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Estimating the Parameters of Undamped Exponential Signals

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A nonlinear eigenvalue method is proposed for estimating the parameters of undamped exponential signals when the parameters are complex-valued. Such data arise in several areas of application, including communications radio location of objects, seismic signal processing, and computer-assisted medical diagnosis. Osborne proposed a method to estimate the parameters of exponential models when the parameters are real-valued. The method is generalized to the complex-parameters case. It is shown to perform better than existing methods due to Tufts and Kumaresan and Bai, Krishnaiah, and Zhao in the sense of having lower mean-squared errors. A simulation study showed that the observed mean-squared errors are close to the Cramer–Rao lower bound for frequency estimates.

KEY WORDS: Consistent estimator; Nonlinear; Normal equation; Separable regression.

1. INTRODUCTION

Signal processing may be considered broadly to involve the recovery of information from physical observations. The received signal is usually disturbed by thermal, electrical, atmospheric, or intentional interferences. The received signal cannot be predicted deterministically, so statistical methods are needed to describe the signal. The problem of detecting the signal in the midst of noise arises in many areas such as communications, radio location of objects, seismic signal processing, and computer-assisted medical diagnosis.

For example, in electromagnetic pulse (EMP) situations (Ricketts, Bridges, and Miletta 1976; Sircar 1987), the EMP pickup can be characterized by a sum of complex exponentials whose parameters are to be determined. The parameters are a means of coding the various pulse wave forms, and the signal approximation thus obtained can be readily employed to analyze responses in various subsystems under EMP environment. In system-identification problems, the characterization of the impulse response of a linear system by a sum of complex exponentials and then identifying/approximating the complex amplitudes and natural frequencies with a high degree of accuracy has its special importance in a wide variety of applications. The study of the transient behavior of a system is one of them.

In this article, I consider estimation for the following undamped exponential model:

$$y_k = \sum_{j=1}^{M} \alpha_j e^{i\beta_j k} + \varepsilon_k, \qquad k = 1, \ldots, n. \qquad (1.1)$$

Here α_j are unknown complex numbers and β_j are real numbers lying between $[0, 2\pi)$, and the ε_k are iid complex-valued random variables with mean 0 and finite variance for both the real and imaginary parts. M is assumed to be known, and $i = \sqrt{-1}$.

This is an old problem (see Kay and Marple 1981) and it is well known to be numerically difficult (Varah 1985). The best known methods of estimation for β's, like the modified FBLP (forward and backward linear prediction) of Tufts and Kumaresan (1982) and EVLP (equivariation linear prediction) discussed by Bai, Krishnaiah, and Zhao (1986) and Rao (1988) have certain deficiencies (see Kundu 1989). Smyth (1985) showed that general-purpose algorithms such as the Gauss–Newton or its variants take a long time to converge to the least squares estimates even from good starting values.

The main aim of this article is to obtain the least squares estimates, which are known to be consistent (Kundu 1991), by using an improved iterative method. I use separable regression technique to separate the linear and the nonlinear parameters. I show that solving the normal equation is equivalent to solving a nonlinear eigenvalue problem. The nonlinear eigenvalue problem can be solved by a simple iteration technique. The proposed method appears to converge even from poor starting values.

2. EXISTING METHODS

Let us denote

$$\mu_k = \sum_{j=1}^{M} \alpha_j e^{i\beta_j k}.$$

Prony (1795) in his classic article showed that there exists a complex vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{M+1})$ such that

$$\sum_{j=1}^{M+1} \gamma_j \mu_{k+j} = 0, \quad k = 0, \ldots, n - M - 1. \quad (2.1)$$

Then, for any $0 \le k \le n - M - 1$,

$$\sum_{j=1}^{M+1} \gamma_j y_{j+k} = \sum_{j=1}^{M+1} \gamma_j \varepsilon_{j+k}. \quad (2.2)$$

The coefficients γ_i are estimated by minimizing

$$\sum_{k=0}^{n-M-1} \left| \sum_{j=1}^{M+1} \gamma_j y_{j+k} \right|^2 \quad (2.3)$$

subject to the conditions $\gamma_1 > 0$ and

$$|\boldsymbol{\gamma}|^2 = \sum_{j=1}^{M} |\gamma_j|^2 = 1.$$

This method is known as the EVLP method. In the LP (linear prediction) and FBLP methods, Expression (2.3) is minimized subject to the condition $\gamma_1 = 1$ (Bai, Rao, and Chow 1989).

