Characterizations of the Proportional (Reversed) Hazard Class

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Abstract

In this paper we provide two simple characterizations of the proportional (reversed) hazard class of distributions based on some conditional expectation and conditional variance. Since exponential, Weibull, generalized exponential, Rayleigh, Burr type X, exponentiated Weibull belong to the proportional (reversed) hazard class of distributions, therefore the present results will provide characterizations of all these special cases.

Key Words and Phrases: Conditional expectation; conditional variance; recurrence relation; exponentiated Weibull distribution.

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1 Introduction

Suppose X is an absolutely continuous random variable with density function, distribution function and survival function as $f_X(.)$, $F_X(.)$ and $S_X(.)$ respectively. For all t such that $S_X(t) > 0$, the hazard function $h_X(t) = f_X(t)/S_X(t)$, similarly for all t such that $F_X(t) > 0$, the reversed hazard function $r_X(t) = f_X(t)/F_X(t)$. Note that, the hazard function or the reversed hazard function uniquely define the distribution function. Now, let us define the proportional hazard class and the proportional reversed hazard class of distributions.

Proportional hazard class with the base hazard function $h_X(.)$, is the class of distributions having hazard functions $\alpha h_X(.)$ for $\alpha > 0$. Similarly the Proportional reversed hazard class with the base reversed hazard function $r_X(.)$, is the class of distributions having reversed hazard functions $\alpha r_X(.)$ for $\alpha > 0$.

Applications of hazard functions are quite well known in the statistical literature. Recently the reversed hazard functions also become quite popular among the statisticians, see for example Gupta and Han (2001). Anderson et al. (1993) show that the reversed hazard function plays the same role in the analysis of left-censored data as the hazard function plays in the analysis of right-censored data. Interestingly, it is observed that there exists a relation between the proportional reversed hazard class of distributions and the exponentiated class of distributions.

A random variable Y is said to be exponentiated random variable with base distribution $F_X(.)$ if Y has distribution function $[F_X(.)]^\alpha$, see Gupta et al. (1998), Mudholkar and Srivastava (1993), Mudholkar et al. (1995). The class of distributions $[F_X(.)]^\alpha$ can be defined as the exponentiated class of distributions with base distribution $F_X(.)$. In this case the reversed hazard functions of X and Y satisfy $r_Y(t) = \alpha r_X(t)$ for all t such that $F_X(t) > 0$. Therefore, we can say that Y belongs to proportional reversed hazard class with base reversed hazard function $r_X(.)$. Similarly, if Z is a random variable whose survival function is $[S_X(.)]^\alpha$, then the hazard functions of X and Z satisfy $h_Z(t) = \alpha h_X(t)$.

In this article we provide two simple characterizations of the proportional hazard class and the proportional reversed hazard class of distributions in terms of the conditional mean and conditional variance. Some new characterizations of the well known distributions like, exponential, Weibull, generalized exponential, Rayleigh, Burr type X, exponentiated Weibull etc. are obtained as special cases.

2 Characterization

For all t such that $F_X(t) > 0$ and for an absolutely continuous random variable Y, let us define the following functions;

$$U(t) = -\ln F_X(t), \;\; a_Y^{(n)}(t) = E[U^n(Y)|Y < t],$$

here $U^n(.)$ denotes the n^{th} power of $U(.)$. Similarly, for all t such that $S_X(t) > 0$ and for an absolutely continuous random variable Z, we define

$$V(t) = -\ln S_X(t), \;\; b_Z^{(n)}(t) = E[V^n(Z)|Z > t],$$

here also $V^n(.)$ denotes the n^{th} power of $V(.)$. Moreover,

$$U'(t) = \frac{d}{dt}U(t) = -r_X(t) \quad \text{and} \quad V'(t) = \frac{d}{dt}V(t) = h_X(t).$$

Theorem 2.1 *For any real number t such that $F_X(t) > 0$, $r_Y(t) = \alpha r_X(t)$ with $\alpha > 0$ iff*

$$a_Y^{(n)}(t) = U^n(t) + \frac{n}{\alpha} a_Y^{(n-1)}(t). \tag{1}$$

Here n is a positive integer.

Proof.

