Amplitude Modulated Model For Analyzing Non Stationary Speech Signals

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Abstract

Recently Amplitude Modulated model in presence of additive white noise was used to analyze certain non-stationary speech data. It is observed that the assumption of white noise may not be proper in many cases. In this paper we consider the Amplitude Modulated signal model in presence of stationary noise. We consider the least squares estimators and the estimators obtained by maximizing the Periodogram function. The two estimators are asymptotically equivalent. We study the theoretical properties of both estimators and observe their performances through numerical simulations. One speech data is analyzed and it is observed that the performance of the proposed estimators is quite satisfactory.

Key Words and Phrases: Strong consistency, frequencies, amplitudes, asymptotic distribution.

Short Running Title: AM signal model.

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1. INTRODUCTION

In signal processing, the signal is often assumed to be stationary. In real life, many signals, like speech are non-stationary in nature. Traditionally, the parametric modeling of a non-stationary signal has been carried out using the quasi-stationary models (McAulay and Quatieri; 1986 and Isaksson, Wennberg and Zetterberg; 1981) where the signal is treated to be stationary only over a short duration of time. The usefulness of these models is restricted due to contradictory requirements for the duration of observations of the signals. On one hand, the duration must be short for the faithfulness of the model; on the other hand, the duration must be long enough to assure accurate estimation of the parameters of the model. It is well known that the time dependent ARMA model provides a general framework for parametric modeling of non-stationary signals (Grenier; 1983). There are several non-linear time series models available in the seminal book of Tong (1990). Unfortunately, these approaches are far too general and often lead to difficult problems when estimating a large number of parameters. Fortunately, by exploiting certain known properties for a particular class of signals often it is possible to find a simple model which serves the purpose of representation of signals efficiently.

One such model was introduced by Sircar and Syali (1996), named as complex Amplitude Modulated (AM) model. It was used to analyze non-stationary speech signals. They proposed certain estimation procedures and the performances were quite satisfactory. They did not study the theoretical properties of the estimators. Moreover, the model validation was also not performed. While re-analyzing the same speech data, we observe that the independent and identically distributed (*i.i.d.*) error assumptions may not be reasonable. It may be more appropriate to assume that the errors are correlated. Unfortunately in that case the estimation procedure proposed by Sircar and Syali (1996) can not be generalized and also obtaining the theoretical properties of these estimators will not be a trivial task. To make the model more general and also at the same time analytically tractable, we assume that the errors are from a stationary distribution.

The main aim of this paper is to define the AM signal model in presence of an additive stationary noise. We propose two estimators. It is observed that both estimators are consistent and we obtain the asymptotic distributions of both the estimators. The asymptotic distribution can be used to construct error bounds, without which the point estimators do not have much value in practice. It is observed that the two estimators are asymptotically equivalent. The small sample performances of the two estimators are compared using numerical simulations. We also analyze a speech data using the proposed method and the performance is quite satisfactory.

The rest of the paper is organized as follows. In section 2, we give the description of the different model assumptions and provide different estimation procedures. The theoretical properties are derived in section 3. A speech data is analyzed in section 4 and finally we conclude the paper in section 5.

2. MODEL DESCRIPTION AND ESTIMATION PROCEDURES

The discrete-time complex random process $y(t)$ consisting of M single-tone AM signals is given by

$$y(t) = \sum_{k=1}^{M} A_k \left[1 + \mu_k e^{i\nu_k t}\right] e^{i\omega_k t} + X(t); \quad t = 1, \ldots, N, \tag{2.1}$$

where A_k is the carrier amplitude of constituent signal, μ_k is the modulation index, ω_k is the carrier angular frequency, ν_k is the modulating angular frequency and $i = \sqrt{-1}$. For physical interpretation of the different parameters see Sircar and Syali (1996). The following assumptions are made on the model parameters;

Assumption 1 $A_k \neq 0$, $\mu_k \neq 0$ and they are bounded and also $0 < \nu_k < \pi$, $0 < \omega_k < \pi$ for all k. Moreover

$$\omega_1 < \omega_1 + \nu_1 < \omega_2 < \omega_2 + \nu_2 < \cdots < \omega_M < \omega_M + \nu_M. \tag{2.2}$$

The additive error $X(t)$ is a stationary sequence and it satisfies assumption 2.

Assumption 2: $X(t)$ has the following representation

$$X(t) = \sum_{k=-\infty}^{\infty} a(k)e(t-k),$$

where $e(t)$'s are *i.i.d.* complex valued random variables with mean zero and variance σ^2 for both the real and imaginary parts. The real and imaginary parts of $e(t)$ are uncorrelated. $a(k)$'s are arbitrary complex-valued constants such that

$$\sum_{k=-\infty}^{\infty} |a(k)| < \infty.$$

The real and imaginary parts of $a(k)$ will be denoted as $a_R(k)$ and $a_I(k)$ and of $e(t)$ as $e_R(t)$ and $e_I(t)$ respectively. We assume M is known. In this paper we mainly consider the estimation of the unknown parameters A_k, μ_k, ν_k and ω_k and study their properties.

We mainly consider two estimators. The first one is the least squares estimators (LSEs), which can be obtained by minimizing

$$Q(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\omega}) = \sum_{t=1}^{N} \left| y(t) - \sum_{k=1}^{M} A_k(1 + \mu_k e^{i\nu_k t})e^{i\omega_k t} \right|^2, \tag{2.3}$$

with respect to $\mathbf{A} = (A_1, \ldots, A_M)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_M)$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_M)$ and subject to the restriction (2.2). We will denote them as $\hat{\mathbf{A}} = (\hat{A}_1, \ldots, \hat{A}_M)$, $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_M)$, $\hat{\boldsymbol{\nu}} = (\hat{\nu}_1, \ldots, \hat{\nu}_M)$ and $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \ldots, \hat{\omega}_M)$ respectively. The second estimator is called the approximate least squares estimators (ALSEs) and it can be obtained by maximizing the Periodogram function, defined as follows;

$$I(\boldsymbol{\nu}, \boldsymbol{\omega}) = \sum_{k=1}^{M} \left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y(t)e^{-i\omega_k t} \right|^2 + \frac{1}{N} \left| \sum_{t=1}^{N} y(t)e^{-i(\omega_k+\nu_k)t} \right|^2 \right\} \tag{2.4}$$

under the restriction (2.2).

