ASYMPTOTIC PROPERTIES OF THE LEAST SQUARES ESTIMATORS OF MULTIDIMENSIONAL EXPONENTIAL SIGNALS

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Abstract: In this paper we investigate the theoretical properties of the least squares estimators of the multidimensional exponential signals under the assumptions of additive errors. The strong consistency and asymptotic normality of the least squares estimators of the different parameters are obtained. We discuss two particular cases in details. It is observed that several one or two dimensional results follow from our results. Finally we discuss some open problems.

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1. INTRODUCTION:

There has been intensive studies recently on estimation of the parameters of multidimensional exponential signals. In many applications, such as synthetic aperture, radar imaging, frequency and wave number estimation in array processing and nuclear magnetic resonance imaging, it is often desired to estimate multidimensional frequencies from multidimensional data. For recent literature on the topic of multidimensional exponential signal parameters estimation, the reference may be made to the works of Bose (1985), McClellan (1982), Barbieri and Barone (1992), Cabrera and Bose (1993), Chun and Bose (1995), Dudgeon and Merserasan (1984), Hua (1992), Kay (1980), Kay and Nakovei (1990) and Lang and McClellan (1982). The multidimensional (M-D) superimposed exponential signal model in additive white noise, in its most general form can be written as follows:

$$y(n_1,\ldots,n_M) = \sum_{j_1=1}^{P_1},\ldots,\sum_{j_M=1}^{P_M} A_{j_1\ldots j_M} e^{i(n_1\omega_{1,j_1}+\ldots+n_M\omega_{M,j_M})} + e(n_1,\ldots,n_M) \tag{1.1}$$

for $n_1 = 1,\ldots,N_1,\ldots,n_M = 1,\ldots,N_M$. Here $y(n_1,\ldots,n_M)$'s are the observed noise corrupted signals, $i = \sqrt{-1}$, $\omega_{1,1},\ldots,\omega_{M,P_M} \in (0, 2\pi)$ are the unknown frequencies. $A_{11\ldots1}$, …, $A_{P_1\ldots P_M}$ are the unknown amplitudes and they can be complex also. It is assumed that the noise random variables $e(n_1,\ldots,n_M)$'s are independent and identically distributed (*i.i.d.*) random variables with zero means and finite variances. The problem is to estimate the unknown parameters assuming $P_1,\ldots,P_M$ are known a priori.

It may be mentioned that different particular cases of this model are well studied in signal processing and time series literature. For example, when $M = 1$, the model can be written as

$$y(n) = \sum_{j=1}^{P} A_j e^{i\omega_j n} + e(n). \tag{1.2}$$

This is a very well discussed model in statistical signal processing, see for example the review articles of Rao (1988) and Prasad, Chakraborty and Parthasarathy (1995). The interested readers may consult Stoica (1993) for an extensive list of references up to that time and Kundu and Mitra (1999) for some recent works. The corresponding real version of this model (1.2) takes the form

$$y(t) = \sum_{j=1}^{P} (A_j cos(\omega_j t) + B_j sin(\omega_j t)) + e(t). \tag{1.3}$$

The model (1.3) is a very well discussed model in time series analysis, particularly when $e(t)$'s are from a stationary sequence, see for example Hannan (1971, 1973), Walker (1971), Rife and Boorstyn (1974), Kundu (1993a, 1997), Kundu and Mitra (1996, 1997) and Quinn (1999) and the references cited there. It plays an important role in analyzing many non stationary time series data. When $M = 2$, the model (1.1) can be written as

$$y(n_1, n_2) = \sum_{j_1=1}^{P_1} \sum_{j_2=1}^{P_2} A_{j_1 j_2} e^{i(n_1 \omega_{1 j_1} + n_2 \omega_{2 j_2})} + e(n_1, n_2). \tag{1.4}$$

One particular case of the model (1.4) is

$$y(n_1, n_2) = \sum_{j=1}^{P} A_j e^{i(n_1 \omega_{1j} + n_2 \omega_{2j})} + e(n_1, n_2). \tag{1.5}$$

Both (1.4) and (1.5) have wide variety of applications in Statistical Signal Processing see Rao *et al.* (1994, 1996). Different estimation procedures and the properties of these estimators are studied quite extensively by Cabrera and Bose (1993), Chun and Bose (1995), Kay and Nekovei (1990), Rao *et al.* (1994, 1996) and Kundu and Mitra (1996). Recently it is observed that the following two dimensional model,

$$y(m, n) = \sum_{k=1}^{P} A_k cos(m\lambda_k + n\mu_k) + e(m, n), \tag{1.6}$$

can be used quite effectively in texture classifications. See for example the works of Manderekar and Zhang (1995) and Francos *et al.* (1990). Different estimation procedures and their properties are studied by Manderakar and Zhang (1995), Kundu and Gupta (1998) and Nandi and Kundu (1999).

