A Note on Estimating the Fundamental Frequency of a Periodic Function

Debasis Kundu † & Swagata Nandi‡

Abstract

In this note we consider the estimation of the fundamental frequency of a periodic function. It is observed that the simple least squares estimators can be used quite effectively to estimate the unknown parameters. The asymptotic distribution of the least squares estimators is provided. Some simulation results are presented and finally we analyze two real life data sets using different methods.

Key Words and Phrases: Fundamental frequency, least squares estimators, spectral density function

Corresponding Author: † Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur - 208016, India, e-mail: kundu@iitk.ac.in

Address ‡Institut für Angewandte Mathematik Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany, e-mail: nandi@statlab.uni-heidelberg.de

1 Introduction

In this note we consider estimating the parameters of the following fundamental frequency model (FFM):

$$y(n) = \mu + \sum_{j=1}^{p} \rho_j \cos(nj\lambda - \phi_j) + X(n); \quad \text{for} \quad n = 1, \ldots, N. \tag{1.1}$$

Here μ is the unknown mean, ρ_j (> 0), $j = 1, \ldots, p$ are unknown amplitudes, $0 < \lambda < \frac{\pi}{p}$ is the fundamental frequency and $-\pi < \phi_j < \pi$, $j = 1, \ldots, p$ are unknown phase components. $\{X(n)\}$ is a sequence of error random variables with mean zero and which satisfies the following assumption 1.

Assumption 1: $X(n)$ has the following representation;

$$X(n) = \sum_{k=-\infty}^{\infty} a(k)e(n-k),$$

where $e(k)$ is independent and identically distributed random variable with zero mean and finite variance σ^2. The constant $a(k)$ satisfies

$$\sum_{k=-\infty}^{\infty} |a(k)| < \infty.$$

The problem is to estimate the unknown parameters, assuming 'p', the number of components, is known.

This is an important problem and it was originally introduced by [1]. It is a particular case of the multiple frequency model (MFM), namely

$$y(n) = \mu + \sum_{j=1}^{p} \rho_j \cos(n\lambda_j - \phi_j) + X(n). \tag{1.2}$$

Assuming p is known the model (1.1) has less number of parameters than (1.2). Quinn and Thomson [4] proposed the following estimation procedure (from now on we call the corresponding estimators as QTEs) of different parameters of the model (1.1). The QTE of λ, say $\tilde{\lambda}$, can be obtained by maximizing

$$Q(\lambda) = \sum_{j=1}^{p} \frac{1}{f(j\lambda)} \left| \frac{1}{N} \sum_{n=1}^{N} y(n) e^{inj\lambda} \right|^2, \tag{1.3}$$

where $f(.)$ is the spectral density function of $X(n)$ and it is assumed that $f(.)$ is known and strictly positive on $[0, \pi]$. If it is unknown, which is usually the case, then $f(j\lambda)$ in (1.3) is replaced by its estimate. The QTEs of the other parameters are as follows;

$$\tilde{\mu} = \sum_{n=1}^{N} y(n), \quad \tilde{\rho}_j = \frac{2}{N} \left| \sum_{n=1}^{N} y(n) e^{inj\tilde{\lambda}} \right|, \quad \tilde{\phi}_j = arg \left\{ \frac{1}{N} \sum_{n=1}^{N} y(n) e^{inj\tilde{\lambda}} \right\}. \tag{1.4}$$

Quinn and Thomson [4] also obtained the asymptotic properties of the QTEs. There are mainly two problems involved in using the QTEs. First of all the QTEs can only be obtained if the spectral density function $f(.)$ is strictly positive. For example if $X(t) = e(t) + e(t-3)$ then QTEs can not be obtained for $\lambda = \pi/3$. The second problem is regarding the asymptotic properties of the estimators. The asymptotic properties of the estimators are obtained assuming the spectral density function is known, therefore they do not work well for small sample sizes when the spectral density function of the error distribution is not known. When the spectral density function is not known, it needs to be estimated at each iteration step for maximizing (1.3), which is quite expensive computationally. Our simulation results indicate that the coverage percentages of the unknown parameters obtained by using the asymptotic properties of the QTEs are far below the nominal level in many situations.