Necessary Part: Suppose $r_Y(t) = \alpha r_X(t)$. Then for all t such that $F_X(t) > 0$. Now,

$$a_Y^{(n)}(t) = E[U^n(Y)|Y < t] = \frac{1}{F_Y(t)} \int_{-\infty}^{t} U^n(y) dF_Y(y). \tag{2}$$

Integrating by parts the right hand side of (2) we get

$$\begin{aligned} a_Y^{(n)}(t) &= U^n(t) + \frac{n}{F_Y(t)} \int_{-\infty}^{t} U^{n-1}(y) r_X(y) F_Y(y) dy \\ &= U^n(t) + \frac{n}{\alpha F_Y(t)} \int_{-\infty}^{t} U^{n-1}(y) f_Y(y) dy \\ &= U^n(t) + \frac{n}{\alpha} a_Y^{(n-1)}(t). \end{aligned}$$

Sufficiency part: Suppose (1) is true. Therefore using (2), (1) can be written as

$$\frac{1}{F_Y(t)} \int_{-\infty}^{t} U^n(y) f_Y(y) dy = U^n(t) + \frac{n}{\alpha} \frac{1}{F_Y(t)} \int_{-\infty}^{t} U^{n-1}(y) f_Y(y) dy$$

or

$$\int_{-\infty}^{t} U^n(y) f_Y(y) dy = F_Y(t) U^n(t) + \frac{n}{\alpha} \int_{-\infty}^{t} U^{n-1}(y) f_Y(y) dy. \tag{3}$$

Differentiating both sides of (3) with respect to t we have

$$U^n(t) f_Y(t) = U^n(t) f_Y(t) - n U^{n-1}(t) F_Y(t) r_X(t) + \frac{n}{\alpha} U^{n-1}(t) f_Y(t).$$

Since $U^{(n-1)}(t) \neq 0$, this implies

$$r_Y(t) = \frac{f_Y(t)}{F_Y(t)} = \alpha r_X(t)$$

for all t such that $F_X(t) > 0$.

Theorem 2.2 *For any real number t such that $F_X(t) > 0$, $r_Y(t) = \alpha r_X(t)$ with $\alpha > 0$ iff*

$$Var(U(Y)|Y < t) = \frac{1}{\alpha^2}.$$

Proof

NECESSARY PART: Suppose $r_Y(t) = \alpha r_X(t)$. Then from theorem (2.1),

$$a_Y^{(1)}(t) = U(t) + \frac{1}{\alpha} \text{ and } a_Y^{(2)}(t) = U^2(t) + \frac{2}{\alpha}a_Y^{(1)}(t) = U^2(t) + \frac{2}{\alpha}U(t) + \frac{2}{\alpha^2} = \left(a_Y^{(1)}(t)\right)^2 + \frac{1}{\alpha^2}.$$

Therefore, the result follows immediately.

SUFFICIENCY PART: Now suppose $Var(U(Y)|Y < t) = \frac{1}{\alpha^2}$ for all t, such that $F_X(t) > 0$. That is,

$$F_Y(t)\int_{-\infty}^{t} U^2(y)f_Y(y)dy - \left[\int_{-\infty}^{t} U(y)f_Y(y)dy\right]^2 = \frac{F_Y^2(t)}{\alpha^2} \tag{4}$$

Differentiating both sides of (4) with respect to t and canceling $f_Y(t)$ from both sides, we get

$$\int_{-\infty}^{t} U^2(y)f_Y(y)dy + F_Y(t)U^2(t) - 2U(t)\int_{-\infty}^{t} U(y)f_Y(y)dy = 2\frac{F_Y(t)}{\alpha^2} \tag{5}$$

Differentiating both sides of (5) with respect to t and arranging terms, we obtain

$$F_Y(t)U(t) - \int_{-\infty}^{t} U(y)f_Y(y)dy = -\frac{f_Y(t)}{r_X(t)\alpha^2}. \tag{6}$$

Again differentiating (6) with respect to t and rearranging, we obtain the differential equation

$$\alpha^2 F_Y(t) = \frac{1}{r_X(t)}\frac{d}{dt}\left(\frac{f_Y(t)}{r_X(t)}\right). \tag{7}$$

A general solution of the differential equation (7) is of the form

$$F_Y(t) = c_1 e^{\alpha U(t)} + c_2 e^{-\alpha U(t)} = c_1 F_X^{-\alpha}(t) + c_2 F_X^{\alpha}(t),$$

where c_1 and c_2 are arbitrary constants. Since $F_X(t)$ and $F_Y(t)$ tend to 0 as t tends to $-\infty$, therefore, $c_1 = 0$. Similarly, as $F_X(t)$ and $F_Y(t)$ tend to 1 as t tends to ∞, therefore, $c_2 = 1$. That is,

$$F_Y(t) = F_X^{\alpha}(t) \quad \text{or} \quad r_Y(t) = \alpha r_X(t).$$

Theorem 2.3 *For any real number t such that $S_X(t) > 0$, $h_Z(t) = \alpha h_X(t)$ with $\alpha > 0$ iff*

$$b_Z^{(n)}(t) = V^n(t) + \frac{n}{\alpha} b_Z^{(n-1)}(t),$$

where n is a positive integer.