In this paper we consider the most general model (1.1), and therefore all the other models mentioned in (1.2), (1.3), (1.4), (1.5) and (1.6) are particular cases of this general model. In this paper, we consider the least squares estimators of the different parameters and study their asymptotic properties. We mainly consider the consistency and the asymptotic normality properties of the unknown parameters. We obtain an explicit expression of the asymptotic distribution of the least squares estimators. It is observed that the asymptotic variance covariance matrix coincides with the Fisher Information matrix even in the general situation as it was observed by Rao and Zhao (1993) and Rao *et al.* (1994) for one and two dimensional cases respectively under the assumptions of normality of the error random variables.

We obtain the consistency of the least squares estimators of the general model. Although it seems that the asymptotic normality results should follow along the same line as Rao *et al.* (1994) or Kundu and Mitra (1995), but obtaining the exact asymptotic variance covariance matrix may not be a trivial task. Looking at the variance covariance matrix of the 2-D case (Rao *et al.*; 1994 or Kundu and Mitra; 1996) it becomes immediate that the exact variance covariance matrix may not be in a simple form for the general case. It is observed that if we arrange the parameters in a different way, then for some special cases the asymptotic variance covariance matrix has a more convenient form. We provide some special cases separately because these cases have some practical applications and these results may not be obtained that easily from the general case.

The rest of the paper is organized as follows. In section 2, we provide two special cases and in section 3, we provide the general form. We consider the estimation of the error variance in Section 4. Finally we draw conclusions from our results in section 5. From now on we will be using the following notations. $N = N_1 \ldots N_M$, $\mathbf{n} = (N_1, \ldots, N_M)$ and $N_{(1)} = min\{N_1, \ldots, N_M\}$, $N_{(M)} = max\{N_1, \ldots, N_M\}$ and almost sure convergence will be denoted by $a.s.$. The constant C may indicate different constant at different places. The parameter vector $\boldsymbol{\theta}$ and the matrix $\mathbf{D}$ might be different at different places, but they will be defined properly at each place. In theorems 2, 4, 6 and 8, the dispersion matrices are evaluated at the true parameter values. We have not make it explicit for brevity. It should be clear from the context.

2. TWO SPECIAL CASES:

In this section we consider two special cases. One is the real valued multidimensional sinusoidal model (Model 1) and the other one is the multidimensional complex valued sum of exponential model (Model 2).

Model 1:

$$y(\mathbf{n}) = \sum_{k=1}^{P} A_k cos(n_1\omega_{1k} + \ldots + n_M\omega_{Mk} + \phi_k) + e(\mathbf{n}). \tag{2.1}$$

Here A_k's are arbitrary real numbers, $\omega_{1k} \in (-\pi, \pi)$, $\omega_{2k}, \ldots, \omega_{Mk}, \phi_k \in (0, \pi)$. Here $e(\mathbf{n})$ is a M-dimensional sequence of $i.i.d.$ random variables with zero means and finite variances and 'P', is assumed to be known. The problem is to estimate the unknown parameters A_k's and ω_{jk}'s for $j = 1, \ldots, M$ and $k = 1, \ldots, P$. The least squares estimators can be obtained by minimizing

$$Q(\boldsymbol{\theta}) = \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} \left(y(\mathbf{n}) - \sum_{k=1}^{P} A_k cos(n_1\omega_{1k} + \ldots + n_M\omega_{Mk} + \phi_k) \right)^2 \tag{2.2}$$

with respect to the unknown parameters. We use the following notations after rearranging the parameters

$$\boldsymbol{\theta_1} = (A_1, \phi_1, \omega_{11}, \ldots, \omega_{M1}), \ldots, \boldsymbol{\theta_p} = (A_P, \phi_P, \omega_{1P}, \ldots, \omega_{MP})$$

$$\boldsymbol{\theta} = (\boldsymbol{\theta_1}, \ldots, \boldsymbol{\theta_P}).$$

Let's consider the following assumptions:

Assumption 1: Let $A_1, \dots, A_P$ be arbitrary real numbers, not any one of them is identically equal to zero, $\omega_{1k} \in (-\pi, \pi)$ and ϕ_k, $\omega_{jk} \in (0, \pi)$ for $k = 1, \dots, P$ and $j = 2, \dots M$, and they are all distinct.

