_k^0) + X(n) \right\} \times \right. \\
&\qquad \left. \left\{ \sin(nj\lambda^0) + n(\tilde{\lambda} - \lambda^0) \cos(nj\tilde{\lambda}) \right\} \right]^2.
\end{aligned}$$

Therefore, using strong consistency of $\tilde{\lambda}$, lemma 1 and lemma 5, we obtain as $N \to \infty$ that

$$\tilde{\rho_j}^2 \longrightarrow 4 \left\{ \frac{\rho_j^0 \cos(\phi_j^0)}{2} \right\}^2 + 4 \left\{ \frac{\rho_j^0 \sin(\phi_j^0)}{2} \right\}^2 = {\rho_j^0}^2 \quad a.s.$$

$j = 1, \ldots, p$. Exactly in the same way, it can be shown that

$$\frac{1}{N} \sum_{n=1}^{N} y(n) \cos(nj\tilde{\lambda}) \longrightarrow \frac{\rho_j^0 \cos(\phi_j^0)}{2} \quad a.s.$$

$$\text{and} \quad \frac{1}{N} \sum_{n=1}^{N} y(n) \sin(nj\tilde{\lambda}) \longrightarrow \frac{\rho_j^0 \sin(\phi_j^0)}{2} \quad a.s.$$

Therefore,

$$\tilde{\phi}_j \to arg \left\{ \frac{\rho_j^0 \cos(\phi_j^0)}{2} + i \frac{\rho_j^0 \sin(\phi_j^0)}{2} \right\} = \phi_j^0 \quad a.s.$$

$j = 1, \ldots, p.$ □

Proof of Theorem 3: Using lemmas 4 and 6, the theorem follows. □

Proof of Theorem 4: Similarly as Hannan (1973), it can be shown that

$$\begin{aligned} \hat{\lambda} &= \tilde{\lambda} + O_p(N^{-2}), \\ \hat{\rho}_j &= \tilde{\rho}_j + O_p(N^{-1}), \\ \hat{\phi}_j &= \tilde{\phi}_j + O_p(N^{-1}). \end{aligned} \tag{11}$$

Here $O_p(N^{-1})$ and $O_p(N^{-2})$ indicate that they converge to zero in probability and also $N\ O_p(N^{-1})$ and $N^2\ O_p(N^{-2})$ are bounded in probability as $N \to \infty$. Using theorem 2 and (11), it follows that LSE's and ALSE's are asymptotically equivalent. Thus, ALSE's have the same asymptotic distribution as LSE's. □

4 A Test for Harmonics

In this section we propose a test to check whether harmonics are present or not in the model (1). This test is very similar to the test proposed by Quinn and Thomson (1991). The only difference is that here we will be using either the LSE's or ALSE's rather than the QT estimators as proposed in Quinn and Thomson (1991).

Consider the more general model (2) where λ_j^0's are not necessarily harmonics of a fundamental frequency λ^0. The other parameters and the errors are same as defined before. We assume that μ is not present and if it is assumed to be present then we work with the transformed data as mentioned earlier. We consider testing of $H_0 : \lambda_j^0 = j\lambda^0$ $(j = 1, \ldots, p)$ vs. H_A: not H_0, here λ^0 is unknown. The likelihood ratio statistic, constructed under the assumption of normality of $X(n)$, is asymptotically equivalent to

$$\chi^2 = \frac{N}{\pi} \left\{ \sum_{j=1}^{p} \frac{J(\hat{\lambda}_j)}{f(\hat{\lambda}_j)} - \sum_{j=1}^{p} \frac{J(j\hat{\lambda})}{f(j\hat{\lambda})} \right\} \tag{12}$$

where

$$J(\lambda) = \left| \frac{1}{N} \sum_{n=1}^{N} y(n) e^{in\lambda} \right|^2.$$

Here $\hat{\lambda}_j$, $j = 1, \ldots, p$ are the least squares estimators of λ_j^0, $j = 1, \ldots, p$ for the model (2) and $\hat{\lambda}$ is the least squares estimator of λ^0 of the model (1).