In this note we suggest to use the least square estimators (LSEs) and it is observed that the performances of the LSEs are quite satisfactory in most cases. We provide the asymptotic distribution of the LSEs. Some simulation results are provided to compare the performances of the two estimators. We also compare the performances of both estimators with the LSEs of the corresponding MFM (1.2). Two real sound data, 'uuu' and 'ahh', are analyzed using different methods.

2 Least Squares Estimators

The LSEs of the unknown parameters of the FFM (1.1), can be obtained by minimizing

$$R(\mu, \boldsymbol{\rho}, \boldsymbol{\phi}, \lambda) = \frac{1}{N} \sum_{n=1}^{N} \left[y(n) - \mu - \sum_{j=1}^{p} \rho_j \cos(nj\lambda - \phi_j) \right]^2, \tag{2.1}$$

with respect to the unknown parameters. Note that there are $2p + 2$ unknown parameters, but λ is the only non-linear parameter. μ and ρ_j are linear parameters and ϕ_j are function of linear parameters. Therefore, using the separable regression technique, first we estimate λ (minimizing a one dimensional function) and then assuming λ is known we can estimate easily the other parameters μ, ρ_j and ϕ_j, $j = 1, \ldots, p$ by linear regression technique. Now we provide the asymptotic properties of the LSEs. The results can be obtained exactly along the same line as [2]. It involves routine calculations. For details one is referred to [3]. The details are not provided here and they can be obtained on requests from the authors. Let us denote the LSEs of μ, ρ_j, ϕ_j and λ as $\hat{\mu}$, $\hat{\rho}_j$, $\hat{\phi}_j$ and $\hat{\lambda}$ respectively. It can be shown that the LSEs are consistent and the vector

$$\sqrt{N} \left[(\hat{\mu} - \mu), (\hat{\rho}_1 - \rho_1), \ldots, (\hat{\rho}_p - \rho_p), (\hat{\phi}_1 - \phi_1), \ldots, (\hat{\phi}_p - \phi_p), N(\hat{\lambda} - \lambda) \right] \to N_{2p+2}(\mathbf{0}, 2\sigma^2 \mathbf{V})$$

as N tends to infinity. The matrix $\mathbf{V}$ is as follows:

$$\mathbf{V} = \begin{bmatrix} \frac{1}{2} \left(\sum_{k=-\infty}^{\infty} a(k) \right)^2 & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\mathbf{D}_\rho^{-1} + \frac{3\delta_G \mathbf{L}\mathbf{L}^T}{(\sum_{j=1}^{p} j^2 \rho_j^2)^2} & \frac{6\delta_G \mathbf{L}}{(\sum_{j=1}^{p} j^2 \rho_j^2)^2} \\ 0 & \mathbf{0} & \frac{6\delta_G \mathbf{L}^T}{(\sum_{j=1}^{p} j^2 \rho_j^2)^2} & \frac{12\delta_G}{(\sum_{j=1}^{p} j^2 \rho_j^2)^2} \end{bmatrix} \tag{2.2}$$

where

$$\delta_G = \mathbf{L}^T \mathbf{D}_\rho \mathbf{C} \mathbf{L} = \sum_{j=1}^{p} j^2 \rho_j^2 c(j), \quad \mathbf{C} = diag\{c(1), \ldots, c(p)\},$$

$$\mathbf{D}_\rho = diag\{\rho_1^2, \ldots, \rho_p^2\}, \quad \mathbf{L} = (1, 2, \ldots, p)^T,$$

$$c(j) = \left\{ \sum_{k=-\infty}^{\infty} a(k) \cos(jk\lambda) \right\}^2 + \left\{ \sum_{k=-\infty}^{\infty} a(k) \sin(jk\lambda) \right\}^2 = \left| \sum_{k=-\infty}^{\infty} a(k) e^{-ijk\lambda} \right|^2.$$

Now we compare the asymptotic variances of the QTEs and the LSEs. Note that the asymptotic distribution of μ and ρ_j are the same for QTEs and LSEs. But the asymptotic distribution of ϕ_j and λ are different for $p \geq 2$. It can be shown by straight forward but lengthy calculations that for any $p \geq 2$, the asymptotic variances of QTEs of ϕ_j and λ are smaller than the corresponding asymptotic variances of LSEs if $\rho_j > 0$ and $c(j)$ are distinct.