Assumption 2: Let $\{e(\mathbf{n})\}$ be a *i.i.d.* sequence of M array real valued random variables with $E(\mathbf{n}) = 0$ and $Var(e(\mathbf{n})) = \sigma^2$.

Theorem 1: Under the assumptions 1 and 2 and as $N_{(M)} \to \infty$, the least squares estimators of the parameters of the model (2.1) which are obtained by minimizing (2.2) are strongly consistent estimators of the corresponding parameters.

We need the following lemmas to prove the above results.

Lemma 1: Let $\{e(\mathbf{n})\}$ be a *i.i.d.* sequence of M-dimensional random variables with mean zero and finite variance, then

$$\lim_{N_{(M)} \to \infty} \sup_{\alpha_1, \dots, \alpha_M} \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \dots \sum_{n_M=1}^{N_M} e(\mathbf{n}) cos(n_1 \alpha_1) \dots cos(n_M \alpha_M) \right| \to 0 \quad a.s. \tag{2.3}$$

Proof of Lemma 1: See Appendix A.

Consider the following set

$$\begin{aligned} S_{\delta,T} = \{\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_M); \quad & |A_1 - A_1^0| > \delta, |\boldsymbol{\theta}| < T \text{ or } \dots |A_M - A_M^0| > \delta, |\boldsymbol{\theta}| < T \text{ or} \\ & |\omega_{11} - \omega_{11}^0| > \delta, |\boldsymbol{\theta}| < T, \text{ or } \dots |\omega_{1P} - \omega_{1P}^0| > \delta, |\boldsymbol{\theta}| < T \text{ or }, \\ & \vdots \\ & |\omega_{M1} - \omega_{M1}^0| > \delta, |\boldsymbol{\theta}| < T, \text{ or } \dots |\omega_{MP} - \omega_{MP}^0| > \delta, |\boldsymbol{\theta}| < T\}. \end{aligned}$$

Here $A_0, \dots, A_M^0, \omega_{11}^0, \dots, \omega_{MP}^0$ are the true value of the parameters.

Lemma 2: If

$$\liminf_{N_{(M)} \to \infty} \inf_{\boldsymbol{\theta} \in S_{\delta,T}} \frac{1}{N} \left(Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta^0})\right) > 0 \qquad a.s.$$

for all $\delta > 0$, then $\hat{\boldsymbol{\theta}}$, the least squares estimator of $\boldsymbol{\theta^0}$, is a strongly consistent estimator of $\boldsymbol{\theta^0}$.

Proof of Lemma 2: The proof is simple see in Appendix C.

Proof of Theorem 1: Expanding the square term of $\boldsymbol{Q(\theta)}$ and using lemma 1, it can be shown that lemma 2 is satisfied. Therefore the result follows. See in Appendix C.

The more important question is what is the asymptotic distribution of the least squares estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ as $N_{(1)} \to \infty$. We use the matrix $\mathbf{D}$ as a $P(M+2)$ diagonal matrix with diagonal elements as

$$\mathbf{D} = diag\{N^{\frac{1}{2}}, N^{\frac{1}{2}}, N_1 N^{\frac{1}{2}}, \ldots, N_M N^{\frac{1}{2}}, \ldots, N^{\frac{1}{2}}, N^{\frac{1}{2}}, N_1 N^{\frac{1}{2}}, \ldots, N_M N^{\frac{1}{2}}\}$$

The following theorem provides the asymptotic distribution of $\hat{\boldsymbol{\theta}}$.