Let us denote the estimators as follows;

$$\tilde{\omega}_1 < \tilde{\omega}_1 + \tilde{\nu}_1 < \tilde{\omega}_2 < \tilde{\omega}_2 + \tilde{\nu}_2 < \cdots < \tilde{\omega}_M < \tilde{\omega}_M + \tilde{\nu}_M.$$

The $(\tilde{\omega}_k, \tilde{\nu}_k)$ is the ALSE of (ω_k, ν_k), for $k = 1, \ldots, M$. The corresponding ALSEs of the linear parameters of A_k and μ_k can be obtained from the following equations;

$$\tilde{A}_k = \frac{1}{N}\sum_{t=1}^{N} y(t) e^{-i\tilde{\omega}_k t}, \qquad \tilde{A}_k \tilde{\mu}_k = \frac{1}{N}\sum_{t=1}^{N} y(t) e^{-i(\tilde{\omega}_k + \tilde{\nu}_k)t}. \tag{2.5}$$

In the next section we consider the estimates of the parameters and study their properties. Note that although maximization of (2.4) is a $2M$ dimensional maximization problem, it can be performed sequentially, *i.e.* the $2M$ dimensional maximization problem can be reduced to $2M$, one dimensional maximization problems. The main idea of using the ALSEs goes back to Walker (1971) and Hannan (1971). Along the same line as Walker (1971) it can be shown by expanding (2.3) that the LSEs and the ALSEs are asymptotically equivalent. It indicates that the ALSEs also can be used as an alternative to the LSEs.

3. THEORETICAL RESULTS

In this section we mainly consider the asymptotic properties of the LSEs and the ALSEs. We state the main results here, proofs of all the results are provided in the appendix. It may be mentioned that the model (2.1) does not satisfy the standard sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991) for the LSEs to be consistent. Therefore, although the least squares method usually provides satisfactory performance, the complexity of the model makes it unclear, how good the LSEs will be in the present situation. It may be mentioned that when the modulation index $\mu_k = 0$ for all k, then the model (2.1) coincides with the sum of complex exponential models. The theoretical properties of the LSEs of the complex exponential model were discussed by Bai *et al.* (1991), Rao and Zhao (1993) and Kundu and Mitra (1999) in great details when the errors are *i.i.d.* random variables. For brevity, first we consider $M = 1$ in (2.1), *i.e.* we have the following model;

$$y(t) = A(1 + \mu e^{i\nu t})e^{i\omega t} + X(t). \tag{3.1}$$

We use the following notation. A_R and A_I denote the real and imaginary parts of A, similarly μ_R and μ_I are defined, $\boldsymbol{\theta} = (A_R, A_I, \mu_R, \mu_I, \nu, \omega)$. The LSE and the ALSE of $\boldsymbol{\theta}$ will be denoted by $\hat{\boldsymbol{\theta}} = (\hat{A}_R, \hat{A}_I, \hat{\mu}_R, \hat{\mu}_I, \hat{\nu}, \hat{\omega})$ and $\tilde{\boldsymbol{\theta}} = (\tilde{A}_R, \tilde{A}_I, \tilde{\mu}_R, \tilde{\mu}_I, \tilde{\nu}, \tilde{\omega})$ respectively. For model (3.1), the assumption 1 is equivalent to the following assumption $1'$.

Assumption $1'$: $A \neq 0$ and $\mu \neq 0$ are bounded and $\nu, \omega \in (0, \pi)$.

We have the following results for model (3.1).

Theorem 1: Under assumptions $1'$ and 2, $\hat{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}$.

Theorem 2: Under assumptions $1'$ and 2, $\tilde{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}$.

Theorem 3: Under assumptions $1'$ and 2,

$$\{N^{\frac{1}{2}}(\hat{A}_R - A_R), N^{\frac{1}{2}}(\hat{A}_I - A_I), N^{\frac{1}{2}}(\hat{\mu}_R - \mu_R), N^{\frac{1}{2}}(\hat{\mu}_I - \mu_I), N^{\frac{3}{2}}(\hat{\nu} - \nu), N^{\frac{3}{2}}(\hat{\omega} - \omega)\}$$

converges to a 6-variate normal distribution with mean vector $\mathbf{0}$ and the dispersion matrix

$$\sigma^2 \boldsymbol{\Sigma}^{-1}(c_1 \boldsymbol{\Sigma}_1 + c_2 \boldsymbol{\Sigma}_2)\boldsymbol{\Sigma}^{-1},$$

where

$$c_1 = \left| \sum_{k=-\infty}^{\infty} a(k) e^{-i\omega k} \right|^2 \quad \text{and} \quad c_2 = \left| \sum_{k=-\infty}^{\infty} a(k) e^{-ik(\omega+\nu)} \right|^2.$$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{A_I}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{A_R}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{A_I}{2} & \frac{A_R}{2} & 0 & 0 & 0 & \frac{1}{3}|A|^2 \end{bmatrix},$$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} |\mu|^2 & 0 & Re(\bar{\mu}A) & -Im(\bar{\mu}A) & -\frac{A_I|\mu|^2}{2} & -\frac{A_I|\mu|^2}{2} \\ 0 & |\mu|^2 & Im(\bar{\mu}A) & Re(\bar{\mu}A) & \frac{A_R|\mu|^2}{2} & \frac{A_R|\mu|^2}{2} \\ Re(\bar{\mu}A) & Im(\bar{\mu}A) & |A|^2 & 0 & -\mu_I\frac{|A|^2}{2} & -\mu_I\frac{|A|^2}{2} \\ -Im(\bar{\mu}A) & Re(\bar{\mu}A) & 0 & |A|^2 & \mu_R\frac{|A|^2}{2} & \mu_R\frac{|A|^2}{2} \\ -\frac{A_I|\mu|^2}{2} & \frac{A_R|\mu|^2}{2} & -\mu_I\frac{|A|^2}{2} & \mu_R\frac{|A|^2}{2} & \frac{1}{3}|\mu|^2|A|^2 & \frac{1}{3}|\mu|^2|A|^2 \\ -\frac{A_I|\mu|^2}{2} & \frac{A_R|\mu|^2}{2} & -\mu_I\frac{|A|^2}{2} & \mu_R\frac{|A|^2}{2} & \frac{1}{3}|\mu|^2|A|^2 & \frac{1}{3}|\mu|^2|A|^2 \end{bmatrix}.$$