Table 1: The average estimates, mean squared errors, average confidence lengths and coverage probability of LSEs and QTEs of Model 1, considering fundamental frequency model.

Para-	LSEs				QTEs			
meters	AEST	MSE	ACL	CP	AEST	MSE	ACL	CP
ρ_1	2.00813	5.540e-2	.91080	.94	2.01330	6.021e-2	0.92307	.93
ρ_2	2.31337	5.192e-2	.84438	.93	2.31892	5.312e-2	0.85852	.93
ρ_3	1.53351	3.876e-2	.74354	.93	1.55502	4.506e-2	0.75811	.92
ϕ_1	0.70682	1.729e-2	.50520	.94	0.77612	2.346e-2	0.50926	.89
ϕ_2	0.61282	2.064e-2	.55326	.94	0.75684	4.616e-2	0.55262	.77
ϕ_3	1.21512	4.177e-2	.79149	.94	1.41991	8.634e-2	0.78627	.79
λ	0.43992	1.147e-6	4.115e-3	.94	0.44138	3.618e-6	4.054e-3	.66

3 Numerical Examples

In this section we present results of numerical experiments based on simulations. We compare the performances of the LSEs and QTEs of unknown parameters of the FFM as defined in (1.1), with the LSEs of the unknown parameters of the corresponding MFM as defined in (1.2), *i.e* when $\lambda_1 = \lambda$, $\lambda_2 = 2\lambda$, $\dots$ $\lambda_p = p\lambda$. We consider the following two models for simulation experiments:

Model 1:

$$p = 3, \mu = 0, \rho_1 = 2.0, \rho_2 = 2.3, \rho_3 = 1.5, \phi_1 = 0.7, \phi_2 = 0.6, \phi_3 = 1.0, \lambda = \frac{14\pi}{100} = 0.4398229,$$

$$\sigma^2 = 0.2, X(n) = e(n) + .5e(n-1) \quad \text{and} \quad N = 100.$$

Model 2:

$$p = 4, \mu = 0, \rho_1 = 4.5, \rho_2 = 3.1, \rho_3 = 2.45, \rho_4 = 4.0, \phi_1 = 0.8, \phi_2 = 2.7, \phi_3 = 1.0, \phi_4 = -2.9,$$

$$\lambda = \frac{12\pi}{100} = 0.3769911, \sigma^2 = 2.0, X(n) = 0.48e(n) - .7e(n-1) + e(n-2) \quad \text{and} \quad N = 100.$$

In each case we generate a data set using the true parameter values. From the data sequence we estimate the unknown parameters by different methods. To evaluate the QTEs, we need to estimate $f(k\lambda)$, the spectral density function at $k\lambda$, for $k = 1, \dots, p$. In our notation, it is equal to $\sigma^2 c(k)$. It can be shown that $f(k\lambda)$ or $\sigma^2 c(k)$ is the expected value of the periodogram function of the noise random variable $X(t)$ at the respective true frequency $k\lambda$. We use the periodogram averaging method, over a window $(-L, L)$ across the point estimate, $k\hat{\lambda}$. We try for different values of L and observe that for $L = 6$ and $L = 4$ provide the best results in case of QTEs for Models 1 and 2 respectively. We also compute the asymptotic 95% confidence bounds for all the parameters by using both methods. For the least squares method we do not need to estimate the spectral density function to estimate the parameters but we need it for constructing the confidence bands. We use the same technique of averaging the periodogram function method in this case also. We report the results for $L = 6$ (Model 1) and $L = 4$ (Model 2). We replicate the process 5000 times and report average estimates (AESTs), mean squared errors (MSEs), average confidence lengths (ACLs) and coverage percentages (CPs). The results are reported in Tables 1 and 2 for Models 1 and 2 respectively.