Theorem 2: Under the assumptions 1 and 2 as $N_{(1)} \to \infty$, the limiting distribution of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\boldsymbol{D}$ is a $P(M+2)$ variate normal with mean zero and dispersion matrix $2\sigma^2\ \boldsymbol{\Sigma^{-1}}$, where

$$\boldsymbol{\Sigma^{-1}} = \begin{bmatrix} \boldsymbol{\Sigma_1^{-1}} & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & \vdots \\ 0 & \ldots & 0 & \boldsymbol{\Sigma_P^{-1}} \end{bmatrix}$$

Here $\boldsymbol{\Sigma_i}$ and $\boldsymbol{\Sigma_i^{-1}}$ are both $(M+2) \times (M+2)$ matrices and

$$\mathbf{\Sigma_i} = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & A_i^2 & \frac{1}{2}A_i^2 & & \dots & \frac{1}{2}A_i^2 \\ 0 & \frac{1}{2}A_i^2 & \frac{1}{3}A_i^2 & \frac{1}{4}A_i^2 & \dots & \frac{1}{4}A_i^2 \\ 0 & \frac{1}{2}A_i^2 & \frac{1}{4}A_i^2 & \frac{1}{3}A_i^2 & \dots & \frac{1}{4}A_i^2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \frac{1}{2}A_i^2 & \frac{1}{4}A_i^2 & \frac{1}{4}A_i^2 & \dots & \frac{1}{3}A_i^2 \end{bmatrix} \qquad \mathbf{\Sigma_i^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_i & b_i & b_i & \dots & b_i \\ 0 & b_i & c_i & 0 & \dots & 0 \\ 0 & b_i & 0 & c_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_i & 0 & 0 & \dots & c_i \end{bmatrix}$$

$$a_i = \frac{3(M-1)+4}{A_i^2}, \qquad b_1 = -\frac{6}{A_i^2}, \qquad c_i = \frac{1}{A_i^2}\frac{36(M-1)+48}{3(M-1)+4}.$$

Proof of Theorem 2: The main idea is to expand the first derivative vector of $Q(\boldsymbol{\theta})$ by Taylor's series and use the proper normalizing constant to obtain the limiting normal distribution. For each $j = 1, \dots, P(M+2)$,

$$0 = \frac{\partial Q(\hat{\boldsymbol{\theta}})}{\partial \theta_j} = \frac{\partial Q(\boldsymbol{\theta^0})}{\partial \theta_j} + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\frac{\partial^2 Q(\bar{\boldsymbol{\theta}}_j)}{\partial \theta_j \partial \boldsymbol{\theta}^T}, \tag{2.4}$$

where $\bar{\boldsymbol{\theta}}_{\boldsymbol{j}}$ lies on the line segment between $\boldsymbol{\theta}^0$ and $\hat{\boldsymbol{\theta}}$. Now let's define the $1 \times P(M+2)$ vector $Q'(\boldsymbol{\theta}^0)$ whose j^{th} column is $\frac{\partial Q(\boldsymbol{\theta^0})}{\partial \theta_j}$ and the $P(M+2) \times P(M+2)$ matrix $Q''(\bar{\boldsymbol{\theta}}_1, \dots, \bar{\boldsymbol{\theta}}_{P(M+2)})$ whose j^{th} column is $\frac{\partial^2 Q(\bar{\boldsymbol{\theta}}_j)}{\partial \theta_j \partial \boldsymbol{\theta}^T}$. Therefore, from (2.4) we obtain,

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) = -Q'(\boldsymbol{\theta}^0)[Q''(\bar{\boldsymbol{\theta}}_1, \dots, \bar{\boldsymbol{\theta}}_{P(M+2)})]^{-1}$$

For notational simplicity let's define $Q''(\bar{\boldsymbol{\theta}}) = Q''(\bar{\boldsymbol{\theta}}_1, \dots, \bar{\boldsymbol{\theta}}_{P(M+2)})$. Therefore,

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta^0})\mathbf{D} = -Q'(\boldsymbol{\theta^0})\mathbf{D}^{-1}[\mathbf{D}^{-1}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}^{-1}]^{-1}.$$

It can be shown (see in Appendix C) that $Q'(\boldsymbol{\theta^0})\mathbf{D}^{-1}$ converges in distribution to a $P(M+2)$ variate normal distribution with mean zero and dispersion matrix $2\sigma^2\ \mathbf{\Sigma}$, using the central limit theorem and the results of the following types,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} sin^2(\beta t) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n} t sin^2(\beta t) = \frac{1}{4}$$

for $\beta \neq 0$ (see Mangulis; 1965). Similarly it can be shown that $\mathbf{D}^{-1}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}^{-1}$ converges almost surely to $\mathbf{\Sigma}$. Therefore the result follows.