Here $\bar{\mu}$ denotes the complex conjugate of μ and

$$\mathbf{\Sigma} = \mathbf{\Sigma}_1 + \mathbf{\Sigma}_2.$$

The matrix $\mathbf{\Sigma}^{-1} = \sigma^{mn}, m, n = 1, \dots 6$ has the following elements.

$$\sigma^{11} = 1 + \frac{3A_I^2}{|A|^2}, \quad \sigma^{12} = \sigma^{21} = \frac{-3A_I A_R}{|A|^2}, \quad \sigma^{13} = \sigma^{31} = \frac{-Re(\bar{\mu}A) - 3\mu_I A_I}{|A|^2},$$

$$\sigma^{14} = \sigma^{41} = \frac{Im(\bar{\mu}A) + 3\mu_R A_I}{|A|^2}, \quad \sigma^{15} = \sigma^{51} = \frac{-6A_I}{|A|^2}, \quad \sigma^{16} = \sigma^{61} = \frac{6A_I}{|A|^2},$$

$$\sigma^{22} = 1 + \frac{3A_R^2}{|A|^2}, \quad \sigma^{23} = \sigma^{32} = \frac{-Im(\bar{\mu}A) + 3A_R\mu_I}{|A|^2}, \quad \sigma^{24} = \sigma^{42} = \frac{-Re(\bar{\mu}A) - 3A_R\mu_R}{|A|^2},$$

$$\sigma^{25} = \sigma^{52} = \frac{6A_R}{|A|^2}, \quad \sigma^{26} = \sigma^{62} = \frac{-6A_R}{|A|^2}, \quad \sigma^{33} = \frac{(1+|\mu|^2)}{|A|^2}(1 + \frac{3\mu_I^2}{|\mu|^2}),$$

$$\sigma^{34} = \sigma^{43} = -\frac{3\mu_R\mu_I(1+|\mu|^2)}{|A|^2|\mu|^2}, \quad \sigma^{35} = \sigma^{53} = \frac{6\mu_I(1+|\mu|^2)}{|A|^2|\mu|^2}, \quad \sigma^{36} = \sigma^{63} = \frac{-6\mu_I}{|A|^2},$$

$$\sigma^{44} = \frac{(1+|\mu|^2)}{|A|^2}(1 + \frac{3\mu_R^2}{|\mu|^2}), \quad \sigma^{45} = \sigma^{54} = -\frac{6\mu_R(1+|\mu|^2)}{|A|^2|\mu|^2}, \quad \sigma^{46} = \sigma^{64} = \frac{6\mu_R}{|A|^2},$$

$$\sigma^{55} = \frac{12(1+|\mu|^2)}{|A|^2|\mu|^2}, \quad \sigma^{56} = \sigma^{65} = \frac{-12}{|A|^2}, \quad \sigma^{66} = \frac{12}{|A|^2}.$$

Theorem 4: Under assumptions $1'$ and 2, the ALSEs have the same asymptotic distributions as the LSEs.

Theorems 1 and 2 indicate that the LSEs and the ALSEs are reasonable estimators of the unknown parameters. Strong consistency ensures that when the sample size is large, then both the LSEs and the ALSEs should be quite close to the corresponding true parameter values. How good or how close the estimators will be can be found from Theorems 3 and 4. Theorems 3 and 4 indicate that both the LSEs and the ALSEs have the same rate of convergence. It is clear that the rate of convergence of the frequencies is higher compared to the rate of convergence of the amplitudes or modulation indexes. Therefore, for a given sample size the frequency estimators will be much better compared to the amplitude and modulation index estimators.

For general M, the results can be easily extended under the assumption that all the frequencies are distinct. Theorems 1 and 2 are still valid replacing $\boldsymbol{\theta}$ by the entire set of parameters. Theorems 3 and 4 also can be extended. For general M, the asymptotic dispersion matrix will be a $6M \times 6M$ matrix, with block diagonal form of M blocks each of size 6×6. Other blocks have only zero entries. Each diagonal block has the same form as defined in Theorem 3.

4. NUMERICAL RESULTS

In this section, first we compare the performances of the LSEs and the ALSEs for finite sample by computer simulations and then we analyze one non-stationary real speech data. All the computations are performed at the Indian Institute of Technology Kanpur using FORTRAN-77 on the Silicon Graphics machine and they can be obtained from the authors. For computer simulations we use the random deviate generator from Press *et al.* (1993).