Table 2: The average estimates, mean squared errors, average confidence lengths and coverage probability of LSEs and QTEs of Model 2, considering fundamental frequency model.

Para-	LSEs				QTEs			
meters	AEST	MSE	ACL	CP	AEST	MSE	ACL	CP
ρ_1	4.4978	1.451e-2	.5123	.95	4.4989	2.035e-2	.5078	.91
ρ_2	3.1007	1.025e-2	.4043	.94	3.1050	1.142e-2	.4009	.93
ρ_3	2.4516	8.152e-3	.3619	.94	2.4519	1.019e-2	.3538	.90
ρ_4	4.0067	1.753e-2	.6077	.96	4.0013	2.702e-2	.6134	.92
ϕ_1	0.7996	9.752e-4	.1250	.94	0.7947	1.258e-3	.1217	.90
ϕ_2	2.7000	1.654e-3	.1657	.95	2.6891	2.295e-3	.1573	.89
ϕ_3	1.0003	2.423e-3	.2122	.95	0.9848	4.089e-3	.1970	.87
ϕ_4	-2.8978	3.314e-3	.2517	.96	-2.8996	0.1221	.2349	.85
λ	0.3770	6.025e-8	1.003e-3	.95	0.3769	1.259e-7	8.856e-4	.78

Table 3: The average estimates, mean squared errors, average confidence lengths and coverage probability of the LSEs of Model 1, considering multiple frequency model.

Para-	LSE's			
meters	AEST	MSE	ACL	CP
ρ_1	2.02663	4.132e-2	0.89806	.96
ρ_2	2.31988	4.237e-2	0.82822	.94
ρ_3	1.52848	3.604e-2	0.73596	.94
ϕ_1	0.69567	4.281e-2	0.89539	.96
ϕ_2	0.59939	2.906e-2	0.71971	.95
ϕ_3	1.19901	5.274e-2	0.97794	.95
λ_1	0.43978	1.451e-5	1.551e-2	.95
λ_2	0.87962	9.654e-6	1.247e-2	.94
λ_3	1.31949	1.749e-5	1.694e-2	.94

We also compare the LSEs of the parameters of the Models 1 and 2 assuming they are MFM as defined in (1.2). The main aim of this experiment is to compare the advantages/ disadvantages of using FFM over MFM in terms of biases and MSEs of the unknown parameters. The results for $L = 6$ and $L = 3$ are provided in Tables 3 and 4 for Models 1 and 2 respectively.

From the simulation study, we observe that LSEs perform better than QTEs in all respect. For all the cases considered here, LSEs have lower MSEs and lower biases than the corresponding QTEs. Most of the LSEs are unbiased, whereas some of the QTEs, mainly the phase estimators are highly biased. The average confidence lengths based on the QTEs are smaller than the corresponding average confidence lengths based on the LSEs for λ and ϕ_j. From the asymptotic variances, it is expected that way also. Interestingly, the coverage percentages based on the LSEs are much higher than the corresponding coverage percentages based on the QTEs in most of the cases. The coverage percentages obtained by the LSEs attain the nominal level but the coverage percentages obtained by the QTEs are quite poor for the phase components and mainly for the fundamental frequency. Most of the times they are lower than 90% and for the fundamental frequency λ, it is only 66% and 78% for Models 1 and 2 respectively when the nominal level is 95%. Therefore, the confidence intervals obtained by using QT method may not be of much use. QTEs may have certain theoretical advantages over LSEs but computational complexity of the QT method makes it difficult to compute the QTEs efficiently and also the asymptotic results may not be useful for finite length data sets.