Comments: Because of the rearrangement of the parameters, the M-dimensional result becomes more convenient than the two dimensional result. Also when $M = 2$, the result matches with that of Kundu and Gupta (1998). It is also interesting to note that the vectors $\hat{\boldsymbol{\theta}}_{\boldsymbol{i}}$ and $\hat{\boldsymbol{\theta}}_{\boldsymbol{j}}$ are asymptotically independent. Also the asymptotic variances of $\hat{\omega}_{1i}, \ldots, \hat{\omega}_{Mi}$ are all same and they are inversely proportional to A_i^2. The asymptotic covariance between $\hat{\omega}_{ji}$ and $\hat{\omega}_{ki}$ is zero and as M increases, the variances of both $\hat{\omega}_{ji}$ and $\hat{\omega}_{ki}$ converge to $\frac{12}{A_i^2}$.

Now we describe another particular model:

Model 2:

$$y(\mathbf{n}) = \sum_{k=1}^{P} A_k e^{i(n_1\omega_{1k} + \ldots + n_M\omega_{Mk})} + e(\mathbf{n}). \tag{2.5}$$

This model has its special importance in statistical signal processing. For $M = 2$, this model coincides with the model (1.5). Here A_k's are arbitrary complex numbers and $\omega_{jk} \in (-\pi, \pi)$ and they are all distinct. The error $e(\mathbf{n})$ is a M-dimensional complex valued random variable with mean zero and finite variance. The least squares estimators are obtained by minimizing

$$Q(\boldsymbol{\theta}) = \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} \left| y(\mathbf{n}) - \sum_{k=1}^{P} A_k e^{i(n_1\omega_{1k} + \ldots + n_M\omega_{n_M})} \right|^2.$$

Similarly as Assumptions 1 and 2, we use Assumptions 3 and 4 for the complex case.

Assumption 3: Let $A_1, \ldots, A_P$ be arbitrary complex numbers, not anyone of them is identically equal to zero, $\omega_{jk} \in (-\pi, \pi)$ for $j = 1, \ldots, M, k = 1, \ldots, P$ and they are all distinct.

Assumption 4: Let $\{e(n_1, \ldots, n_M)\}$ be a *i.i.d.* sequence of complex valued random variables, with

$$E(e(\mathbf{n})) = 0, \qquad Var(Re\{e(\mathbf{n})\}) = Var(Im\{e(\mathbf{n})\}) = \frac{\sigma^2}{2}.$$

The $Re(e(n))$ and $Im(e(n))$ are independently distributed. Now we state the following consistency results:

Theorem 3: Under the assumptions 3 and 4 as $N_{(M)} \to \infty$, the least squares estimators of the parameters of the model (2.5) are strongly consistent estimators of the unknown parameters.

Proof of Theorem 3: The proof can be obtained similarly as theorem 1 (see also the proof of theorem 1 of Kundu and Mitra; 1996), therefore it is omitted.

To obtain the asymptotic distribution of the least squares estimators, we rearrange the parameters and use the following notations:

$$\boldsymbol{\theta_1} = (A_{1R}, A_{1I}, \omega_{11}, \ldots, \omega_{M1}), \ldots \boldsymbol{\theta_P} = (A_{PR}, A_{PI}, \omega_{1P}, \ldots, \omega_{MP}),$$

$$\boldsymbol{\theta} = (\boldsymbol{\theta_1},\ \ldots,\ \boldsymbol{\theta_P}).$$

A_{kR}, A_{kI} denote the real and imaginary part of A_k for $k = 1, \ldots P$. We use $\hat{\boldsymbol{\theta}}$ for the least squares estimator of $\boldsymbol{\theta}$. Now we use $\mathbf{D}$ as a $P(M+2)$ diagonal matrix with the diagonal elements as follows:

$$\mathbf{D} = diag\{N^{\frac{1}{2}}, N^{\frac{1}{2}}, N_1 N^{\frac{1}{2}}, \ldots, N_M N^{\frac{1}{2}}, \ldots, N^{\frac{1}{2}}, N^{\frac{1}{2}}, N_1 N^{\frac{1}{2}}, \ldots N_M N^{\frac{1}{2}}\}.$$

Similarly as theorem 2, we have the following result for the asymptotic distributions of the least squares estimators.