It can be shown along the same line as Quinn and Thomson (1991) that χ^2 is asymptotically equivalent to

$$\chi_*^2 = \frac{N^3}{48\pi} \sum_{j=1}^{p} \frac{\hat{\rho}_j^2 (\hat{\lambda}_j - j\hat{\lambda})^2}{f(j\hat{\lambda})}. \quad (13)$$

Furthermore, under H_0

$$\chi^2 = \chi_{QT}^2 + o(1).$$

Here χ_{QT}^2 is the χ^2 defined in equation no. (19) of Quinn and Thomson (1991). χ_{QT}^2 is asymptotically distributed as a χ_{p-1}^2 random variable, therefore χ^2 is also asymptotically distributed as a χ_{p-1}^2 random variable. In (12), the LSE's can be replaced by ALSE's and asymptotic property of χ^2 remains unchanged. In practice, we should use (13). It may be mentioned that χ_*^2 as defined in (13) is much easier to compute than the corresponding χ_*^2 defined in Quinn and Thomson (1991).

5 Numerical Examples

In this section we present numerical results based on simulations to compare performance of the different estimators for finite samples. We consider the model (1) with $p = 3$ and three different sets of parameters. Thus the models are as follows:

Model 1:

$$\rho_1 = 1.0, \rho_2 = 0.5, \rho_3 = 0.5, \phi_1 = 0.8, \phi_2 = 0.6, \phi_3 = 1.0,$$

$$\lambda = \frac{14\pi}{100} = 0.4398229, \sigma^2 = 0.2, X(n) = e(n) + .5e(n-1).$$

Model 2:

$$\rho_1 = 2.2, \rho_2 = 1.5, \rho_3 = 3.0, \phi_1 = -1.2, \phi_2 = 0.5, \phi_3 = 0.9,$$

$$\lambda = \frac{20\pi}{100} = 0.62831854, \sigma^2 = 0.5, X(n) = e(n) + 0.7e(n-1).$$

Model 3:

$$\rho_1 = 1.0, \rho_2 = 2.6, \rho_3 = 3.5, \phi_1 = 0.9, \phi_2 = 1.0, \phi_3 = 2.1,$$

$$\lambda = \frac{30\pi}{100} = 0.94247781, \sigma^2 = 0.5, X(n) = e(n) - 1.2e(n-1) + 0.48e(n-2).$$

For all three models the sample size is taken as 100. For each model first we generate a data set and using that data sequence we estimate the unknown parameters by different methods. To evaluate the QT estimators, we need to estimate the spectral density function at $j\lambda^0$, $j = 1, \ldots, p$. In our notation the spectral density $f(j\lambda^0)$ is equal to $\sigma^2 c(j)$. It can be shown that

$$\sigma^2 c(j) = E\left(\frac{1}{N}\left|\sum_{n=1}^{N} X(n)e^{-inj\lambda^0}\right|^2\right),$$

i.e. the expected value of the periodogram function of the noise at the respective frequency. For numerical experiments we use the method of averaging the periodogram function over a window $(-L, L)$ across the point estimate, as proposed in Quinn and Thomson (1991). We try for different L and observe that for $L = 4, 3$ and 5, it provides the best results in this case for Models 1, 2 and 3 respectively. It has been observed in the simulation study that the performance of the QT estimators depend on the choice of L. We also compute asymptotic 95% confidence bounds for all the parameters by using different methods. For our proposed methods we do not need to estimate the spectral density function for estimation purposes but we need it for constructing confidence intervals. Here also we use the same technique by averaging the periodogram function of the estimated residuals over a window $(-L, L)$. For the LSE's, we report the results using $L = 6, 8$, and 7 for Models 1, 2 and 3 respectively and for ALSE's they are 6, 6 and 5. We replicate the process 5000 times and report the average estimates (EST), mean squared errors (MSE), average confidence lengths (ACL) and coverage percentages (CP) over 5000 replications. For comparison purposes we also report asymptotic variances (ASYV) and expected confidence lengths (ECL) for all parameter estimators. The results for Models 1, 2 and 3 are reported in Tables 1, 2 and 3 respectively.