Now let us compare models (1.1) and (1.2) based on our numerical experiments. We observe

Table 4: The average estimates, mean squared errors, average confidence lengths and coverage probability of LSEs of Model 2 considering multiple frequency model.

Parameters	LSEs			
	AEST	MSE	ACL	CP
ρ_1	4.50006	6.163e-3	0.49669	.99
ρ_2	3.10094	7.753e-3	0.38728	.95
ρ_3	2.45371	5.794e-3	0.37850	.96
ρ_4	4.00012	9.751e-3	0.58815	.98
ϕ_1	0.80159	2.870e-3	0.22081	.94
ϕ_2	2.70063	4.093e-3	0.25002	.93
ϕ_3	1.00035	3.556e-3	0.30883	.97
ϕ_4	-2.90004	3.688e-3	0.29428	.97
λ_1	0.37703	9.243e-7	3.824e-3	.93
λ_2	0.75400	1.328e-6	4.330e-3	.92
λ_3	1.13098	1.294e-6	5.349e-3	.96
λ_4	1.50796	1.382e-6	5.097e-3	.95

that the linear parameters can be estimated more accurately in the case of MFM than FFM in terms of MSEs. The LSEs of ϕ_j and λ_1 of the MFM have higher MSEs than the corresponding LSEs of the FFM. The average confidence lengths for linear parameters are slightly higher for the FFM. On the other hand, for phase parameters and fundamental frequency estimators, the average confidence lengths are much lower in the case of FFM as compared to MFM.

Thus in the case of fundamental frequency model (1.1) we use the additional information that the frequencies of the observed periodic function are in the form of harmonics and this enables us to estimate the phase components and fundamental frequency more accurately as compared to the general multiple frequency model (1.2).

4 Data Analysis

In this section we analyze two sound data sets "uuu" and "ahh" using FFM and MFM. We apply both Least Squares (LS) and QT methods to estimate the unknown parameters in the case of FFM and LS method in the case of MFM.

"uuu" data set: The data set contains 512 signal values sampled at 10kHz frequency. The observed data and its periodogram function are plotted in Figures 1 and 2 respectively. The periodogram plot indicates that $p = 4$ and an initial estimate of λ can obtained from the periodogram plot. We fit the model (1.1) with $p = 4$ and obtain LSEs and QTEs and their 95% confidence intervals of all the parameters. The results are presented in Table 5.

Using the run test (see Table 6) we observe that errors are not independent in both cases. The autocorrelation and partial auto-correlation functions suggest that in the case of LSEs the estimated errors are of the following form: $X(t) = 1.08X(t-1) - 0.40X(t-2) + e(t)$, and for the QTEs, the estimated errors are of the form; $X(t) = 1.08X(t-1) - 0.39X(t-2) + e(t)$. If the MFM is used then the estimated error is $X(t) = 1.09X(t-1) - 0.50X(t-2) + 0.10X(t-3) + e(t)$.

Performing the run test (see Table 6) on $\hat{e}(t)$, we observe that the independence assumption on

Table 5: LSEs, QTEs and corresponding confidence intervals for "uuu" data with the fundamental frequency model.

Para- meters	LSEs			QTEs		
	LSE	Lower Limit	Upper Limit	QTE	Lower Limit	Upper Limit
μ	0.06814	0.02870	0.10759	0.06814	0.02810	0.10819
ρ_1	0.63516	0.57906	0.69126	0.59103	0.53922	0.64284
ρ_2	1.71760	1.66072	1.77448	1.69948	1.65698	1.74198
ρ_3	0.43157	0.37413	0.48901	0.45359	0.41935	0.48783
ρ_4	0.34866	0.29202	0.40529	0.38883	0.36073	0.41693
ϕ_1	-2.38314	-2.47487	-2.29140	-2.34016	-2.42934	-2.25098
ϕ_2	1.54111	1.48152	1.60069	1.61571	1.57448	1.65693
ϕ_3	-2.73531	-2.88774	-2.58288	-2.71112	-2.80121	-2.62104
ϕ_4	-2.56195	-2.75222	-2.37169	-2.40214	-2.49971	-2.30457
λ	0.11404	0.11394	0.11414	0.11418	0.11412	0.11425

Table 6: Test statistic values for run test and square root of average residual sum of squares (AVRSS) for "uuu" data.