Example 1:

First we consider the data generated from the model (3.1), with $A = A_R + iA_I = 5 + i1.0$, $\mu = \mu_R + i\mu_I = .5 + i1.0$, $\nu = .50286$, $\omega = 2.01143$. Here $X(t)$ is a stationary sequence which has the following form;

$$X(t) = a_0 e(t) + a_1 e(t-1),$$

where $a_0 = 0.2 + i0.4$ and $a_1 = 0.3 + i0.5$. The real and the imaginary parts of $e(t)$ are independent and normally distributed each with mean zero and variance one. $e(t)$'s are *i.i.d.* The data is generated at 50 points. We compute the LSEs and the ALSEs of the unknown parameters and also compute the 95% confidence bound for each parameter. The process is repeated 5000 times and we compute the average estimates, average biases, variances, the average confidence lengths and the coverage percentages over five thousand replications for all the unknown parameters. The results are reported in Tables 1 and 2. For comparison purposes we also report the asymptotic variances and the expected confidence lengths, as obtained from Theorems 3 and 4. Note that to compute the confidence intervals of the different parameters, we need to estimate σ^2, c_1 and c_2. Although, we can not estimate σ^2, c_1 and c_2 separately, but it is possible to estimate $\sigma^2 c_1$ and $\sigma^2 c_2$, which are needed to compute the confidence bands. By straight forward and lengthy calculations, it can be shown that

$$\sigma^2 c_1 = E\left(\frac{1}{2N}\left|\sum_{t=1}^{N} X(t)e^{-i\omega t}\right|^2\right), \qquad \sigma^2 c_2 = E\left(\frac{1}{2N}\left|\sum_{t=1}^{N} X(t)e^{-i(\omega+\nu)t}\right|^2\right).$$

Since $\sigma^2 c_1$ and $\sigma^2 c_2$ are the expected values of the Periodogram function at ω and $(\omega + \nu)$ respectively, we estimate $\sigma^2 c_1$ and $\sigma^2 c_2$ by averaging the Periodogram function over a window (-L, L) across the point estimates of ω and $(\omega + \nu)$. This estimator has been proposed by Hannan (1970, page 470) in a different context but it was exploited later on by Quinn and Thomson (1991). This estimator works reasonably well. We present the results for LSEs and ALSEs in Tables 1 and 2 respectively.

Some of the points are quite clear from Tables 1 and 2. Both the LSEs and ALSEs work reasonably well even for small samples but the biases and the variances of the ALSEs are slightly larger than the corresponding biases and variances of the LSEs. As the theory suggests, it is observed that for both the LSEs and ALSEs the frequency estimates are much better than the amplitude and modulation index estimates in terms of the biases and variances. The variances of the LSEs are quite close to the asymptotic variances, but the same thing can not be said for the ALSEs. From the results, it is clear that the estimation of $\sigma^2 c_1$ and $\sigma^2 c_2$ are also quite good at least when the LSEs are used. It reflects in the average confidence length calculations and in the coverage percentages. For the LSEs the average confidence lengths are closer to the expected confidence lengths and also the coverage percentages are quite close to the nominal level. Interestingly most of the times the average confidence lengths based on the ALSEs are larger than the corresponding confidence lengths based on the LSEs but the coverage probabilities for the LSEs are higher than the ALSEs. Note that the expected confidence lengths are based on the true values of $\sigma^2 c_1$ and $\sigma^2 c_2$. It may be mentioned that computationally LSEs are more involved than the ALSEs. Comparing all the points we recommend to use the LSEs to estimate the unknown parameters for the AM model if the sample size is not very large even if it is computationally more expensive. If the sample size is large we can use the ALSEs. For better performance, when the sample size is large, LSE can be computed using ALSE as an initial estimate.

For illustration purpose, we consider one particular realization of the model presented in example 1. The real and imaginary parts of the data are plotted in Figure 1 and Figure 2 respectively. The Periodogram function (2.4) of the data is provided in Figure 3. The Periodogram function clearly indicates that $M = 1$. Assuming $M = 1$, from the Periodogram function the initial estimates of ω and ν are 2.0097 and 0.5063 respectively. Using these initial estimates, the LSEs of A_R, A_I, μ_R, μ_I, ω and ν become 5.04679, 1.02681, 0.49275, 1.00665, 2.01139 and 0.50297 respectively. The real and imaginary parts of the estimated signal are plotted in Figures 4 and 5 respectively. The confidence intervals of A_R, A_I, μ_R, μ_I, ω and ν are (4.90734, 5.18623), (0.76715, 1.28647), (0.43581, 0.54968), (0.96795, 1.04534), (2.00961, 2.01316) and (0.50107, 0.50487) respectively.

Example 2:

In this example we re-analyze the sustained vowel sound of '*uuu*'. It was analyzed by Sircar and Syali (1996) also. A total of 512 signal values sampled at 10kHz frequency is available. Sircar and Syali (1996) used the model (2.1) while analyzing the data assuming that $X(t)$'s are *i.i.d.* random variables. They did not study the residuals to verify the model assumptions. While re-analyzing the data, we observe that the residuals are correlated, therefore the assumptions of *i.i.d.* errors may not be reasonable.

The plot of the original data is provided in Figure 6 and the plot of the Periodogram function is provided in Figure 7. The Periodogram function clearly indicates that $M = 2$, therefore, the model is of the form;

$$y(t) = A_1(1 + \mu_1 e^{i\nu_1 t})e^{i\omega_1 t} + A_2(1 + \mu_2 e^{i\nu_2 t})e^{i\omega_2 t} + X(t). \tag{4.1}$$

We obtain the estimates of the different parameters and also the 95% confidence intervals for all the parameters. They are presented in Table 3. Now we obtain the predicted value of $y(t)$ as

$$\hat{y}(t) = \hat{A}_1(1 + \hat{\mu}_1 e^{i\hat{\nu}_1 t})e^{i\hat{\omega}_1 t} + \hat{A}_2(1 + \hat{\mu}_2 e^{i\hat{\nu}_2 t})e^{i\hat{\omega}_2 t} \tag{4.2}$$

and the estimated error as

$$\hat{X}(t) = y(t) - \hat{y}(t). \tag{4.3}$$

The $\hat{y}(t)$'s are plotted in Figure 8 and the residuals (4.3) are plotted in Figure 9. The predicted values match quite well with the true values. We want to test whether the residuals are independent or not. We use the run test (Draper and Smith; 1981) and z = -11.892 confirms that the residuals are dependent. The autocorrelation function and the partial autocorrelation function suggest that the residuals should be an AR(3) process and the parameter can be estimated as