First we compare the performance of the LSE's and QT estimators. Comparing these two estimators it is clear that the LSE's perform better than QT estimators for all the three models in all respects. In each case the LSE's have lower MSE's and lower biases than the corresponding QT estimators. Most of the LSE's are unbiased, whereas some of the QT estimators are highly biased, mainly phase estimators obtained using the QT method are highly biased. The confidence lengths based on the QT estimators are smaller than the corresponding confidence lengths based on LSE's for λ and ϕ_j's. It should be that way also, because asymptotic variances of the LSE's are larger for these parameters than the QT estimators. The coverage percentages of the LSE's are much higher than the corresponding QT estimators in most cases. The coverage percentages obtained by the LSE's do not attain the 95% nominal level all the times but all of them are quite close to the nominal value. On the other hand the coverage percentages obtained by the QT estimators are quite poor. In case of Model 1, for ρ_j's the QT method works well but for the phase components and for the fundamental frequency, the coverage percentages are lower than 90%. For Model 2, the coverage percentages are quite small as compared to the nominal value for all the parameters and for the fundamental frequency it is only 52%. In case of Model 3, for some parameters, the coverage percentages are almost 100% and for others they are below 70%.

Comparing the performance of the ALSE's and QT estimators, it is clear that the two estimators behave quite similarly as far as MSE's and biases are concerned at least for Models 1 and 2. For Model 3, the MSE's and biases of the ALSE's of phase components and the fundamental frequency are higher than the QT's estimators. Both the ALSE's and QT estimators are highly biased for phase components for Models 1 and 2, otherwise they are more or less unbiased. The confidence lengths of ALSE's are larger than the QT estimators for almost all the parameters and for all three models but the coverage percentages of the ALSE's are much better than the QT estimators. Although coverage percentages of the amplitudes based on the ALSE's are very close to the nominal level, the same is not true for phase components or for the fundamental frequency. The coverage percentage of the fundamental frequency based on ALSE's is quite poor and it is as low as 74%, 81% and 87% for Models 1, 2 and 3 respectively.

Finally we compare the LSE's and ALSE's. Comparing the two it is observed that the LSE's have lower biases in all the cases. The ALSE's have higher MSE's for all the parameters and for all the models except for the parameter ρ_1 in Model 3. The coverage percentages of the LSE's are quite close to the nominal level. For the ALSE's only the amplitude estimates cover almost 95% of times but for other parameters the coverage percentages are lower than the nominal level. In simulation study we observe that there are minor differences between average confidence lengths obtained using the LSE's and ALSE's. It may not be very surprising because confidence lengths are obtained using the estimated errors. We have shown that theoretically the LSE's and ALSE's are equivalent for large sample sizes, and this different behavior may be a finite-sample phenomenon.

Comparing all three estimators, we come to the following conclusions. The confidence intervals obtained by using the QT method may not be of much use. QT estimators have certain theoretical advantages over LSE's and ALSE's but computational complexity of the QT method makes it difficult to compute QT estimators and also the asymptotic results may not be used for finite length data. In practice LSE's can be used to estimate the unknown parameters of the model (1).

6 Data Analysis

In this section we analyze one sustained vowel sound 'uuu' using the least squares and QT methods. The computations of this section have been performed using Silicon Graphics Machine. The data set contains 512 signal values sampled at 10kHz frequency. It is presented in Figure 1. Simple periodogram plot (Figure 2) indicates that $p = 4$. The estimate of μ is 0.0681 and the 95% confidence interval for μ is (.0126, 0.1407). So we fit the model (1) to the mean corrected data using LSE's and QT estimators. The results

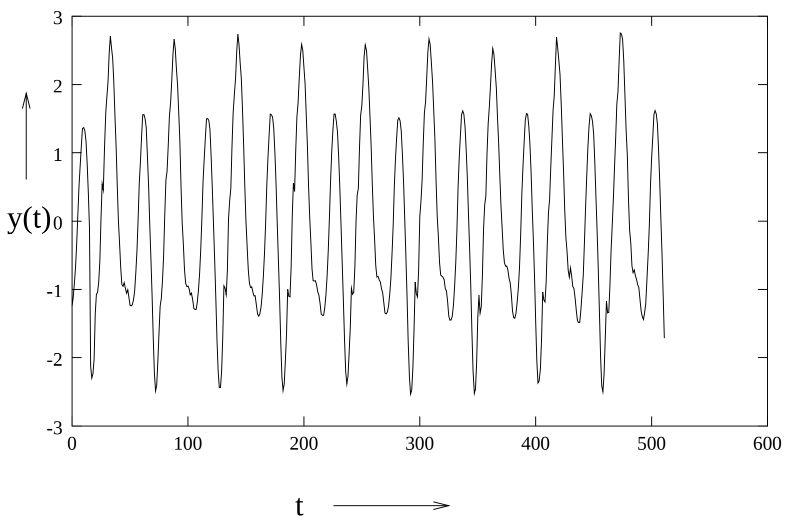

Fig. 1 The plot of the observed “uuu” vowel sound.

