Estimation of P$[Y < X]$ for Generalized Exponential Distribution

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Abstract

This paper deals with the estimation of $P[Y < X]$ when X and Y are two independent generalized exponential distributions with different shape parameters but having the same scale parameters. The maximum likelihood estimator and its asymptotic distribution is obtained. The asymptotic distribution is used to construct an asymptotic confidence interval of $P[Y < X]$. Assuming that the common scale parameter is known, the maximum likelihood estimator, uniformly minimum variance unbiased estimator and Bayes estimator of $P[Y < X]$ are obtained. Different confidence intervals are proposed. Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a simulated data set has also been presented for illustrative purposes.

Key Words and Phrases: Stress-Strength model; maximum likelihood estimator; Bayes Estimator; Bootstrap Confidence intervals; Credible intervals; Asymptotic distributions.

Short Running Title: Generalized exponential distributions.

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1 Introduction

Recently the two-parameter generalized exponential (GE) distribution has been proposed by the authors. It has been studied extensively by Gupta and Kundu (1999, 2001a, 2001b, 2002, 2003a, 2003b, 2004), Raqab (2002), Raqab and Ahsanullah (2001), Zheng (2002) and Kundu, Gupta and Manglick (2004). Note that the generalized exponential distribution is a sub-model of the exponentiated Weibull distribution introduced by Mudholkar and Srivastava (1993) and later studied by Mudholkar, Srivastava and Freimer (1995) and Mudholkar and Hutson (1996).

The two-parameter GE distribution is an alternative to the well known two-parameter gamma, two-parameter Weibull or two parameter log-normal distributions. The two-parameter GE distribution has the following density function

$$f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}; \qquad \text{for} \quad x>0, \tag{1.1}$$

and the distribution function

$$F_{GE}(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha} \qquad \text{for} \quad x>0. \tag{1.2}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively. For different values of the shape parameter, the density function can take different shapes. For detailed description of the distribution, one is referred to the original paper of Gupta and Kundu (1999). From now on GE distribution with the shape parameter α and scale parameter λ will be denoted by GE(α,λ).

The main aim of this paper is to focus on the inference of $R = P[Y < X]$, where $Y \sim$ GE(α,λ), $X \sim$ GE(β,λ) and they are independently distributed. Here the notation $\sim$ means 'follows' or 'has the distribution'. The estimation of R is very common in the statistical literature. For example, if X is the strength of a component which is subject to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. We obtain the maximum likelihood estimator (MLE) of R and obtain its asymptotic distribution. The asymptotic distribution has been used to construct an asymptotic confidence interval. Two bootstrap confidence intervals of R are also proposed. Assuming that the common scale parameter is known, we obtain the MLE and the uniformly minimum variance unbiased estimator (UMVUE) of R. Bayes estimators of R assuming gamma priors on the shape parameters are obtained under different loss functions. A confidence interval based on the exact distribution of the MLE of R is obtained. Different methods are compared using Monte Carlo simulations and one data set has been used for illustrative purposes.

It may be mentioned here that related problems have been widely used in the statistical literature. The MLE of $P[Y < X]$, when X and Y have bivariate exponential distribution, has been considered by Awad *et al.* (1981). Church and Harris (1970), Downtown (1973), Govidarajulu (1967), Woodward and Kelley (1977) and Owen, Craswell and Hanson (1977) considered the estimation of $P[Y < X]$, when X and Y are normally distributed. Similar problem for the multivariate normal distribution has been considered by Gupta and Gupta (1990). Kelley, Kelley and Schucany (1976), Sathe and Shah (1981), Tong (1974, 1977) considered the estimation of $P(Y < X)$ when X and Y are independent exponential random variables. Constantine and Karson (1986) considered the estimation of $P[Y < X]$, when X and Y are independent gamma random variables. Ahmad, Fakhry and Jaheen (1997) and Surles and Padgett (2001, 1998) considered the estimation of $P[Y < X]$, where X and Y are Burr Type X random variables.

The rest of the paper is organized as follows. In Section 2, we derive the MLE of R. The asymptotic distribution of the MLE of R is given and different confidence intervals are proposed in Section 3. In Section 4, we consider different estimation procedures of R if λ is known. The different proposed methods have been compared using Monte Carlo simulations and the results have been reported in Section 5. Analysis of a real life data set has been presented in Section 6 and finally we draw conclusions in Section 7.

2 Maximum Likelihood Estimator of R

Let $Y \sim \text{GE}(\alpha, \lambda)$, $X \sim \text{GE}(\beta, \lambda)$, where X and Y are independently distributed. Therefore,

$$R = P[Y < X] = \int_0^\infty \int_0^x \alpha\beta\lambda^2 e^{-\lambda(y+x)}(1-e^{-\lambda x})^{\beta-1}(1-e^{\lambda y})^{\alpha-1} dy dx$$
$$= \int_0^\infty \beta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha+\beta-1} dx = \frac{\beta}{\alpha+\beta}. \tag{2.1}$$

Now to compute the MLE of R, first we obtain the MLEs of α and β. Suppose $X_1, \ldots, X_n$ is a random sample from $\text{GE}(\beta, \lambda)$ and $Y_1, \ldots, Y_m$ is a random sample from $\text{GE}(\alpha, \lambda)$. Therefore, the log-likelihood function of the observed samples is

$$L(\alpha, \beta, \lambda) = m \ln \alpha + n \ln \beta + (m+n) \ln \lambda - \lambda(\sum_{i=1}^m Y_i + \sum_{j=1}^n X_j) + (\alpha - 1) \sum_{i=1}^m \ln\left(1 - e^{-\lambda Y_i}\right)$$
$$+(\beta-1)\sum_{j=1}^n \ln\left(1 - e^{-\lambda X_j}\right). \tag{2.2}$$

The MLE's of α, β and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ respectively, can be obtained as the solutions of

$$\frac{\partial L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^m \ln\left(1 - e^{-\lambda Y_i}\right) = 0 \tag{2.3}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{j=1}^n \ln\left(1 - e^{-\lambda X_j}\right) = 0 \tag{2.4}$$

$$\frac{\partial L}{\partial \lambda} = \frac{m+n}{\lambda} - \left(\sum_{i=1}^m Y_i + \sum_{j=1}^n X_j\right) + (\alpha-1)\sum_{i=1}^m \frac{Y_i e^{-\lambda Y_i}}{(1-e^{-\lambda Y_i})} + (\beta-1)\sum_{j=1}^n \frac{X_j e^{-\lambda X_j}}{(1-e^{-\lambda X_j})} = 0. \tag{2.5}$$