$$X(t) = 1.0904X(t-1) - 0.5067X(t-2) + 0.1065X(t-3) + e(t). \quad (4.4)$$

Performing the run test on $\hat{e}(t)$, we obtain z = -1.32973. So it does not reject the independent assumptions on $e(t)$'s. Since all the roots of the polynomial equation;

$$z^3 - 1.0904z^2 + 0.5067z - 0.1065 = 0$$

are less than one in absolute value, therefore, $X(t)$ can be modeled as a stationary AR(3) process, which satisfies assumption 2 and clearly it does not satisfy the error assumption of Sircar and Syali (1996). From this data analysis it is clear that the AM model can be used quite effectively for modeling sustained vowel sound 'uuu' with the proper error assumptions. It may be mentioned that without the proper error assumptions the confidence intervals of the unknown parameters will not be correct.

5. CONCLUSIONS

In this paper we consider the AM signal model originally proposed by Sircar and Syali (1996) to analyze certain non-stationary speech data. We assume the errors are from a stationary distribution. It is observed that the usual LSEs and the ALSEs work quite well even when the errors are correlated. The estimated signal matches quite well with the original one. We have the asymptotic distribution of the different estimators and it was used to construct the asymptotic confidence intervals of the different unknown parameters. Note that we have used the Periodogram function to estimate M but no formal result is obtained.

It seems some of the model selection technique like information theoretic criteria or cross validation approach can be used to estimate M. Further work is needed in that direction.

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Appendix

In the Appendix we denote $\boldsymbol{\theta}^0 = (A_R^0, A_I^0, \mu_R^0, \mu_I^0, \nu^0, \omega^0)$ as the true parameter value of $\boldsymbol{\theta} = (A_R, A_I, \mu_R, \mu_I, \nu, \omega)$. To prove the different results we need the following lemmas.

Lemma 1: Let $U(t)$ be a real valued stationary sequence such that

$$U(t) = \sum_{k=-\infty}^{\infty} \alpha(k)\epsilon(t-k), \qquad (A.1)$$

where $\epsilon(t)$'s are $i.i.d.$ real valued random variables with mean zero and finite variance σ^2 and $\sum_{k=-\infty}^{\infty} |\alpha(k)| < \infty$, then

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N} \sum_{t=1}^{N} U(t)\cos(\theta t) \right| = 0 \qquad a.s.,$$

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N} \sum_{t=1}^{N} U(t)\sin(\theta t) \right| = 0 \qquad a.s.$$

Proof of Lemma 1: See Kundu (1997). The lemma also follows from Theorem 4.5.1 in Brillinger (1981; page 98).

Note that using Lemma 1, the following results can be obtained along the same line.

Lemma 2: If $X(t)$ satisfies assumption 1, then

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N^{L+1}} \sum_{t=1}^{N} U(t)t^L\cos(\theta t) \right| = 0 \qquad a.s.,$$

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N^{L+1}} \sum_{t=1}^{N} U(t)t^L\sin(\theta t) \right| = 0 \qquad a.s.,$$

for $L = 1, 2, \ldots$.

Lemma 3: Define $S_c = \{\boldsymbol{\theta} : |\boldsymbol{\theta} - \boldsymbol{\theta}^0| > c\}$, then $\hat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}^0$, obtained by minimizing (2.3) (when $M = 1$), is a strongly consistent estimator of $\boldsymbol{\theta}^0$ provided

$$\underline{\lim} \inf_{\boldsymbol{\theta}\in S_c} \frac{1}{N} \left\{ Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0) \right\} > 0 \qquad a.s.$$

for all $c > 0$.

Proof of Lemma 3: The proof is quite simple and can be obtained along the same line as Wu (1981).

Proof of Theorem 1: Let us write $S_c = A_{Rc} \cup A_{Ic} \cup M_{Rc} \cup M_{Ic} \cup N_c \cup W_c$, where

$$\begin{aligned} A_{Rc} &= \left\{\boldsymbol{\theta} : |A_R - A_R^0| > c\right\}, \qquad A_{Ic} = \left\{\boldsymbol{\theta} : |A_I - A_I^0| > c\right\}, \\ M_{Rc} &= \left\{\boldsymbol{\theta} : |\mu_R - \mu_R^0| > c\right\}, \qquad M_{Ic} = \left\{\boldsymbol{\theta} : |\mu_R - \mu_I^0| > c\right\}, \\ N_c &= \left\{\boldsymbol{\theta} : |\nu - \nu^0| > c\right\}, \qquad W_c = \left\{\boldsymbol{\theta} : |\omega - \omega^0| > c\right\}. \end{aligned}$$

Now observe that

$$\begin{aligned} \frac{1}{N}\left\{Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right\} &= \frac{1}{N}\sum_{t=1}^{N}\left\{\left|y(t) - A(1+\mu e^{i\nu t})e^{i\omega t}\right|^2 - |X(t)|^2\right\} \\ &= \frac{1}{N}\sum_{t=1}^{N}\left|A^0(1+\mu^0 e^{i\nu^0 t})e^{i\omega^0 t} - A(1+\mu e^{i\nu t})e^{i\omega t}\right|^2 \\ &\quad + \frac{2}{N} Re\left\{\sum_{t=1}^{N} X(t)\left(\bar{A}^0(1+\bar{\mu}^0 e^{-i\nu^0 t})e^{-i\omega^0 t} - \bar{A}(1+\bar{\mu} e^{-i\nu t})e^{-i\omega t}\right)\right\} \\ &= f_N(\boldsymbol{\theta}) + g_N(\boldsymbol{\theta}). \end{aligned}$$