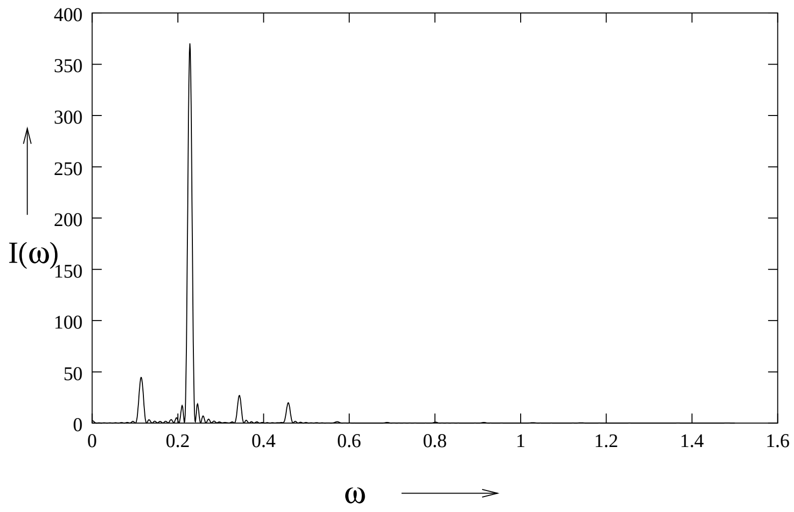

Fig. 2 The plot of the periodogram function of “uuu” vowel sound.

are presented in Table 4. We perform the χ^2 test proposed in section 4 for testing whether harmonics are present or not. We obtain the value of $\chi^2_* < 1$ and thus the null hypothesis (harmonics are present) is retained. Therefore, fitting of the model (1) is justified to the present data set. Analyzing the errors it is observed that in both cases the estimated errors are stationary. In case of the LSE's the estimated error is of the following form:

$$X(t) = a_1 X(t-1) + a_2 X(t-2) + a_3 X(t-2) + e(t)$$

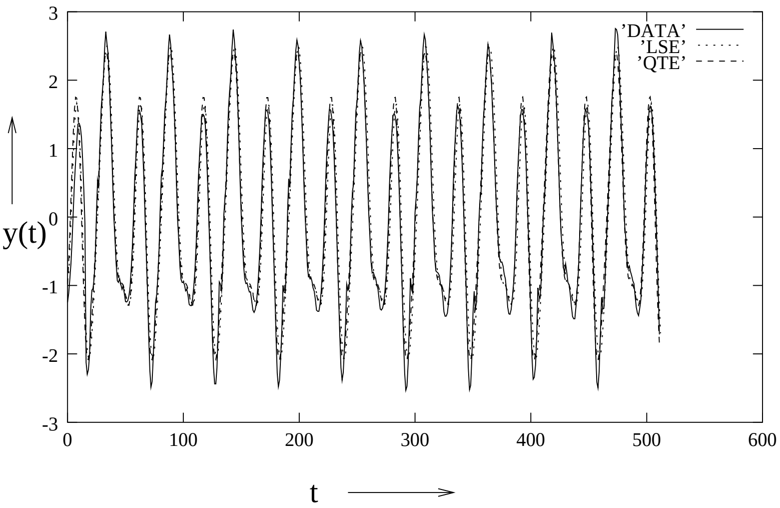

Fig. 3 The observed "uuu" vowel sound (DATA), the predicted values using least squares estimators (LSE's) and the predicted values using QT estimators (QTE).

with $a_1 = 1.1229$, $a_2 = -0.5140$ and $a_3 = 0.1012$ and for the QT estimators, the estimated error is of the form:

$$X(t) = a_1 X(t-1) + a_2 X(t-2) + e(t)$$

with $a_1 = 1.0765$ and $a_2 = -0.3936$. But the stationary assumption is not satisfied for the whole data set for the QT estimators. We obtain it for different segments of the data set and the error is estimated using one of such segments. The fitted models are provided in Figure 3. Figure 3 indicates that it is a very good fit for both cases. The square root of the residual sum of squares for the QT estimators and for LSE's are 0.2522245 and 0.2520563 respectively. Therefore, the LSE's provide a marginally better fit than QT estimators.