From (2.3), (2.4) and (2.5), we obtain

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^m \ln\left(1 - e^{-\hat{\lambda} Y_i}\right)}, \tag{2.6}$$

$$\hat{\beta} = -\frac{n}{\sum_{j=1}^{n} \ln\left(1 - e^{-\hat{\lambda} X_j}\right)}, \tag{2.7}$$

and $\hat{\lambda}$ can be obtained as the solution of the non-linear equation

$$g(\lambda) = \frac{m+n}{\lambda} - \frac{n}{\sum_{k=1}^{n} \ln(1 - e^{-\lambda X_k})} \times \sum_{j=1}^{n} \frac{X_j e^{-\lambda X_j}}{(1 - e^{-\lambda X_j})} - \frac{m}{\sum_{k=1}^{m} \ln(1 - e^{-\lambda Y_k})} \times \sum_{i=1}^{m} \frac{Y_i e^{-\lambda Y_i}}{(1 - e^{-\lambda Y_i})}$$
$$- \sum_{j=1}^{n} \frac{X_j}{(1 - e^{-\lambda X_j})} - \sum_{i=1}^{m} \frac{Y_i}{(1 - e^{-\lambda Y_i})} = 0. \tag{2.8}$$

Therefore, $\hat{\lambda}$ can be obtained as a solution of the non-linear equation of the form:

$$h(\lambda) = \lambda, \tag{2.9}$$

where

$$h(\lambda) = (m+n) \left[\frac{n}{\sum_{k=1}^{n} \ln(1 - e^{-\lambda X_k})} \times \sum_{j=1}^{n} \frac{X_j e^{-\lambda X_j}}{(1 - e^{-\lambda X_j})} + \frac{m}{\sum_{k=1}^{m} \ln(1 - e^{-\lambda Y_k})} \times \sum_{i=1}^{m} \frac{Y_i e^{-\lambda Y_i}}{(1 - e^{-\lambda Y_i})} \right.$$
$$\left. + \sum_{j=1}^{n} \frac{X_j}{(1 - e^{-\lambda X_j})} + \sum_{i=1}^{m} \frac{Y_i}{(1 - e^{-\lambda Y_i})} \right]^{-1}.$$

Since $\hat{\lambda}$ is a fixed point solution of the non-linear equation (2.9), therefore, it can be obtained by using a simple iterative scheme as follows:

$$h(\lambda_{(j)}) = \lambda_{(j+1)}, \tag{2.10}$$

where $\lambda_{(j)}$ is the j^{th} iterate of $\hat{\lambda}$. The iteration procedure should be stopped when $|\lambda_{(j)} - \lambda_{(j+1)}|$ is sufficiently small. Once we obtain $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from (2.6) and (2.7) respectively. Therefore, the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}}. \tag{2.11}$$

3 Asymptotic Distribution and Confidence Intervals

In this section first we obtain the asymptotic distribution of $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and then we derive the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, we obtain the asymptotic confidence interval of R. Let us denote the Fisher information matrix of $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$ as $I(\boldsymbol{\theta}) = (I_{ij}\ (\boldsymbol{\theta}))$; i,j = 1,2,3. Therefore,

$$I(\boldsymbol{\theta}) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 L}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \beta^2}\right) & E\left(\frac{\partial^2 L}{\partial \beta \partial \lambda}\right) \\ E\left(\frac{\partial^2 L}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad \text{(say)}. \tag{3.1}$$

Moreover,

$$E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) = -\frac{m}{\alpha^2}, \quad E\left(\frac{\partial^2 L}{\partial \beta^2}\right) = -\frac{n}{\beta^2}, \quad E\left(\frac{\partial^2 L}{\partial \alpha \partial \beta}\right) = E\left(\frac{\partial^2 L}{\partial \beta \partial \alpha}\right) = 0.$$

Also,

$$\begin{aligned} E\left(\frac{\partial^2 L}{\partial \alpha \partial \lambda}\right) &= \frac{m}{\lambda}\left[\frac{\alpha}{\alpha - 1}(\psi(\alpha) - \psi(1)) - (\psi(\alpha+1) - \psi(1))\right] \quad \text{if} \quad \alpha \neq 1 \\ &= \frac{m}{\lambda}\sum_{k=0}^{\infty}\frac{1}{(i+2)^2} \quad \text{if} \quad \alpha = 1, \end{aligned}$$

$$\begin{aligned} E\left(\frac{\partial^2 L}{\partial \beta \partial \lambda}\right) &= \frac{n}{\lambda}\left[\frac{\beta}{\beta - 1}(\psi(\beta) - \psi(1)) - (\psi(\beta+1) - \psi(1))\right] \quad \text{if} \quad \beta \neq 1 \\ &= \frac{n}{\lambda}\sum_{k=0}^{\infty}\frac{1}{(i+2)^2} \quad \text{if} \quad \beta = 1, \end{aligned}$$

$$\begin{aligned} E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) &= -\frac{m}{\lambda^2}\left[1 + \frac{\alpha(\alpha-1)}{\alpha-2}(\psi'(1) - \psi'(\alpha-1) + (\psi(\alpha-1) - \psi(1))^2)\right] \\ &\quad + \frac{m\alpha}{\lambda^2}\left[(\psi'(1) - \psi'(\alpha)) + (\psi(\alpha) - \psi(1))^2\right] \\ &\quad - \frac{n}{\lambda^2}\left[1 + \frac{\beta(\beta-1)}{\beta-2}(\psi'(1) - \psi'(\beta-1) + (\psi(\beta-1) - \psi(1))^2)\right] \\ &\quad + \frac{n\beta}{\lambda^2}\left[(\psi'(1) - \psi(\beta)) + (\psi(\beta) - \psi(1))^2\right] \quad \text{if} \quad \alpha \neq 2, \beta \neq 2, \\ &= -\frac{m+n}{\lambda^2} - \frac{4(m+n)}{\lambda^2} \times \sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha = 2, \beta = 2, \\ &= -\frac{m+n}{\lambda^2} - \frac{m}{\lambda^2}\left[\frac{\alpha(\alpha-1)}{\alpha-2}(\psi'(1) - \psi'(\alpha-1) + (\psi(\alpha-1) - \psi(1))^2)\right] \\ &\quad + \frac{m\alpha}{\lambda^2}\left[(\psi'(1) - \psi'(\alpha)) + (\psi(\alpha) - \psi(1))^2\right] - \frac{4n}{\lambda^2} \times \sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha \neq 2, \beta = 2, \end{aligned}$$

$$= \quad -\frac{m+n}{\lambda^2} - \frac{n}{\lambda^2}\left[\frac{\beta(\beta-1)}{\beta-2}(\psi'(1)-\psi'(\beta-1)+(\psi(\beta-1)-\psi(1))^2)\right]$$
$$+\frac{n\beta}{\lambda^2}\left[(\psi'(1)-\psi'(\beta))+(\psi(\beta)-\psi(1))^2\right] - \frac{4m}{\lambda^2}\times\sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha=2,\beta\neq 2.$$

Since GE family satisfies all the regularity conditions (Gupta and Kundu; 2001b), therefore we have the following result:

Theorem 1: As $m \to \infty$ and $n \to \infty$ and $\frac{m}{n} \to p$, then

$$\left[\sqrt{m}(\hat{\alpha}-\alpha), \sqrt{n}(\hat{\beta}-\beta), \sqrt{m}(\hat{\lambda}-\lambda)\right] \to N_3(\mathbf{0}, \mathbf{A}^{-1}(\alpha,\beta,\lambda)),$$

where,

$$\mathbf{A}(\alpha,\beta,\lambda) = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$a_{11} = \frac{1}{\alpha^2} = -\frac{1}{m}I_{11}, \quad a_{13} = a_{31} = -\frac{1}{m}I_{13} = -\frac{1}{m}I_{31}, \quad a_{22} = \frac{1}{\beta^2} = -\frac{1}{n}I_{22},$$
$$a_{23} = a_{32} = -\frac{\sqrt{p}}{m}I_{23} = -\frac{\sqrt{p}}{m}I_{32}, \quad a_{33} = -\lim_{m,n\to\infty}\frac{1}{m}I_{33}.$$

Note that, a_{33} can be written as follows;

$$\begin{aligned}
a_{33} \quad = \quad & \frac{1}{\lambda^2}\left[1+\frac{\alpha(\alpha-1)}{\alpha-2}(\psi'(1)-\psi'(\alpha-1)+(\psi(\alpha-1)-\psi(1))^2)\right] - \\
& \frac{\alpha}{\lambda^2}\left[(\psi'(1)-\psi'(\alpha))+(\psi(\alpha)-\psi(1))^2\right] + \\
& \frac{1}{p\lambda^2}\left[1+\frac{\beta(\beta-1)}{\beta-2}(\psi'(1)-\psi'(\beta-1)+(\psi(\beta-1)-\psi(1))^2)\right] - \\
& \frac{\beta}{p\lambda^2}\left[(\psi'(1)-\psi'(\beta))+(\psi(\beta)-\psi(1))^2\right] \quad \text{if} \quad \alpha\neq 2,\beta\neq 2, \\
= \quad & \frac{p+1}{p\lambda^2} + \frac{4(p+1)}{p\lambda^2}\times\sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha=2,\beta=2, \\
= \quad & \frac{p+1}{p\lambda^2} + \frac{1}{\lambda^2}\left[\frac{\alpha(\alpha-1)}{\alpha-2}(\psi'(1)-\psi'(\alpha-1)+(\psi(\alpha-1)-\psi(1))^2)\right] - \\
& \frac{\alpha}{\lambda^2}\left[(\psi'(1)-\psi'(\alpha))+(\psi(\alpha)-\psi(1))^2\right] + \frac{4}{p\lambda^2}\times\sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha\neq 2,\beta=2, \\
= \quad & \frac{p+1}{p\lambda^2} + \frac{1}{p\lambda^2}\left[\frac{\beta(\beta-1)}{\beta-2}(\psi'(1)-\psi'(\beta-1)+(\psi(\beta-1)-\psi(1))^2)\right]
\end{aligned}$$

$$-\frac{\beta}{p\lambda^2}\left[(\psi'(1)-\psi'(\beta))+(\psi(\beta)-\psi(1))^2\right]+\frac{4}{\lambda^2}\times\sum_{k=0}^{\infty}\frac{1}{(2+k)^3} \quad \text{if} \quad \alpha=2, \beta\neq 2.$$

Proof of Theorem 1: The proof follows by expanding the derivative of the log-likelihood function using Taylor series and using the Central Limit theorem.

Now, we have the main result:

Theorem 2: As $m \to \infty$ and $n \to \infty$ so that $\frac{m}{n} \to p$, then

$$\sqrt{m}\left(\hat{R}-R\right) \to N(0,B), \tag{3.3}$$

where

$$B=\frac{1}{u(\alpha+\beta)^4}\left[\beta^2(a_{22}a_{33}-a_{23}^2)-2\alpha\beta\sqrt{p}a_{23}a_{31}+\alpha^2 p(a_{11}a_{33}-a_{13}^2)\right]$$

and

$$u=a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{31}.$$

Proof of Theorem 2: It follows using Theorem 1.

Remark 1: In Theorem 1, the normalizing constants $\sqrt{m}$ and $\sqrt{n}$ can be interchanged and the necessary changes are required in the corresponding dispersion matrix.

Remark 2: Theorem 2 can be used to construct asymptotic confidence intervals. To compute the confidence interval of R, the variance B needs to be estimated. We recommend to use the empirical Fisher information matrix and the MLE estimates of α, β and λ to estimate B, which is very convenient. Use,

$$\hat{a}_{11}=\frac{1}{\hat{\alpha}^2}, \quad \hat{a}_{22}=\frac{1}{\hat{\beta}^2}, \quad \hat{a}_{13}=\hat{a}_{31}=-\frac{1}{m}\sum_{i=1}^{m}\frac{y_i e^{-\hat{\lambda}y_i}}{(1-e^{-\hat{\lambda}y_i})}, \quad \hat{a}_{23}=\hat{a}_{32}=-\frac{\sqrt{p}}{m}\sum_{j=1}^{n}\frac{x_j e^{-\hat{\lambda}x_j}}{(1-e^{-\hat{\lambda}x_j})}$$

$$\hat{a}_{33}=\frac{1}{\lambda^2}\left(1+\frac{1}{p}\right)+(\hat{\alpha}-1)\frac{1}{m}\sum_{i=1}^{m}\frac{y_i^2 e^{-\hat{\lambda}y_i}}{(1-e^{-\hat{\lambda}y_i})^2}+(\hat{\beta}-1)\frac{1}{m}\sum_{j=1}^{n}\frac{x_j^2 e^{-\hat{\lambda}x_j}}{(1-e^{-\hat{\lambda}x_j})^2}.$$

It is observed that the confidence intervals based on the asymptotic result do not perform very well for small sample sizes. We propose the following two confidence intervals mainly for small sample sizes, which might be computationally very demanding for large samples.

3.1 Bootstrap Confidence Intervals

In this subsection, we propose to use two confidence intervals based on the parametric bootstrap methods; (i) percentile bootstrap method (we call it from now on as Boot-p) based on the idea of Efron (1982), (ii) bootstrap-t method (we refer it as Boot-t from now on) based on the idea of Hall (1988). We illustrate briefly how to estimate confidence intervals of R using both methods.

Boot-p Methods:

Step 1: From the sample $\{y_1, \ldots, y_m\}$ and $\{x_1, \ldots x_n\}$, compute $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

Step 2: Using $\hat{\alpha}$ and $\hat{\lambda}$ generate a bootstrap sample $\{y_1^*, \ldots, y_m^*\}$ and similarly using $\hat{\beta}$ and $\hat{\lambda}$ generate a bootstrap sample $\{x_1^*, \ldots x_n^*\}$. Based on $\{y_1^*, \ldots, y_m^*\}$ and $\{x_1^*, \ldots x_n^*\}$ compute the bootstrap estimate of R using (2.11), say $\hat{R}^*$.