Let us write $X(t) = X_R(t) + iX_I(t)$, where

$$\begin{aligned} X_R(t) &= \sum_{k=-\infty}^{\infty}\left\{a_R(k)e_R(t-k) - a_I(k)e_I(t-k)\right\}, \\ X_I(t) &= \sum_{k=-\infty}^{\infty}\left\{a_R(k)e_I(t-k) + a_I(k)e_R(t-k)\right\}. \end{aligned}$$

So both $X_R(t)$ and $X_I(t)$ are in the form $U_1(t) + U_2(t)$ where $U_k(t),\ k = 1, 2$ are real-valued stationary sequence satisfying equation (A.1) stated in Lemma 1. Now using Lemma 1, we have,

$$\underline{\lim}_{N\to\infty} \inf_{\boldsymbol{\theta}} g_N(\boldsymbol{\theta}) = 0 \qquad \text{a.s.,} \tag{A.2}$$

and for any $c > 0$,

$$\underline{\lim} \inf_{\boldsymbol{\theta}\in A_{Rc}} f_N(\boldsymbol{\theta}) = \underline{\lim} \inf_{\boldsymbol{\theta}\in A_{Rc}} \frac{1}{N}\sum_{t=1}^{N}\left|A^0(1+\mu^0 e^{i\nu^0 t})e^{i\omega^0 t} - A(1+\mu e^{i\nu t})e^{i\omega t}\right|^2$$

$$= \ \underline{\lim} \inf_{|A_R - A_R^0| > c} \frac{1}{N} \sum_{t=1}^{N} |A^0 - A|^2 \ |(1 + \mu^0 e^{i\nu^0 t}) e^{i\omega^0 t}|^2$$

$$\geq \ c^2 \underline{\lim} \frac{1}{N} \sum_{t=1}^{N} |1 + \mu_0 e^{i\nu_0 t}|^2 \ \geq \ c^2(1 + |\mu^0|^2) \ > \ 0 \quad a.s.$$

Similarly it can be proved for A_{Ic}, M_{Rc}, M_{Ic}, N_c and W_c which implies that

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_c} f_N(\boldsymbol{\theta}) \ > \ 0 \ a.s. \tag{A.3}$$

So using (A.2), (A.3) and Lemma 3, the theorem follows.

To prove Theorem 2, we need the following lemmas.

Lemma 4: If $\tilde{\boldsymbol{\eta}} = (\tilde{\nu}, \tilde{\omega})$ is the ALSE of $\boldsymbol{\eta}^0 = (\nu^0, \omega^0)$ obtained by maximizing (2.4) (for $M = 1$) with respect to ν and ω then $(\tilde{\nu}, \tilde{\omega})$ is a strongly consistent estimator of (ν^0, ω^0), provided

$$\overline{\lim} \sup_{|\boldsymbol{\eta}^0 - \boldsymbol{\eta}| > \delta} \frac{1}{N} \left\{ I(\nu, \omega) - I(\nu^0, \omega^0) \right\} < 0 \qquad a.s.$$

for all $\delta > 0$.

Proof of Lemma 4: The proof is quite simple and can be obtained along the same line as Lemma 3.

Lemma 5: Under assumptions $1'$ and 2, $\tilde{\boldsymbol{\eta}} = (\tilde{\nu}, \tilde{\omega})$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

Proof of Lemma 5: Define $S_\delta = \{\boldsymbol{\eta} : |\boldsymbol{\eta} - \boldsymbol{\eta}^0| > 2\delta\} = S_\delta^\nu \cup S_\delta^\omega$, where

$$S_\delta^\nu = \left\{ \boldsymbol{\eta} : |\nu - \nu^0| > \delta \right\} \quad \text{and} \quad S_\delta^\omega = \left\{ \boldsymbol{\eta} : |\omega - \omega^0| > \delta \right\}.$$

Note that because of Lemma 1, expanding $I(\boldsymbol{\eta})$, we have

$$\overline{\lim} \sup_{S_\delta^\nu} \frac{1}{N} I(\boldsymbol{\eta}) \ = \ \overline{\lim} \sup_{S_\delta^\nu} \left[\left| \frac{1}{N} \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2 + \left| \frac{1}{N} \sum_{t=1}^{N} y(t) e^{-i(\omega+\nu)t} \right|^2 \right]$$

$$\leq \ \overline{\lim} \sup_{S_\delta^\nu} \frac{1}{N^2} \left[\left| \sum_{t=1}^{N} A^0 e^{-i(\omega-\omega^0)t} \right|^2 + \left| \sum_{t=1}^{N} A^0 \mu^0 e^{-i(\omega-\nu^0-\omega^0)t} \right|^2 \right.$$

$$+\left|\sum_{t=1}^{N} A^0 e^{-i(\omega+\nu-\omega^0)t}\right|^2 + \left|\sum_{t=1}^{N} A^0\mu^0 e^{-i(\omega+\nu-\omega^0-\nu^0)t}\right|^2\Bigg]$$

$$= \overline{\lim} \sup_{|\nu-\nu^0|>\delta} \frac{1}{N^2}\left[\left|\sum_{t=1}^{N} A^0\right|^2 + \left|\sum_{t=1}^{N} A^0\mu^0 e^{i\nu^0 t}\right|^2\right.$$

$$\left.+\left|\sum_{t=1}^{N} A^0 e^{-i\nu t}\right|^2 + \left|\sum_{t=1}^{N} A^0\mu^0 e^{-i(\nu-\nu^0)t}\right|^2\right] = |A^0|^2.$$

Similarly using Lemma 1 and expanding $I(\boldsymbol{\eta}^0)$, we have

$$\overline{\lim} \sup_{S_\delta^\nu} \left[\left|\frac{1}{N}\sum_{t=1}^{N} y(t)e^{-i\omega^0 t}\right|^2 + \left|\frac{1}{N}\sum_{t=1}^{N} y(t)e^{-i(\omega^0+\nu^0)t}\right|^2\right] = |A^0|^2 + |A^0\mu^0|^2 > 0.$$