	FFM (LSE)	FFM (QTE)	MFM
Z for $X(t)$	-11.5389566	-13.2993221	-12.1590595
Z for $e(t)$	-1.08383095	-1.48203266	-1.48203266
AVRSS	0.25882411	0.259211272	0.249162257

$e(t)$ is satisfied in all cases. The roots of the characteristic equation in each case are less than one in absolute value, so estimated errors can be modeled as stationary processes in all cases considered above. Note that the confidence length of λ in the case of QTE is smaller than that of LSE but it is much higher in the case of LSE with MFM, which is expected from the theory also. The fitted $y(t)$ values are provided in Figure 3. For comparison, we have plotted the predicted values using LSEs and QTEs with FFM , LSEs with MFM and the original signal in the same figure. The fitted values match quite well in each case.

"ahh" data set The data set contains 340 signal values. The observed data set and its periodogram function are plotted in Figures 4 and 5 respectively. For this data set we use $p = 6$. Using the initial estimate of λ (obtained from periodogram function) we estimate LSEs and QTEs using FFM and LSEs using MFM and obtain the 95% confidence intervals of all the parameters in each case. The results are reported in Tables 7 and 8 respectively. Similarly as for the "uuu" data we analyze the errors and for LSEs with FFM, the time series can be modelled as

$X(t) = 0.67X(t-1) + e(t)$, and for QTEs as: $X(t) = 0.92X(t-1) - 0.47X(t-2) + e(t)$. The estimated error in the case LSEs with MFM can be modeled as $X(t) = 1.02X(t-1) - 0.64X(t-2) + 0.18X(t-3) + e(t)$.

In each case it has been observed that errors are stationary. The summary statistics are provided in Table 10. As observed in "uuu" data, in the case of "ahh" data also the confidence interval of λ in the case of QTE is smaller than that of LSE but it is much higher in the case of LSE with MFM. The predicted $y(t)$ values using LSEs and QTEs with FFM and using LSEs with MFM along with the observed $y(t)$ values are plotted in Figure 6.

Table 7: LSEs, QTEs and corresponding confidence intervals for "ahh" data with the fundamental frequency model.

Para-	LSE's			QTE		
meters	LSE	Lower Limit	Upper Limit	QTE	Lower Limit	Upper Limit
μ	0.03711	-0.09135	0.16557	0.03711	-0.02731	0.10153
ρ_1	0.15635	-0.02066	0.33335	0.15554	0.06617	0.24491
ρ_2	0.11424	-0.05070	0.27917	0.11486	0.03014	0.19958
ρ_3	0.20047	0.05104	0.34990	0.19604	0.11764	0.27443
ρ_4	0.24276	0.10904	0.37648	0.25639	0.18481	0.32798
ρ_5	1.03037	0.91092	1.14983	1.05295	0.98790	1.11799
ρ_6	1.47253	1.36540	1.57966	1.48329	1.42416	1.54242
ϕ_1	1.69557	0.56328	2.82785	1.75676	1.18211	2.33141
ϕ_2	2.08789	0.64359	3.53220	2.16462	1.42674	2.90251
ϕ_3	2.76189	2.01441	3.50937	2.94120	2.54013	3.34227
ϕ_4	-2.70619	-3.26205	-2.15032	-2.50757	-2.78968	-2.22547
ϕ_5	-1.93713	-2.08590	-1.78836	-1.69181	-1.77156	-1.61206
ϕ_6	1.12935	0.99590	1.26280	1.44840	1.37593	1.52088
λ	0.09262	0.09251	0.09273	0.09294	0.09288	0.09300

Table 8: LSE's and confidence intervals for "ahh" data with multiple frequency model.