Step 3: Repeat step 2, NBOOT times.

Step 4: Let $G(x) = P(\hat{R}^* \le x)$, be the cumulative distribution function of $\hat{R}^*$. Define $\hat{R}_{Boot-p}(x) = G^{-1}(x)$ for a given x. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-p}(\frac{\gamma}{2}), \hat{R}_{Boot-p}(1-\frac{\gamma}{2})\right). \tag{3.4}$$

Bootstrap-t Confidence Interval

Step 1: From the sample $\{y_1, \ldots, y_m\}$ and $\{x_1, \ldots x_n\}$, compute $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

Step 2: Using $\hat{\alpha}$ and $\hat{\lambda}$ generate a bootstrap sample $\{y_1^*, \ldots, y_m^*\}$ and similarly using $\hat{\beta}$ and $\hat{\lambda}$ generate a bootstrap sample $\{x_1^*, \ldots x_n^*\}$ as before. Based on $\{y_1^*, \ldots, y_m^*\}$ and $\{x_1^*, \ldots x_n^*\}$ compute the bootstrap estimate of R using (2.11), say $\hat{R}^*$ and the following statistic:

$$T^* = \frac{\sqrt{m}(\hat{R}^* - \hat{R})}{\sqrt{V(\hat{R}^*)}}.$$

Compute $V(\hat{R}^*)$ using Remark 2.

Step 3: Repeat step 2, NBOOT times.

Step 4: From the NBOOT T* values obtained, determine the upper and lower bound of the $100(1-\gamma)\%$ confidence interval of R as follows: Let $H(x) = P(T^* \leq x)$ be the cumulative distribution function of T^*. For a given x, define

$$\hat{R}_{Boot-t} = \hat{R} + m^{-\frac{1}{2}}\sqrt{V(\hat{R})}H^{-1}(x).$$

Here also, $V(\hat{R})$ can be computed as mentioned in Remark 2. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-t}(\frac{\gamma}{2}), \hat{R}_{Boot-t}(1-\frac{\gamma}{2})\right).$$

4 Estimation of R if λ is Known

In this section we consider the estimation of R when λ is known. Without loss of generality, we can assume that $\lambda = 1$. Therefore, in this section it is assumed that $X_1, \ldots X_n$ is a random sample from $GE(\beta, 1)$ and $Y_1, \ldots, Y_m$ is a random sample from $GE(\alpha, 1)$ and based on the samples we want to estimate R. First we consider the MLE of R and its distributional properties.

4.1 MLE of R

Based on the above samples, it is clear that the MLE of R, say $\hat{R}$ will be

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}}, \tag{4.1}$$

where

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^{m} \ln(1 - e^{-Y_i})}, \tag{4.2}$$

$$\hat{\beta} = -\frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-X_j})}. \tag{4.3}$$

Therefore,

$$\hat{R} = \frac{n \sum_{i=1}^{m} \ln(1 - e^{-Y_i})}{n \sum_{i=1}^{m} \ln(1 - e^{-Y_i}) + m \sum_{j=1}^{n} \ln(1 - e^{-X_j})}. \tag{4.4}$$

It is observed in Gupta and Kundu (2002), that

$$-2\beta \sum_{j=1}^{n} \ln(1 - e^{-X_j}) \sim \chi^2_{2n} \quad \text{and} \quad -2\alpha \sum_{i=1}^{m} \ln(1 - e^{-Y_i}) \sim \chi^2_{2m}.$$

Therefore,

$$\hat{R} \stackrel{d}{=} \frac{V}{V + cU} \stackrel{d}{=} \frac{1}{1 + \frac{\alpha}{\beta} Z}, \quad \text{or} \quad \frac{R}{1 - R} \times \frac{1 - \hat{R}}{\hat{R}} \stackrel{d}{=} Z, \tag{4.5}$$

here $\stackrel{d}{=}$ indicates equivalent in distribution and $c = \frac{m\alpha}{n\beta}$. The random variables U and V are independent and follow χ^2 distribution, with $2n$ and $2m$ degrees of freedom respectively. Moreover, Z has an F distribution with $2n$ and $2m$ degrees of freedom. Therefore, the PDF of $\hat{R}$ is as follows;

$$f_{\hat{R}}(x) = k \times \frac{\left(\frac{1-x}{x}\right)^{n-1}}{\left(1 + \frac{n\beta(1-x)}{m\alpha x}\right)^{m+n}}, \qquad 0 < x < 1,$$

where

$$k = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \left(\frac{\beta}{\alpha}\right)^{n-1} \left(\frac{n}{m}\right)^{n}.$$

The 100(1-γ)% confidence interval of R can be obtained as

$$\left[\frac{1}{1 + F_{2m,2n;1-\frac{\alpha}{2}} \times \left(\frac{1}{\hat{R}} - 1\right)}, \frac{1}{1 + F_{2m,2n;\frac{\alpha}{2}} \times \left(\frac{1}{\hat{R}} - 1\right)}\right], \tag{4.6}$$

where, $F_{2m,2n;\frac{\gamma}{2}}$ and $F_{2m,2n;1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}^{th}$ percentile points of a F distribution with $2m$ and $2n$ degrees of freedom.

4.2 UMVUE of R

In this subsection we obtain the UMVUE of R using the results of Tong (1974, 1975). When the common scale parameter λ is known, $\left(\sum_{i=1}^{n}\ln(1-e^{-X_i}), \sum_{i=1}^{m}\ln(1-e^{-Y_i})\right)$ is a jointly sufficient statistic for (β, α). Let us define

$$\phi(X_1, Y_1) = 1 \quad \text{if} \quad \ln(1-e^{-Y_1}) > \ln(1-e^{-X_1})$$

and 0 otherwise. Note that $R = P(Y < X) = E(1-\phi(X_1, Y_1))$. Clearly, $1-\phi(X_1, Y_1)$ is an unbiased estimator of R. Therefore, the UMVUE of R, say $\tilde{R}$, can be obtained as

$$\tilde{R} = E\left[1-\phi(X_1, Y_1)|\left(\sum_{i=1}^{n}\ln(1-e^{-X_i}), \sum_{i=1}^{m}\ln(1-e^{-Y_i})\right)\right].$$

Since $-\ln\left(1-e^{-Y_1}\right)$ and $-\ln(1-e^{-X_1})$ are exponential random variables with mean $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ respectively, therefore using the results of Tong (1974, 1975) it follows that

$$\tilde{R} = 1 - \sum_{i=0}^{n-1}(-1)^i \frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}\left(\frac{T_2}{T_1}\right)^i, \quad \text{if} \quad T_2 \leq T_1$$

or

$$\tilde{R} = \sum_{i=0}^{m-1}(-1)^i \frac{(m-1)!(n-1)!}{(m-i-1)!(n+i-1)!}\left(\frac{T_1}{T_2}\right)^i, \quad \text{if} \quad T_1 \leq T_2,$$

where $T_1 = -\sum_{i=1}^{n}\ln(1-e^{-X_i})$ and $T_2 = -\sum_{i=1}^{m}\ln(1-e^{-Y_i})$.