7 Conclusions

In this paper we consider the problem of estimating the unknown parameters of a periodic signal when harmonics of a fundamental frequency are present. We propose to use two simple estimators and have studied their large and small sample performance with the present estimators. It is observed that the proposed LSE's and ALSE's are computationally very efficient and both of them involve only one dimensional optimization, whereas the QT method proposed in Quinn and Thomson (1991) is quite involved computationally. Theoretically it is observed that the QT estimators have lower asymptotic variances for certain parameters than the LSE's or ALSE's but small sample performance indicate that the proposed LSE's behave better than the estimators proposed by Quinn and Thomson in terms of MSE's, biases and coverage probabilities. In the simulation study it is also observed that asymptotic confidence intervals do not work well for the QT estimators but the corresponding performance of the LSE's are quite satisfactory. Moreover, the LSE's and ALSE's can be used for larger class of errors than the QT estimators. Considering all these points we recommend to use the LSE's to estimate the parameters of a periodic signal when harmonics of a fundamental frequency are present.

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Table 1 Average estimates, MSEs, average confidence lengths and coverage percentages of LSE's, QTE's and ALSE's for Model 1.

Parameters	LSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	1.0065	8.43e-03	8.62e-03	.3400	.3639	.93
ρ_2	.5078	7.46e-03	7.55e-03	.3194	.3406	.93
ρ_3	.5102	6.11e-03	5.99e-03	.2853	.3035	.92
ϕ_1	.8015	1.23e-02	1.35e-02	.4275	.4562	.94
ϕ_2	.6080	4.29e-02	4.99e-02	.8297	.8757	.94
ϕ_3	1.0045	5.56e-02	6.83e-02	.9593	1.0245	.94
λ	.4399	1.62e-06	1.97e-06	5.11e-03	5.50e-03	.94

Parameters	QTEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	1.0000	8.33e-03	8.62e-03	.3639	.3639	.95
ρ_2	.5149	7.72e-03	7.55e-03	.3406	.3406	.95
ρ_3	.5205	6.27e-03	5.99e-03	.3035	.3035	.95
ϕ_1	.8882	1.97e-02	1.34e-02	.4533	.4543	.89
ϕ_2	.7733	7.78e-02	4.94e-02	.8661	.8716	.87
ϕ_3	1.2606	.1267	6.73e-02	.9940	1.0166	.83
λ	.4416	4.91e-06	1.92e-06	5.30e-03	5.43e-03	.74

Parameters	ALSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	1.0048	8.45e-03	8.62e-03	.3570	.3639	.94
ρ_2	.5133	7.65e-03	7.55e-03	.3342	.3406	.94
ρ_3	.5212	6.56e-03	5.99e-03	.2988	.3035	.93
ϕ_1	.8811	1.87e-02	1.35e-02	.4468	.4562	.89
ϕ_2	.7688	7.77e-02	4.99e-02	.8577	.8757	.86
ϕ_3	1.2425	.1204	6.83e-02	.9878	1.0245	.83
λ	.4415	4.65e-06	1.97e-06	5.28e-03	5.50e-03	.74

Table 2 Average estimates, MSE's, average confidence lengths and coverage percentages of LSE's, QTE's and ALSE's for Model 2.

Parameters	LSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	2.2087	2.30e-02	2.62e-02	.6044	.6348	.95
ρ_2	1.5083	1.58e-02	1.92e-02	.5209	.5435	.95
ρ_3	2.9999	9.87e-03	1.06e-02	.3913	.4031	.94
ϕ_1	-1.2025	3.26e-03	5.80e-03	.2852	.2987	.97
ϕ_2	.4971	6.96e-03	1.01e-02	.3790	.3937	.96
ϕ_3	.8967	4.27e-03	4.65e-03	.2593	.2672	.95
λ	.6283	1.40e-07	1.54e-07	1.49e-03	1.54e-03	.95

(continue...)