Therefore,

$$\overline{\lim} \sup_{S_\delta^\nu} \frac{1}{N}\left\{I(\boldsymbol{\eta}) - I(\boldsymbol{\eta}^0)\right\} = -|A^0\mu^0|^2 < 0 \qquad a.s. \qquad (A.4)$$

Similarly it can be shown that

$$\overline{\lim} \sup_{S_\delta^\omega} \frac{1}{N}\left\{I(\boldsymbol{\eta}) - I(\boldsymbol{\eta}^0)\right\} = -(1+|\mu^0|^2)|A^0|^2 < 0 \qquad a.s. \qquad (A.5)$$

(A.4) and (A.5) imply that

$$\overline{\lim} \sup_{S_\delta} \frac{1}{N}\left\{I(\boldsymbol{\eta}) - I(\boldsymbol{\eta}^0)\right\} = -(1+|\mu^0|^2)|A^0|^2 < 0 \qquad a.s.$$

and so using Lemma 4, the result follows.

Lemma 6: If $\tilde{\boldsymbol{\eta}} = (\tilde{\nu}, \tilde{\omega})$ is the ALSE of $\boldsymbol{\eta}^0 = (\nu^0, \omega^0)$ of the model (2.1) (for $M = 1$), then under assumptions $1'$ and 2,

$$N(\tilde{\nu} - \nu^0) \rightarrow 0 \qquad a.s.$$

$$N(\tilde{\omega} - \omega^0) \rightarrow 0 \qquad a.s.$$

Proof of Lemma 6: Expanding $I'(\tilde{\nu}, \tilde{\omega}) = I'(\tilde{\boldsymbol{\eta}})$ around $\boldsymbol{\eta}^0$, using multivariate Taylor series expansion up to first order term

$$I'(\tilde{\boldsymbol{\eta}}) - I'(\boldsymbol{\eta}^0) = (\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)I''(\bar{\boldsymbol{\eta}}), \qquad (A.6)$$

where $\bar{\boldsymbol{\eta}}$ is a point between $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}^0$. $I'(\boldsymbol{\eta})$ and $I''(\boldsymbol{\eta})$ are the vector of first derivatives and the matrix of second derivatives of $I(\boldsymbol{\eta})$ *w.r.t.* $\boldsymbol{\eta}$ respectively. Note that $I'(\tilde{\boldsymbol{\eta}}) = 0$, so from (A.6), we have

$$\begin{aligned} (\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}^0) &= -I'(\boldsymbol{\eta}^0)\left[I''(\bar{\boldsymbol{\eta}})\right]^{-1} \\ \Rightarrow N(\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}^0) &= -\left[\frac{1}{N^2}I'(\boldsymbol{\eta}^0)\right]\left[\frac{1}{N^3}I''(\bar{\boldsymbol{\eta}})\right]^{-1}. \end{aligned}$$

Using Lemma 1 and Lemma 2, it can be shown that $\frac{1}{N^3}I''(\boldsymbol{\eta}^0) \to \Gamma$, where

$$\Gamma = \begin{pmatrix} |A^0|^2|\mu^0|^2 & |A^0|^2|\mu^0|^2 \\ |A^0|^2|\mu^0|^2 & |A^0|^2 + |A^0|^2|\mu^0|^2 \end{pmatrix},$$

which is an invertible matrix because of the assumptions $1'$. Elements of $I''(\boldsymbol{\eta})$ are continuous functions of ν and ω and $\bar{\boldsymbol{\eta}}$ is a point between $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}^0$. So using the fact that $\tilde{\boldsymbol{\eta}} \to \boldsymbol{\eta}^0$ a.s., we have

$$\lim_{N\to\infty} \frac{1}{N^3}I''(\bar{\boldsymbol{\eta}}) = \lim_{N\to\infty} \frac{1}{N^3}I''(\boldsymbol{\eta}^0) = \Gamma.$$

Also using Lemma 1, it can be shown that $\frac{1}{N^2}I'(\boldsymbol{\eta}^0) \to 0$ *a.s.* Hence

$$N(\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}^0) \to 0 \qquad a.s.$$

which implies that $N(\tilde{\nu} - \nu^0) \to 0$ a.s. and $N(\tilde{\omega} - \omega^0) \to 0$ a.s.

Lemma 7: $\tilde{A}$ and $\tilde{\mu}$, as given in (2.5) (for $M = 1$) are strongly consistent estimators of A^0 and μ^0.

Proof of Lemma 7: Let us denote $y_R(t)$, $y_I(t)$ as the real and imaginary parts of $y(t)$. Therefore,

$$\tilde{A} = \frac{1}{N}\left[\sum_{t=1}^{N}\{y_R(t)\cos(\tilde{\omega}t) + y_I(t)\sin(\tilde{\omega}t)\}\right] + \frac{i}{N}\left[\sum_{t=1}^{N}\{y_I(t)\cos(\tilde{\omega}t) - y_R(t)\sin(\tilde{\omega}t)\}\right].$$

Expanding $\cos(\tilde{\omega}t)$, $\sin(\tilde{\omega}t)$ by Taylor series around ω^0 and using Lemmas 1, 2 and 6, we get

$$\tilde{A} \to {A_R}^0 + i{A_I}^0 = A^0 \quad a.s. \qquad \text{and} \qquad \tilde{A}\tilde{\mu} \to A^0\mu^0 \quad a.s.$$

which proves the lemma.

Proof of Theorem 2: Combining Lemmas 5 and 7, the result follows immediately.