Proof: It follows along the same line as the proof of theorem 2.1.

Theorem 2.4 *For any real number t such that $S_X(t) > 0$, $h_Z(t) = \alpha h_X(t)$ with $\alpha > 0$ iff*

$$Var(V(Z)|Z > t) = \frac{1}{\alpha^2}.$$

Proof: It follows along the same lines as of theorem 2.2.

3 Examples

Example 1. The probability density function (pdf) of Z is of the form

$$f_Z(z) = \lambda \alpha z^{\alpha-1} e^{-\lambda z^{\alpha}}, \quad \lambda, \alpha, z > 0,$$

i.e. Z follows Weibull distribution with the shape parameter α and scale parameter λ, iff either of the following is true for all $t > 0$,

(i) $E[Z^{n\alpha}|Z > t] = t^{n\alpha} + \frac{n}{\lambda} E[Z^{(n-1)\alpha}|Z > t]$ for any positive integer n;

(ii) $Var(Z^{\alpha}|Z > t) = \frac{1}{\lambda^2}$.

Comments: Note that in Example 1, for $\alpha = 1$, we get the corresponding characterization of the exponential distribution function.

Example 2. The pdf of Z is of the form

$$f_Z(z) = \alpha(1+z)^{-(1+\alpha)}, \quad \alpha > 0, z > 0,$$

i.e. Z follows Pareto distribution, iff any one of the following is true for all $t > 0$,

(i) $E[(\ln(1+Z))^n|Z > t] = (\ln(1+t))^n + E[(\ln(1+Z))^{n-1}|Z > t]$

(ii) $Var(\ln(1+Z)|Z > t) = \frac{1}{\alpha^2}$.

Example 3. The random variable Y has the cumulative distribution function (cdf)

$$F_Y(y) = \left(1 - e^{(\lambda y)^\beta}\right)^\alpha, \quad \alpha, \beta, \lambda, y > 0,$$

i.e. Y follows an exponentiated Weibull distribution, with the scale parameter λ and the shape parameters α and β, iff any one of the following is true for all $t > 0$

(i) $E[(-\ln(1-e^{-(\lambda y)^\beta}))^n|Y < t] = (-\ln(1-e^{-(\lambda y)^\beta}))^n + E[(-\ln(1-e^{-(\lambda y)^\beta}))^{n-1}|Y < t]$

(ii) $Var(-\ln(1-e^{-(\lambda y)^\beta}))^n|Y < t) = \frac{1}{\alpha^2}$

Comments: Note that the exponential ($\alpha = 1, \beta = 1$) and Weibull ($\alpha = 1$) are special cases of the exponentiated Weibull distribution. Several other known distributions like generalized exponential ($\beta = 1$), Rayleigh ($\beta = 2, \alpha = 1$) distributions are also special cases of the exponentiated Weibull distribution.

Example 4 The random variable Y has the cdf

$$F_Y(y) = \left[\Phi\left(\frac{y-\mu}{\sigma}\right)\right]^\alpha, \quad \alpha, \sigma > 0, -\infty < x, \mu < \infty$$

i.e. Y follows an exponentiated normal distribution with the scale, location and shape parameters as σ, μ and α respectively, iff any one of the following holds for all t,

(i) $E[(-\ln\Phi(\frac{(y-\mu)}{\sigma}))^n|Y < t] = (-\ln\Phi(\frac{(y-\mu)}{\sigma}))^n + E[(-\ln\Phi(\frac{(y-\mu)}{\sigma}))^{n-1}|Y < t]$

(ii) $Var((-\ln\Phi(\frac{(y-\mu)}{\sigma}))^n|Y < t) = \frac{1}{\alpha^2}$.

4 Conclusions

In this paper we provide two simple characterizations of the proportional hazard class and proportional reversed hazard class of distributions based on conditional mean and conditional variance. It is observed that these two simple results can be used for characterizing several well known distributions, and these characterizations were not used before. Apparently, these characterizations can be used for goodness of fit test for some of the distributions. Work is in progress and it will be published elsewhere.

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