Theorem 4: Under assumptions 3 and 4 as $N_{(1)} \to \infty$, the limiting distribution of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\boldsymbol{D}$ is a $P(M+2)$ variate complex normal distribution with mean zero and dispersion matrix $\sigma^2 \boldsymbol{\Sigma}^{-1}$, where

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma_1^{-1}} & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & \vdots \\ 0 & \ldots & 0 & \boldsymbol{\Sigma_P^{-1}} \end{bmatrix}.$$

Here $\boldsymbol{\Sigma_j}$ and $\boldsymbol{\Sigma_j^{-1}}$ are $M + 2 \times M + 2$ matrices as follows:

$$
\mathbf{\Sigma_j} = \begin{bmatrix} 2 & 0 & -A_{jI} & \dots & \dots & -A_{jI} \\ 0 & 2 & A_{jR} & \dots & \dots & A_{jR} \\ -A_{jI} & A_{jR} & \frac{2}{3}|A_j|^2 & \frac{1}{2}|A_j|^2 & \dots & \frac{1}{2}|A_j|^2 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ -A_{jI} & A_{jR} & \frac{1}{2}|A_j|^2 & \dots & \frac{1}{2}|A_j|^2 & \frac{2}{3}|A_j|^2 \end{bmatrix}
$$

and

$$
\mathbf{\Sigma_j^{-1}} = \begin{bmatrix} a & b & d & d & \dots & d \\ b & c & e & e & \dots & e \\ d & e & f & 0 & \dots & 0 \\ \vdots & \vdots & 0 & f & \dots & \vdots \\ d & e & 0 & \dots & \dots & f \end{bmatrix}
$$

Here

$$
a = \frac{1}{2} + \frac{3}{2}\frac{A_{jI}^2}{|A_j|^2}M, \qquad b = -\frac{3}{2}M\frac{A_{jI}A_{jR}}{|A_j|^2}, \qquad c = \frac{1}{2} + \frac{3}{2}M\frac{A_{jR}^2}{|A_j|^2}
$$

$$
d = \frac{3A_{jI}}{|A_j|^2}, \qquad e = -\frac{3A_{jR}}{|A_j|^2}, \qquad f = \frac{6}{|A_j|^2}
$$

Proof of Theorem 4: The proof follows quite similarly as Kundu and Mitra (1996) and therefore it is omitted. The important point is to rearrange the parameters in the specified manner then the asymptotic dispersion matrix can be expressed in a convenient form.

Comments: Note that the results of Rao and Zhao (1993) and Kundu and Mitra (1999) follow from theorems 3 and 4. It is interesting to see that all the frequency estimators are asymptotically independent and the asymptotic variances of the frequency estimators are inversely proportional to the corresponding amplitude estimators. The asymptotic variances of the frequency estimators are independent of M, whereas the asymptotic variances of the amplitude estimators are directly proportional to M. Therefore, as M increases the asymptotic variances of the amplitude estimators increase. Interestingly if the real part or the imaginary part of the amplitude is zero then the asymptotic variance of the corresponding imaginary part or the real part of the amplitude estimator is independent of M.

3. M-DIMENSIONAL CASE:

In this section we present the asymptotic distribution of the least squares estimators of the general M-dimensional model (1.1), under the following assumptions.

Assumption 5: $\{\ \omega_{1,1},\ldots,\omega_{1,P_1},\ldots,\omega_{M,1},\ldots,\omega_{M,P_M}\ \}$ are all distinct and they lie between $(0,2\pi)$.

Assumption 6: $||A_{j_1\ldots}||^2 = \sum_{j_2=1}^{P_2}\ldots\sum_{j_M=1}^{P_M}|A_{j_1\ldots j_M}|^2 > 0$. Similarly $||A_{.j_2\ldots}||^2\ldots,||A_{\ldots j_M}||^2$ are defined $||A_{.j_2\ldots}||^2 > 0\ldots||A_{\ldots j_M}||^2 > 0$ for $j_1 = 1,\ldots,P_1\ \ldots\ j_M = 1,\ldots,P_M$.