Now write

$$\hat{\gamma}_{rs} = \frac{1}{n - M} \sum_{t=1}^{n-M} \bar{y}_{t+r} y_{t+s},$$

$$r, s = 0, 1, \ldots, M, \quad (2.4)$$

and construct the $(M + 1) \times (m + 1)$ matrix

$$\hat{\Gamma} = [(\hat{\gamma}_{rs})]. \quad (2.5)$$

It is easily seen that the EVLP estimate $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is the unit eigenvector with a nonnegative first element providing the smallest eigenvector of $\hat{\Gamma}$. Construct the polynomial equation

$$\hat{\gamma}_1 + \hat{\gamma}_2 z + \cdots + \hat{\gamma}_{M+1} z^M = 0, \quad (2.6)$$

obtain the solutions in the form

$$\hat{\rho}_1 e^{i\hat{\beta}_1}, \ldots, \hat{\rho}_{M+1} e^{i\hat{\beta}_M}, \quad (2.7)$$

and take $\hat{\beta}_1, \ldots, \hat{\beta}_M$ as estimates of $\beta_1, \ldots, \beta_M$. It was shown by Bai et al. (1986) that $\hat{\boldsymbol{\beta}}$ is a consistent estimator of $\boldsymbol{\beta}$ with a convergence rate of $O_p(n^{-1/2})$. Rao (1988) pointed out that the FBLP or the modified FBLP of Tufts and Kumaresan (1982) does not give consistent estimates, although simulation results show that they perform well in small samples.

3. SEPARABLE REGRESSION

We can write the model (1.1) in the following matrix form:

$$Y = A(\beta)\alpha + E, \quad (3.1)$$

where $Y^T = (y_1, \ldots, y_n)$, $\alpha^T = (\alpha_1, \ldots, \alpha_M)$, $E^T = (\varepsilon_1, \ldots, \varepsilon_n)$, and $\beta^T = (\beta_1, \ldots, \beta_M)$. $A(\beta)$ is an $n \times M$ matrix, and its (p, q)th element is $e^{i\beta_q p}$, $p = 1, \ldots, n$, $q = 1, \ldots, M$. The least squares estimate of (α, β) can be obtained by minimizing $R(\alpha, \beta)$ with respect to (α, β), where

$$R(\alpha, \beta) = \frac{1}{n} |Y - A(\beta)\alpha|^2. \quad (3.2)$$

Here the linear parameters α are separable from the nonlinear parameters β. For a fixed value of β, the minimization of R with respect to α is a simple linear regression problem. The solution of $\hat{\alpha}(\beta)$ is

$$\hat{\alpha}(\beta) = [A^*(\beta)A(\beta)]^{-1}A^*(\beta)Y, \quad (3.3)$$

where * denotes the conjugate transpose of a matrix or of a vector.

Substituting $\hat{\alpha}(\beta)$ in (2.2) and denoting $Q(\beta) = nR(\hat{\alpha}, \beta)$, we obtain

$$Q(\beta) = Y^*(I - P_A)Y, \quad (3.4)$$

where $P_A = A(\beta)[A^*(\beta)A(\beta)]^{-1}A^*(\beta)$ is the projection operator on the range space spanned by the columns of the matrix $A(\beta)$.

Therefore, the least squares estimator, $\hat{\beta}$, of β can be obtained by minimizing $Q(\beta)$ with respect to β, so the least squares estimator of α becomes

$$\hat{\alpha} = [A^*(\hat{\beta})A(\hat{\beta})]^{-1}A^*(\hat{\beta})Y.$$

4. NORMAL EQUATION

From (2.1) we can conclude that there exists an $n \times n - M$ matrix X such that $X^*\mu = 0$, where

$$X^* = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_{M+1} & 0 & 0 \ldots & 0 \\ 0 & \gamma \ldots & & \gamma_{M+1} & 0 \ldots & 0 \\ \ldots & & & & & \\ 0 & 0 \ldots & & 0 & \gamma_1 & \gamma_{M+1} \end{bmatrix}. \quad (4.1)$$

Since $X^*\mu = 0$, this implies that $X^*A(\beta) = 0$; that is, the columns of A are orthogonal to the rows of X^*. Therefore, we can write $Q(\beta)$ of (3.4) as

$$Q(\beta) = Y^* P_x Y, \quad (4.2)$$

where $P_x = X(X^*X)^{-1}X^*$ is the projection matrix on the space spanned by the columns of X.

The elements of X^* depend only on $\gamma_1, \ldots, \gamma_{M+1}$. Since $e^{i\beta_1}, \ldots, e^{i\beta_M}$ are the roots of the polynomial equation

$$\gamma_{M+1}Z^M + \gamma_M z^{M-1} + \cdots + \gamma_1 = 0, \quad (4.3)$$

it follows that

$$\min_{\gamma_1,\ldots,\gamma_{M+1}} Y^*PY \le \min_{0\le\beta_1,\ldots,\beta_M<2\pi} Y^*PY. \quad (4.4)$$

For brevity, let us denote

$$Q(\gamma) = Y^*P_xY, \quad (4.5)$$

where $\gamma^T = (\gamma_1, \ldots, \gamma_{m+1})$.