Para-	LSEs		
meters	LSE	Lower Limit	Upper Limit
μ	0.03711	-0.04130	0.11552
ρ_1	0.15321	0.04255	0.26386
ρ_2	0.12292	0.01293	0.23291
ρ_3	0.19762	0.08844	0.30680
ρ_4	0.26646	0.15817	0.37476
ρ_5	1.05137	0.94365	1.15908
ρ_6	1.47706	1.36958	1.58455
ϕ_1	1.57744	0.13292	3.02196
ϕ_2	2.56365	0.77409	4.35321
ϕ_3	2.45601	1.35103	3.56099
ϕ_4	-1.93182	-2.74466	-1.11898
ϕ_5	-1.58160	-1.78651	-1.37670
ϕ_6	1.38753	1.24199	1.53307
λ_1	0.09222	0.08486	0.09958
λ_2	0.18906	0.17995	0.19818
λ_3	0.27671	0.27108	0.28234
λ_4	0.37503	0.37089	0.37917
λ_5	0.46543	0.46439	0.46647
λ_6	0.55728	0.55654	0.55802

Table 9: LSEs and confidence intervals for "uuu" data with the multiple frequency model.

Para-	LSEs		
meters	LSE	Lower Limit	Upper Limit
μ	0.06814	0.02750	0.10879
ρ_1	0.63175	0.57488	0.68861
ρ_2	1.71710	1.66195	1.77225
ρ_3	0.43196	0.37941	0.48452
ρ_4	0.35917	0.30972	0.40861
ϕ_1	-2.41835	-2.59838	-2.23832
ϕ_2	1.52894	1.46471	1.59318
ϕ_3	-2.30741	-2.55075	-2.06407
ϕ_4	-2.08317	-2.35851	-1.80782
λ_1	0.11390	0.11329	0.11451
λ_2	0.22804	0.22782	0.22825
λ_3	0.34376	0.34294	0.34458
λ_4	0.45793	0.45700	0.45886

Table 10: Test statistic values for run test and square root of average residual sum of squares (AVRSS) for "ahh" data.

	FFM (LSE)	FFM (QTE)	MFM
Z for $X(t)$	-8.10970116	-8.01139259	-7.69534779
Z for $e(t)$	-1.45976138	1.15754771	-0.594967306
AVRSS	0.53678745	0.482003003	0.48966378

Remark 1: For numerical experiments based on simulations we used the method of averaging the periodogram function over a window $(-L, L)$ across the point estimate of the frequency to estimate $\sigma^2 c(k)$, $k = 1, \ldots, p$. In these calculation the choice of L is a difficult problem. It can be interpreted as the problem of bandwidth selection in non-parametric setup. We have reported the best results among different choices of L. We observe for LSE that, if we increase L, after a certain value the results do not change much. But in the case of QTEs, as L is needed in estimation also, the choice of L affects the performance of QTEs.

Remark 2: To analyze the data sets, we do not use the averaging method to obtain the confidence intervals. First we estimate the errors with its parameters. Next we use the estimated errors to estimate the spectral density or equivalently $\sigma^2 c(k)$. For real data how L can be chosen may be a topic of further research.

ACKNOWLEDGEMENTS: The authors would like to thank Professor G.C. Ray of Indian Institute of Technology Kanpur, for providing the data. The authors are grateful to the referees for some valuable comments.

REFERENCES

[1] Hannan, E.J. (1974), "Time series analysis", IEEE Trans. Auto. Control, vol. 19, 706-715.

[2] Kundu, D. (1997), "Asymptotic properties of the least squares estimators of sinusoidal signals", Statistics, vol. 30, 221-238.

[3] Nandi, S. (2001), "Analyzing some non-stationary signal processing models", Ph. D. Thesis, Indian Institute of Technology, Kanpur.

[4] Quinn, B.G. and Thomson, P.J. (1991), "Estimating the frequency of a periodic function", Biometrika, vol. 78, 1, 65-74.