Proof of Theorem 3: Let us denote

$$Q'(\boldsymbol{\theta}) = \left[\frac{\partial Q(\boldsymbol{\theta})}{\partial A_R}, \frac{\partial Q(\boldsymbol{\theta})}{\partial A_I}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \mu_R}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \mu_I}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \nu}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \omega}\right]$$

and $Q''(\boldsymbol{\theta})$ denotes the corresponding 6×6 double derivative matrix of $Q(\boldsymbol{\theta})$. Now expanding $Q'(\hat{\boldsymbol{\theta}})$ around $\boldsymbol{\theta}^0$ by multivariate Taylor series up to the first order term, we get

$$Q'(\hat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)Q''(\bar{\boldsymbol{\theta}}), \tag{A.7}$$

where $\bar{\boldsymbol{\theta}}$ is a point between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Since $Q'(\hat{\boldsymbol{\theta}}) = 0$, (A.7) implies

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) = -Q'(\boldsymbol{\theta}^0)[Q''(\bar{\boldsymbol{\theta}})]^{-1}.$$

The main idea to prove that $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)$ converges to a normal distribution is as follows. Consider the following 6×6 diagonal matrix $\mathbf{D}$;

$$\mathbf{D} = \text{diag}\left\{N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{3}{2}}\right\}.$$

Therefore

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} = -Q'(\boldsymbol{\theta}^0)\mathbf{D}[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}]^{-1}.$$

It can be shown by the straight forward but lengthy calculations that

$$\lim_{N\to\infty}[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}] = \lim_{N\to\infty}[\mathbf{D}Q''(\boldsymbol{\theta}^0)\mathbf{D}] = 2\boldsymbol{\Sigma}, \tag{A.8}$$

where $\boldsymbol{\Sigma}$ is same as defined in the statement of Theorem 3. Using the central limit theorem of stochastic processes (Fuller; 1976, page 251), it can be shown that

$$Q'(\boldsymbol{\theta}^0)\mathbf{D} \to N_6\left[\mathbf{0}, 4\sigma^2(c_1\boldsymbol{\Sigma}_1 + c_2\boldsymbol{\Sigma}_2)\right], \tag{A.9}$$

where c_1, c_2, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are same as defined in the statement of Theorem 3. Now combining (A.8) and (A.9), the result follows immediately.

Proof of Theorem 4: It can be shown similarly as Hannan (1971) or Walker (1971) that

$$\hat{A}_R - \tilde{A}_R = O_p(N^{-1}), \quad \hat{A}_I - \tilde{A}_I = O_p(N^{-1}), \quad \hat{\mu}_R - \tilde{\mu}_R = O_p(N^{-1}),$$

$$\hat{\mu}_I - \tilde{\mu}_I = O_p(N^{-1}), \quad \hat{\omega} - \tilde{\omega} = O_p(N^{-2}), \quad \hat{\nu} - \tilde{\nu} = O_p(N^{-2}). \tag{A.10}$$

Here the terms $O_p(N^{-1})$ and $O_p(N^{-2})$ indicate that they converge to zero in probability and also $NO_p(N^{-1})$ and $N^2O_p(N^{-2})$ are both bounded in probability as $N \to \infty$. Therefore, using Theorem 3 and (A.10) the result follows.

Table 1: **The average LSEs, biases, variances, confidence lengths and coverage probabilities of different parameters.**

Parameter	Average LSE (Bias)	Variance (Asymp. Var.)	Average Conf. Length (Expected Conf. Length)	Cov. Prob (Nominal Level)
A_R	4.99960 (-0.00040)	5.2096e-3 (7.9057e-3)	0.29360 (0.34854)	0.94 (0.95)
A_I	1.00121 (0.00121)	1.2422e-2 (2.7534e-2)	0.54750 (0.65046)	0.95 (0.95)
μ_R	0.50004 (0.00004)	6.20334e-4 (1.5312e-3)	0.12819 (0.15339)	0.96 (0.95)
μ_I	0.99958 (-0.00042)	4.9372e-4 (7.2058e-4)	8.7987e-2 (0.10523)	0.94 (0.95)
ω	2.01143 (0.00000)	6.7122e-07 (1.3085E-06)	3.7752e-3 (4.4841e-3)	0.95 (0.95)
ν	0.50285 (-0.00001)	1.0174e-06 (1.7294e-06)	4.3085e-3 (5.1551e-3)	0.95 (0.95)

Table 2: **The average ALSEs, biases, variances, confidence lengths and coverage probabilities of different parameters.**

Parameter	Average LSE (Bias)	Variance (Asymp. Var.)	Average Conf. Length (Expected Conf. Length)	Cov. Prob (Nominal Level)
A_R	4.99958 (-0.00042)	5.8642e-3 (7.9057e-3)	0.33484 (0.34854)	0.96 (0.95)
A_I	0.97714 (-0.02286)	2.4587e-2 (2.7534e-2)	0.62577 (0.65046)	0.94 (0.95)
μ_R	0.54447 (0.04447)	1.2956e-3 (1.5312e-3)	0.15338 (0.15339)	0.80 (0.95)
μ_I	0.97679 (-0.02321)	6.7943e-4 (7.2058e-4)	0.11073 (0.10523)	0.88 (0.95)
ω	2.01145 (0.00002)	1.1478e-6 (1.3085e-06)	5.2387e-3 (4.4841e-3)	0.75 (0.95)
ν	0.50463 (0.00177)	1.4023e-6 (1.7294e-6)	4.3154e-3 (5.1551e-3)	0.95 (0.95)

Table 3: **The least squares estimates and the confidence lengths of the different parameters of the sustained vowel sound 'uuu'.**

Parameter	Estimate	Lower Bound	Upper Bound
A_{R_1}	-521.0508	-553.7765	-488.3251
A_{I_1}	378.3746	338.8855	417.8636
μ_{R_1}	-1.60671	-1.79414	-1.41928
μ_{I_1}	2.09645	1.93629	2.25660
ω_1	0.11352	0.11328	0.11376
ν_1	0.11457	0.11429	0.11485
A_{R_2}	-334.4524	-369.4416	-299.4631
A_{I_2}	290.0626	251.9665	328.1586
μ_{R_2}	0.75906	0.70911	0.80901
μ_{I_2}	-0.28176	-0.36335	-0.20018
ω_2	0.34336	0.34301	0.34372
ν_2	0.11462	0.11426	0.11498