4.3 Bayes Estimation of R

In this subsection, we obtain the Bayes estimation of R under the assumptions that the shape parameters α and β are random variables for both the populations. It is assumed that α and β have independent gamma priors with the PDF's;

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}; \quad \alpha > 0, \tag{4.7}$$

$$\pi(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta}; \quad \beta > 0, \tag{4.8}$$

respectively. Here $a_1, b_1, a_2, b_2 > 0$. Therefore, α and β follow Gamma(a_1, b_1) and Gamma(a_2, b_2) respectively. The posterior PDF's of α and β are as follows:

$$\alpha|\text{data} \sim \text{Gamma}(a_1 + m, b_1 + T_1), \tag{4.9}$$

$$\beta|\text{data} \sim \text{Gamma}(a_2 + n, b_2 + T_2), \tag{4.10}$$

where

$$T_1 = -\sum_{i=1}^{m} \ln(1 - e^{-Y_i}), \quad \text{and} \quad T_2 = -\sum_{j=1}^{n} \ln(1 - e^{-X_j}).$$

Since *a priori* α and β are independent, using (4.9) and (4.10), the posterior PDF of R becomes

$$f_R(r) = C\frac{r^{a_2+n-1}(1-r)^{a_1+m-1}}{((b_1+T_1)(1-r)+(b_2+T_2)r)^{m+n+a_1+a_2}}, \quad \text{for} \quad 0 < r < 1, \tag{4.11}$$

and 0 otherwise, where

$$C = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(a_1+m)\Gamma(a_2+n)}(b_1+T_1)^{a_1+m}(b_2+T_2)^{a_2+n}.$$

It is important to mention that, it is not possible to obtain the explicit expressions for the posterior mean or median. On the other hand the posterior mode can be easily obtained. Note that, $\frac{d}{dr}\ln f_R(r) = 0$ has only two roots. Using the fact that $\lim_{r\to 0+}\frac{d}{dr}\ln f_R(r) > 0$ and $\lim_{r\to 1-}\frac{d}{dr}\ln f_R(r) < 0$, it easily follows that the density function $f_R(r)$ has a unique mode. The posterior mode can be obtained as the unique root which lies between 0 and 1 of the following quadratic equation:

$$2r^2(B_1 - B_2) + r(2B_2 - 2B_1 + A_1B_2 + A_2B_1) - A_2B_1 = 0, \tag{4.12}$$

where $B_1 = b_1 + T_1$, $B_2 = b_2 + T_2$, $A_1 = a_1 + m - 1$, $A_2 = a_2 + n - 1$.

Now, consider the following loss function:

$$L(a,b) = \begin{cases} 0 & \text{if} \quad |a-b| \le c \\ 1 & \text{if} \quad |a-b| > c. \end{cases} \tag{4.13}$$

It is known that the Bayes estimate with respect the above loss function (4.13) is the midpoint of the 'modal interval' of length $2c$ of the posterior distribution, see (Ferguson, T.S.; 1967, page 51, problem 5). Therefore, the posterior mode is an approximate Bayes estimator of R with respect to the loss function (4.13) when the constant c is small.

The credible interval of R can be obtained by using numerical integration. Alternatively, the MCMC method proposed by Chen and Shao (1999) can be used to compute the highest posterior density (HPD) interval. It is not pursued here.

As we had mentioned before, the Bayes estimate of R under squared error loss can not be computed analytically. Alternatively, using the approximation of Lindley (1980) and following the approach of Ahmad, Fakhry and Jaheen (1997), it can be easily seen that the approximate Bayes estimate of R, say $\hat{R}_{BS}$, under squared error loss is

$$\hat{R}_{BS} = \tilde{R}\left[1 + \frac{\tilde{\alpha}\tilde{R}^2}{\tilde{\beta}^2(n+a_2-1)(m+b_1-1)} \times \Big(\tilde{\alpha}(m+a_1-1) - \tilde{\beta}(n+a_2-1)\Big)\right], \qquad (4.14)$$

where

$$\tilde{\beta} = \frac{n+a_2-1}{b_2+T_2}, \quad \tilde{\alpha} = \frac{m+a_1-1}{b_1+T_1}, \quad \tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha}+\tilde{\beta}}.$$

5 Numerical Experiments and Discussions

In this section we present some results based on Monte Carlo simulations to compare the performance of the different methods mainly for small sample sizes. All computations were performed at the Indian Institute of Technology Kanpur using Pentium III processor. All the programs are written in FORTRAN-77 and these are available on request from the authors. We have used the random deviate generator of Press *et al.* (1992) for all cases.

We consider two cases separately to draw inference on R, namely when (i) λ is unknown and (ii) λ is known. In both cases we consider the following small sample sizes; $(m, n) =$

(15,15), (20,20), (25,25), (15,20), (20,15), (15,25), (25,15), (20,25), (25,20). In both cases we take $\alpha = 1.50$ and $\beta = 2.00$, 2.50, 3.00, 3.50 and 4.00 respectively. Without loss of generality we take $\lambda = 1$ in all cases considered. All the results are based on 1000 replications.

First we consider the case when the common scale parameter λ is unknown. From the sample, we compute the estimate of λ using the iterative algorithm (2.9). We have used the initial estimate to be 1 and the iterative process stops when the difference between the two consecutive iterates are less than 10^{-6}. Once we estimate λ, we estimate α and β using (2.6) and (2.7) respectively. Finally we obtain the MLE of R using (2.11). We report the average biases and mean squared errors (MSEs) over 1000 replications. We also compute the 95% confidence intervals based on the asymptotic distribution of $\hat{R}$ and based on the estimation of B using two different methods. First we estimate B using Theorem 2 and replacing the parameters by their estimates. We also estimate B as we had suggested in Remark 2. We report the average confidence lengths and the coverage percentages based on 1000 replications. We report the 95% confidence intervals based on Boot-p and Boot-t methods. The bootstrap intervals are obtained using 100 bootstrap replications in both cases. For both bootstrap methods, we report the average confidence lengths and the coverage percentages. All the results are reported in Table 1.