Para-meters	QTEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	2.2074	2.69e-02	2.62e-02	.3639	.6348	.73
ρ_2	1.5061	1.86e-02	1.92e-02	.3406	.5435	.78
ρ_3	3.0088	1.05e-02	1.06e-02	.3035	.4031	.86
ϕ_1	-1.1750	6.71e-03	5.78e-03	.1747	.2980	.71
ϕ_2	.5476	1.25e-02	9.99e-03	.2532	.3918	.74
ϕ_3	.9749	1.03e-02	4.42e-03	.1930	.2607	.61
λ	.6288	4.03e-07	1.44e-07	1.10e-03	1.49e-03	.53

Para-meters	ALSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	2.2081	2.69e-02	2.62e-02	.6215	.6348	.93
ρ_2	1.5072	1.86e-02	1.92e-02	.5352	.5435	.94
ρ_3	3.0081	1.05e-02	1.06e-02	.4009	.4031	.94
ϕ_1	-1.1798	6.50e-03	5.80e-03	.2936	.2987	.92
ϕ_2	.5381	1.18e-02	1.01e-02	.3901	.3937	.91
ϕ_3	.9604	8.53e-03	4.65e-03	.2654	.2672	.84
λ	.6287	3.24e-07	1.54e-07	1.53e-03	1.54e-03	.81

Table 3 Average estimates, MSE's, average confidence lengths and coverage percentages of LSE's, QTE's and ALSE's for Model 3.

Para-meters	LSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	.9946	5.36e-03	2.86e-03	.2591	.2096	.92
ρ_2	2.6415	2.31e-02	2.99e-02	.6203	.6780	.95
ρ_3	3.5078	3.96e-02	6.83e-02	.9084	1.0241	97
ϕ_1	.8944	6.17e-03	4.17e-03	.2919	.2530	93
ϕ_2	.9965	7.34e-03	9.66e-03	.3464	.3852	.95
ϕ_3	2.0961	1.48e-02	1.73e-02	.4585	.5162	.93
λ	.9425	4.13e-07	5.23e-07	2.52e-03	2.84e-03	.94

Para-meters	QTEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	1.0025	3.18e-03	2.86e-03	.3639	.2096	1.0
ρ_2	2.6146	3.14e-02	2.99e-02	.3406	.6780	.66
ρ_3	3.5224	6.63e-02	6.83e-02	.3035	1.0241	.44
ϕ_1	.9088	3.73e-03	3.90e-03	.3670	.2450	1.0
ϕ_2	1.0200	8.35e-03	8.61e-03	.1596	.3637	.61
ϕ_3	2.1296	1.366e-02	1.50e-02	.1618	.4798	.51
λ	.9427	4.66e-07	4.18e-07	9.11e-04	2.54e-03	.48

(continue...)

Para-meters	ALSEs					
	EST	MSE	ASYV	ACL	ECL	CP
ρ_1	1.0013	3.18e-03	2.86e-03	.2371	.2096	.95
ρ_2	2.6129	3.14e-02	2.99e-02	.6717	.6780	.93
ρ_3	3.5283	6.63e-02	6.83e-02	.9989	1.0241	94
ϕ_1	.9259	4.62e-03	4.17e-03	.2765	.2530	.95
ϕ_2	1.0547	1.20e-02	9.66e-03	.3804	.3852	.91
ϕ_3	2.1804	2.04e-02	1.73e-02	.5028	.5162	.92
λ	.9430	8.13e-07	5.23e-07	2.76e-03	2.84e-03	.87

Table 4 LSE's and QT estimators of different parameters and their 95% confidence intervals of "uuu" vowel sound.

Para-meters	LSE's			QT estimators		
	LSE	Lower Limit	Upper Limit	QTE	Lower Limit	Upper Limit
ρ_1	0.6387	0.5750	0.7023	0.5948	0.5806	0.6090
ρ_2	1.7156	1.6541	1.7772	1.6973	1.6840	1.7105
ρ_3	0.4312	0.3728	0.4896	0.4538	0.4415	0.4657
ρ_4	0.3493	0.2948	0.4039	0.3899	0.3790	0.4009
ϕ_1	-2.3816	-2.4847	-2.2786	-2.3364	-2.3608	-2.3120
ϕ_2	1.5412	1.4775	1.6049	1.6201	1.6067	1.6335
ϕ_3	-2.7337	-2.8905	-2.5770	-2.7037	-2.7349	-2.6724
ϕ_4	-2.5603	-2.7488	-2.3720	-2.3913	-2.4269	-2.3558
λ	0.1140	0.1139	0.1141	0.1142	0.1142	0.1142