To minimize $Q(\gamma)$ with respect to γ, we differentiate $Q(\gamma)$ with respect to the real and imaginary parts of γ and equate them to 0. This leads to solving a matrix equation of the following form:

$$B(\gamma)\gamma = 0, \quad (4.6)$$

where B is an $M + 1 \times M + 1$ matrix whose (p, q)th element is

$$Y^*X_p(X^*X)^{-1}X_q^TY - Y^*X(X^*X)^{-1}X_q^TX_P(X^*X)^{-1}X^*Y.$$

This is a nonlinear eigenvalue problem. I discuss in detail the solution procedure of such a problem in Section 5.

5. THE MODIFIED PRONY ALGORITHM

Observe that $Q(\gamma)$ is invariant under scalar multiplication; that is,

$$Q(\gamma) = Q(C\gamma) \quad (5.1)$$

for any complex constant C. Therefore,

$$\min_{\gamma:|\gamma|=1} Q(\gamma) = \min_{\gamma} Q(\gamma). \quad (5.2)$$

It is also clear that γ satisfying (4.6) should be an eigenvector corresponding to the zero eigenvalue of the matrix $B(\gamma)$.

I suggest the following iterative technique similar to that of Osborne (1975) to solve (4.6):

$$[B(\gamma^k) - \lambda^{k+1}I]\gamma^{k+1} = 0, \quad |\gamma| = 1, \quad (5.3)$$

where λ^{k+1} is the eigenvalue closest to 0 and γ^{k+1} is the corresponding eigenvector after normalization. The iterative procedure should be stopped when λ^{k+1} is small compared to $\|B\|$ (see Kundu 1989 for details).

The suggested modified Prony algorithm has the following form:

Step 1: Set an initial value γ^1 and normalize it; that is,

$$\gamma^1 = \frac{\gamma^1}{|\gamma|}, \quad i := 1.$$

Step 2: Calculate the matrix $B(\gamma^k)$.

Step 3: Find the eigenvalue λ^{k+1} of $B(\gamma^k)$ closest to 0 and normalize the corresponding eigenvector γ^{k+1}.

Step 4: Test for convergence by testing if

$$|\lambda^{k+1}| < \in \|B\|.$$

Step 5: If the test in Step 4 fails, set $i = i + 1$ and go to Step 2.

Once $\hat{\gamma}$ is obtained, the estimate of β can be obtained using (2.6) and (2.7).

6. NUMERICAL EXPERIMENTS

I performed a simulation study to compare the behavior of the modified Prony algorithm with EVLP and modified FBLP in finite samples. All simulations were done on the IBM 4381 computer at the Pennsylvania State University, using the IMSL random deviate generator. I considered the following model:

$$y(t) = e^{2.0it} + e^{3.5it} + 2e^{4.0it} + 2e^{5.5it} + \varepsilon(t), \quad t = 1, \ldots, n \quad (6.1)$$

Here $\varepsilon(t)$ are iid complex-valued random variables with mean 0 and standard deviation σ for both the real and the imaginary part. The real and the imaginary parts are independent and normally distributed. One hundred different sets of data were generated for $n = 20, 30, 40$, and 50 and $\sigma = .5, .1$, and .01.

For each data set, I estimated β by the modified Prony (MP) algorithm as described in Section 5, by the modified FBLP (MF) as described by Tufts and Kumaresan (1982), and by the modified EVLP (ME) as described by Bai et al. (1989). For MP, I used $\varepsilon = 10^{-6}$ and took $\|B\|$ to be the largest eigenvalue of the matrix B.

The eigenvector corresponding to the smallest eigenvalue of Y^*Y, where

$$Y = \begin{bmatrix} y_1 & \cdots & y_{M+1} \\ y_{n-M} & \cdots & y_n \end{bmatrix},$$

was used as an initial value in each iteration. For ME, I used 10-step refinement over ordinary EVLP. In all cases, the iterations converged within six steps. The mean estimate of the β's and their mean-squared errors (MSE's) over 100 replications were calculated.

The numerical results confirmed the satisfactory performance of the proposed algorithm. The MP estimates had the lowest MSE in all of the cases studied. The performance of the ME estimates was better than that of MF estimates for small samples. The Cramer–Rao (CR) bound of the parameter β_j is $(6/(2\pi n)^2) \cdot (1/\rho)$, where ρ is the spectral signal-to-noise power ratio and $\rho = |\alpha_j|^2n/2\sigma^2$. So, for example, when $\sigma = .5$ and $n = 20$, the CR bound for β_1 is

.0194. For this case, the MSE of MP estimates was .0207 and of MF estimates was .3390 and of ME estimates was .0431. The ratios, MSE/CR bound, were 1.067, 17.474, and 2.222. As n increases or σ decreases, the MSE of the estimates decreases. The asymptotic stability of the proposed method was proved by Kundu (1989).

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