Some of the points are quite clear from this experiment. Even for small sample sizes, the performance of the MLEs are quite satisfactory in terms of biases and MSEs. It is observed that when $m = n$ and m, n increase then MSEs decrease. It verifies the consistency property of the MLE of R. For fixed n, as m increases, the MSEs decreases. Similarly, for fixed m, as n increases the MSEs decrease as expected. Comparing the average confidence lengths and coverage percentages, it is observed that estimating B either by substituting the parameters by their estimates in Theorem 2 or by using Remark 2 lead to very similar estimates. The confidence intervals based on the MLEs work quite well unless the sample size is very small, say (15,15). It is observed (not reported here) that when the sample size is (30,30) then the coverage percentages of the confidence intervals based on the asymptotic results reach the nominal level. The performance of the bootstrap confidence intervals are quite good. Particularly, Boot-p intervals perform very well. It reaches the nominal level even when the sample size is small namely (15,15). It also has the shorter average confidence lengths than the corresponding Boot-t confidence intervals. But construction of bootstrap confidence intervals are computationally more demanding than the asymptotic confidence intervals. Based on all these, we recommend to use the parametric percentile bootstrap confidence intervals if min{m,n} is less than or equal to 20, otherwise use the asymptotic confidence intervals and estimate B using Remark 2 in those cases.

Now let us consider the case when the common scale parameter is known. In this case we obtain the estimates of R by using the MLE and UMVUE. We do not have any prior information on R, and therefore, we prefer to use the non-informative prior to compute different Bayes estimates. Since the non-informative prior, *i.e.* $a_1 = a_2 = b_1 = b_2 = 0$ provides prior distributions which are not proper, we adopt the suggestion of Congdon (2001, page 20), *i.e.*, choose $a_1 = a_2 = b_1 = b_2 = 0.0001$, which are almost like Jeffreys priors, but they are proper. Under the same prior distributions, we compute Bayes estimates and approximate Bayes estimates of R under different loss functions. We report the average estimates and the MSEs based on 1000 replications. The results are reported in Table 2. In this case as expected for all the methods when $m = n$ and m, n increase then the average biases and the MSEs decrease. For fixed m as n increases the MSEs decrease and also for fixed n as m decreases the MSEs decrease. Therefore, if we do not have any prior information about α and β then using Bayes estimates we may not gain much as expected. Since for the MLE, the exact distribution is also known and it can be used for constructing confidence intervals also, we recommend to use MLEs in this case.

6 Data Analysis

In this section, we present a complete analysis of a simulated data. The data has been generated using $m = n = 20$, $\alpha = 1.5$, $\beta = 2.5$ and $\lambda = 0.5$. Therefore, $R = 0.625$. The data has been truncated after two decimal places and it has been presented below. The Y values are 2.58, 3.61, 0.96, 5.55, 6.31, 0.47, 2.30, 0.08, 0.88, 2.90, 2.13, 4.01, 2.01, 1.22, 2.51, 0.92, 1.06, 1.02, 0.66, 1.76 and the corresponding X values are 1.70, 2.11, 2.50, 3.77, 1.41, 3.67, 3.00, 2.59, 1.29, 1.86, 0.64, 0.93, 3.28, 2.69, 0.64, 5.17, 12.24, 1.91, 3.09, 3.21.

First, we estimate the parameters assuming λ is unknown. We plot the profile likelihood function of λ in Figure 1. It is an upside down function and it has a unique minimum. The proposed iterative procedure should work with a reasonable starting value. We obtain the MLE of λ using the iterative procedure (2.9). We use the stopping criterion that the iteration stops whenever two consecutive values are less than 10^{-6}. Starting with the initial guess 1.00, and using the above stopping criterion, the iteration stops after 7 steps and it provides $\hat{\lambda} = 0.623$. Now, using (2.6) and (2.7), we obtain $\hat{\alpha} = 1.62$ and $\hat{\beta} = 2.77$. Therefore, $\hat{R} = 0.631$. The 95% asymptotic confidence interval becomes (0.484, 0.778). Using the parametric percentile bootstrap method, we obtain the 95% confidence interval as (0.489, 0.780) and similarly using the percentile bootstrap-t confidence interval becomes (0.497, 0.826).

Now, we estimate the parameters assuming λ is known to be 0.5. We obtain the MLE estimates of α and β as, 1.35 and 2.15 respectively. Therefore, the MLE of R becomes $\hat{R} = 0.615$. The corresponding 95% confidence interval based on (4.6) is (0.460, 0.750). The UMVUE of R is 0.617. Assuming the $0-1$ loss function and under the same prior distribution as proposed before, the approximate Bayes estimate is 0.609. We plot the posterior PDF, (4.11), of R for the given data in Figure 2. For computational simplicity, in this case we assume the posterior PDF to be symmetric and based on that numerically we obtain the

95% credible interval of R as (0.459, 0.749). Interestingly, the asymptotic confidence interval and the corresponding credible interval are almost identical.

Under squared error loss function, based on (4.14), the approximate Bayes estimate is 0.612. Numerically we compute the exact Bayes estimate, which is the mean of the posterior PDF, and it is also 0.612. Since the posterior PDF is almost symmetric in nature, therefore, in this case the 95% credible interval of R, based on squared error loss function will be same as above. Hence, for the given data set, the Bayes estimates are almost identical. They match up to second decimal places.

7 Conclusions

In this paper, we have addressed the problem of estimating $P(Y < X)$ for the Generalized Exponential distribution when they have the same scale parameter. We consider both the cases, namely the common scale parameter is known or unknown. When the common scale parameter is unknown, it is observed that the maximum likelihood estimator works quite well. We can use the asymptotic distribution of the maximum likelihood estimator to construct confidence intervals and that also works quite well, unless when the sample size is very small. When the sample size is very small we recommend to use the parametric percentile Bootstrap confidence interval, whose performance is very good even for very small sample sizes.

When the common scale parameter is known we propose maximum likelihood estimator and uniformly minimum variance unbiased estimator. We also propose two approximate Bayes estimators based on 0-1 and squared error loss functions. It is observed that their performance are quite similar in nature, although based on mean squared errors, the performance of the MLEs are marginally better than the rest. The data analysis indicates that the confidence interval based on the exact distribution of the MLE and the corresponding credible intervals based on non-informative priors are almost identical at least for equal sample sizes, although we could not prove it theoretically. More work is needed in that direction.

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Table 1

Biases, MSEs, Confidence Lengths and Coverage Percentages*

S.S.	2.00	2.50	3.00	3.50	4.00
(15,15)	- 0.0016(0.0089)	-0.0002(0.0083)	0.0008(0.0076)	0.0016(0.0070)	0.0022(0.0064)
	0.3424(91%)	0.3308(91%)	0.3171(91%)	0.3032(91%)	0.2898(91%)
	0.3405(96%)	0.3202(96%)	0.3007(95%)	0.2832(96%)	0.2677(95%)
	0.3847(92%)	0.3620(92%)	0.3409(91%)	0.3228(88%)	0.3073(87%)
(20,20)	0.0002(0.0065)	0.0009(0.0060)	0.0018(0.0055)	0.0025(0.0050)	0.0031 (0.0045)
	0.2987(93%)	0.2884(93%)	0.2763(92%)	0.2641(92%)	0.2523(92%)
	0.3013(95%)	0.2829(95%)	0.2657(95%)	0.2510(94%)	0.2385(94%)
	0.3361(95%)	0.3191(95%)	0.3030(94%)	0.2898(94%)	0.2794(93%)
(25,25)	-0.0007(0.0052)	0.0002(0.0049)	0.0010(0.0045)	0.0015(0.0041)	0.0019(0.0038)
	0.2680(93%)	0.2592(94%)	0.2489(93%)	0.2374(92%)	0.2275(93%)
	0.2683(94%)	0.2590(94%)	0.2483(93%)	0.2373(92%)	0.2268(92%)
	0.3013(95%)	0.2858(95%)	0.2698(96%)	0.2552(95%)	0.2423(95%)
	0.3321(95%)	0.3183(95%)	0.3037(95%)	0.2909(95%)	0.2800(95%)
(15,20)	0.0012(0.0076)	0.0000(0.0071)	0.0009(0.0065)	0.0016(0.0060)	0.0021(0.0054)
	0.3215(92%)	0.3103(92%)	0.2973(91%)	0.2840(92%)	0.2712(91%)
	0.3003(94%)	0.2816(95%)	0.2643(95%)	0.2488(94%)	0.2347(94%)
	0.3312(92%)	0.3139(90%)	0.2983(90%)	0.2845(88%)	0.2720(88%)
(20,15)	0.0031(0.0075)	0.0042(0.0071)	0.0050(0.0065)	0.0055(0.0060)	0.0059(0.0054)
	0.3211(92%)	0.3097(92%)	0.2965(91%)	0.2833(91%)	0.2706(91%)
	0.3118(93%)	0.2891(94%)	0.2700(93%)	0.2543(94%)	0.2410(95%)
	0.3582(90%)	0.3380(90%)	0.3216(89%)	0.3085(89%)	0.2972(89%)
(15,25)	-0.0050(0.0067)	-0.0037(0.0063)	-0.0027(0.0058)	-0.0019(0.0053)	–0.0013(0.0048)
	0.3085(93%)	0.2980(93%)	0.2857(93%)	0.2730(93%)	0.2607(93%)
	0.2972(91%)	0.2861(93%)	0.2734(94%)	0.2612(94%)	0.2497(94%)
	0.3222(91%)	0.3093(94%)	0.2963(95%)	0.2851(95%)	0.2748(95%)
(25,15)	0.0031(0.0068)	0.0042(0.0062)	0.0049(0.0057)	0.0055(0.0052)	0.0059(0.0047)
	0.3077(93%)	0.2968(93%)	0.2842(93%)	0.2715(92%)	0.2593(92%)
	0.3123(95%)	0.2981(95%)	0.2838(95%)	0.2698(95%)	0.2561(95%)
	0.3539(93%)	0.3441(93%)	0.3337(92%)	0.3224(92%)	0.3108(92%)
(20,25)	0.0030(0.0052)	0.0039(0.0048)	0.0045(0.0045)	0.0049(0.0041)	0.0051(0.0038)
	0.2839(93%)	0.2737(93%)	0.2620(92%)	0.2501(92%)	0.2388(92%)
	0.2840(92%)	0.2676(91%)	0.2517(92%)	0.2370(93%)	0.2239(91%)
	0.2815(91%)	0.2711(92%)	0.2610(91%)	0.2415(93%)	0.2267(91%)
(25,20)	0.0032(0.0049)	0.0042(0.0046)	0.0048(0.0043)	0.0053(0.0039)	0.0055(0.0036)
	0.2843(94%)	0.2741(93%)	0.2625(93%)	0.2506(93%)	0.2394(93%)
	0.2889(93%)	0.2727(91%)	0.2575(92%)	0.2435(91%)	0.2310(92%)
	0.2799(92%)	0.2743(92%)	0.2611(92%)	0.2511(93%)	0.2355(93%)

* In each cell the first row represents the average biases of the MLEs and the corresponding mean squared error is reported within bracket. Similarly the second, third and fourth rows represent the average confidence lengths and coverage percentages based on estimated asymptotic distribution, Boot-p and Boot-t methods.

Table 2

Biases and Mean Squared Errors of the MLEs, UMVUEs, and Two Approximate Bayes Estimates**

S.S.	2.00	2.50	3.00	3.50	4.00
(15,15)	-0.0037(0.0074)	-0.0050(0.0067)	-0.0058(0.0059)	-0.0062(0.0053)	-0.0065(0.0046)
	-0.0029(0.0078)	-0.0027(0.0070)	-0.0025(0.0061)	-0.0024(0.0054)	-0.0022(0.0048)
	-0.0010(0.0088)	0.0060(0.0081)	0.0017(0.0073)	0.0024(0.0065)	0.0030(0.0058)
	-0.0072(0.0070)	-0.0097(0.0064)	-0.0113(0.0057)	-0.0122(0.0052)	-0.0127(0.0047)
(20,20)	-0.0028(0.0055)	-0.0037(0.0049)	-0.0043(0.0043)	-0.0046(0.0038)	-0.0048(0.0034)
	-0.0015(0.0057)	-0.0014(0.0050)	-0.0013(0.0044)	-0.0012(0.0038)	-0.0011(0.0033)
	0.0000(0.0064)	0.0012(0.0058)	0.0020(0.0052)	0.0026(0.0046)	0.0030(0.0041)
	-0.0054(0.0053)	-0.0073(0.0047)	-0.0085(0.0042)	-0.0091(0.0038)	-0.0094(0.0034)
(25,25)	-0.0012(0.0043)	-0.0020(0.0038)	-0.0025(0.0034)	0.0028(0.0030)	-0.0030(0.0026)
	-0.0020(0.0046)	-0.0019(0.0041)	-0.0018(0.0036)	0.0017(0.0032)	-0.0016(0.0028)
	-0.0008(0.0052)	0.0001(0.0047)	0.0008(0.0042)	0.0013(0.0038)	0.0016(0.0034)
	-0.0033(0.0042)	-0.0049(0.0037)	-0.0058(0.0033)	-0.0064(0.0029)	-0.0068(0.0026)
(15,20)	-0.0055(0.0064)	-0.0064(0.0057)	-0.0070(0.0051)	-0.0072(0.0045)	-0.0073(0.0040)
	-0.0005(0.0067)	-0.0005(0.0059)	-0.0004(0.0052)	-0.0004(0.0046)	-0.0004(0.0040)
	-0.0008(0.0075)	0.0006(0.0069)	0.0016(0.0062)	0.0023(0.0055)	0.0028(0.0049)
	-0.0067(0.0061)	-0.0088(0.0055)	-0.0102(0.0050)	-0.0109(0.0044)	-0.0113(0.0040)
(20,15)	-0.0022(0.0063)	-0.0033(0.0056)	-0.0040(0.0049)	-0.0045(0.0044)	-0.0047(0.0039)
	-0.0007(0.0067)	-0.0006(0.0059)	-0.0006(0.0052)	-0.0005(0.0045)	-0.0005(0.0040)
	0.0032(0.0075)	0.0044(0.0068)	0.0052(0.0061)	0.0057(0.0055)	0.0059(0.0049)
	-0.0071(0.0061)	-0.0093(0.0055)	-0.0106(0.0049)	-0.0113(0.0044)	-0.0117(0.0039)
(15,25)	-0.0059(0.0059)	-0.0067(0.0053)	-0.0072(0.0047)	-0.0074(0.0042)	-0.0074(0.0037)
	-0.0024(0.0060)	-0.0022(0.0054)	-0.0020(0.0047)	-0.0019(0.0042)	-0.0018(0.0037)
	-0.0042(0.0067)	-0.0027(0.0062)	-0.0016(0.0056)	-0.0008(0.0050)	-0.0002(0.0045)
	-0.0057(0.0056)	-0.0077(0.0051)	-0.0090(0.0046)	-0.0097(0.0041)	-0.0100(0.0037)
(25,15)	-0.0006(0.0059)	-0.0017(0.0052)	-0.0025(0.0046)	-0.0030(0.0041)	-0.0033(0.0036)
	-0.0014(0.0059)	-0.0013(0.0052)	-0.0012(0.0045)	-0.0011(0.0039)	-0.0010(0.0034)
	0.0035(0.0067)	0.0046(0.0060)	0.0053(0.0054)	0.0057(0.0047)	0.0060(0.0042)
	-0.0063(0.0057)	-0.0084(0.0051)	-0.0096(0.0046)	-0.0103(0.0041)	-0.0107(0.0037)
(20,25)	-0.0042(0.0049)	-0.0049(0.0044)	-0.0053(0.0039)	-0.0055(0.0034)	-0.0056(0.0030)
	0.0028(0.0046)	0.0027(0.0041)	0.0026(0.0035)	0.0025(0.0031)	0.0024(0.0027)
	0.0030(0.0052)	0.0041(0.0047)	0.0048(0.0042)	0.0052(0.0037)	0.0054(0.0033)
	-0.0054(0.0047)	-0.0071(0.0043)	-0.0081(0.0038)	-0.0086(0.0034)	-0.0089(0.0030)
(25,20)	0.0014(0.0048)	-0.0023(0.0043)	-0.0028(0.0037)	-0.0032(0.0033)	-0.0034(0.0029)
	0.0007(0.0044)	0.0009(0.0039)	0.0009(0.0034)	0.0009(0.0030)	0.0009(0.0026)
	0.0034(0.0049)	0.0044(0.0045)	0.0051(0.0040)	0.0055(0.0036)	0.0057(0.0032)
	-0.0049(0.0046)	-0.0066(0.0041)	-0.0076(0.0037)	-0.0082(0.0033)	-0.0085(0.0030)

** In each cell the first, second, third and fourth rows represent the average biases and mean squared errors of the MLEs, UMVUEs, approximate Bayes estimates based on 0-1 loss function (Mode) and approximate Bayes estimate bases on squared error loss functions respectively (App).

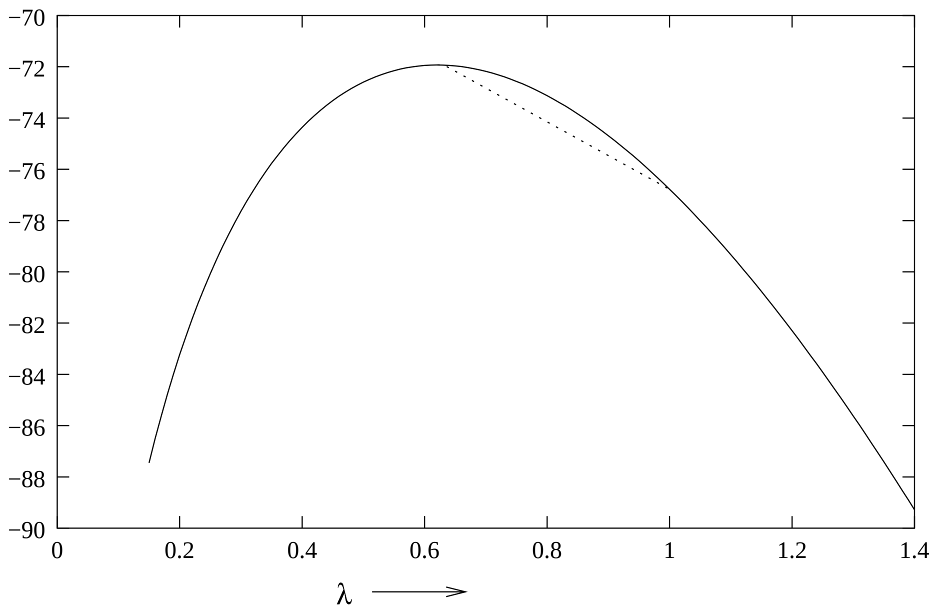

Figure 1: The profile likelihood of λ for the given data set and it shows how the iterative process works

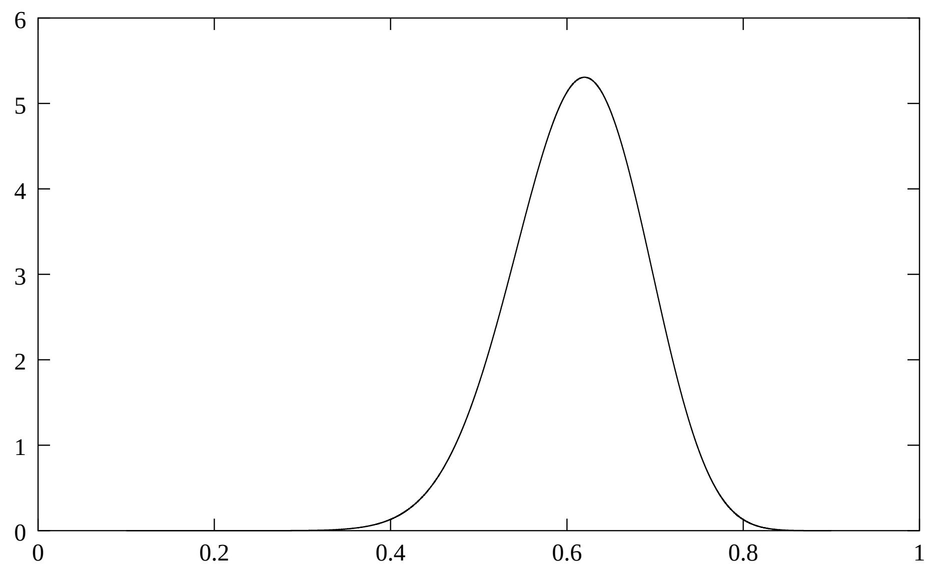

Figure 2: Posterior probability density function of R for the given data set