We use the following notations:

$$\omega_1 = (\omega_{1,1},\ldots,\omega_{1,P_1}),\ldots,\omega_M = (\omega_{M,1},\ldots,\omega_{M,P_M})$$

$$A = (A_{1\ldots1},\ldots,A_{P_1\ldots P_M}),\qquad \boldsymbol{\theta} = (\boldsymbol{\omega_1,\ldots,\omega_M},\ \mathrm{Re(A)},\ \mathrm{Im(A)})$$

Note that the LSE's of $\boldsymbol{\theta}$ of the model (1.1) can be obtained by minimizing

$$Q(\boldsymbol{\theta}) = \sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}\left|y(\mathbf{n}) - \sum_{j_1=1}^{P_1}\ldots\sum_{j_M=1}^{P_M}A_{j_1\ldots j_M}e^{i(n_1\omega_{1,j_1}+\ldots+n_M\omega_{M,j_M})}\right|^2 \tag{3.1}$$

We define the matrices $\mathbf{C^T_{j,M+1}} = \mathbf{C_{M+1,j}} = \mathbf{C^T_{j,M+2}} = \mathbf{C_{M+2,j}}$ of order $P_1\ldots P_M \times P_j$ as follows. Label the $P_1\ldots P_M$ rows of the matrix $\mathbf{C^T_{j,M+1}}$ as $(1\ldots1),(1\ldots2),\ldots,(P_1\ldots P_M)$ respectively and the P_j columns as $1,\ldots,P_j$. Then the k th. $(=1,\ldots,P_j)$ column of $\mathbf{C^T_{j,M+1}}$ has non zero entries at the rows

$$(\underbrace{1,1,\ldots,1}_{j-1},k,\underbrace{1,\ldots,1}_{M-j}),(\underbrace{1,\ldots1}_{j-1},k,\underbrace{1,\ldots,1,2}_{M-j}),\ldots,(P_1,\ldots,P_{j-1},k,P_{j+1},\ldots,P_M)$$

and they are $A_{1,\ldots,1,k,1,\ldots,1},A_{1,\ldots,1,k,1,\ldots,1,2},\ldots,A_{P_1,\ldots,P_{j-1},k,P_{j+1},\ldots,P_M}$ respectively. For better understanding we provide $\mathbf{C^T_{1,M+1}}$, $\mathbf{C^T_{2,M+1}}$ and $\mathbf{C^T_{M,M+1}}$ in the Appendix B..

The matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma^{-1}}$ are of the order $(P_1+\ldots+P_M+2P_1\ldots P_M)\times(P_1+\ldots+P_M+2P_1\ldots P_M)$ and are defined as follows:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma_{11}} & \dots & \boldsymbol{\Sigma_{1,M+2}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma_{M+2,1}} & \dots & \boldsymbol{\Sigma_{M+2,M+2}} \end{pmatrix} \qquad \boldsymbol{\Sigma^{-1}} = \begin{pmatrix} \boldsymbol{\Sigma^{11}} & \dots & \boldsymbol{\Sigma^{1,M+2}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma^{M+2,1}} & \dots & \boldsymbol{\Sigma^{M+2,M+2}} \end{pmatrix}$$

where $\boldsymbol{\Sigma_{ii}}$ and $\boldsymbol{\Sigma^{ii}}$ are both $P_i \times P_i$ matrices for $i = 1, \dots M$ and $P_1 \dots P_M \times P_1 \times P_M$ matrices for $i = M+1, M+2$. The elements of $\boldsymbol{\Sigma_{ij}}$ and $\boldsymbol{\Sigma^{ij}}$ are defines as follows:

$$\begin{aligned} \boldsymbol{\Sigma_{11}} &= \frac{2}{3} diag\{||A_{1\dots}||^2, \dots, ||A_{P_1\dots}||^2\} \\ \vdots &= \qquad\qquad \vdots \\ \boldsymbol{\Sigma_{MM}} &= \frac{2}{3} diag\{||A_{\dots 1}||^2, \dots, ||A_{\dots P_M}||^2\} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Sigma_{12}} &= ((\sigma_{ij}^{12})) = \boldsymbol{\Sigma_{21}^T}, \qquad \sigma_{ij}^{12} = \frac{1}{2}\sum_{j_3=1}^{P_3} \dots \sum_{j_M=1}^{P_M} |A_{ijj_3\dots j_M}|^2 \\ \vdots &= \vdots \\ \boldsymbol{\Sigma_{M-1,M}} &= ((\sigma_{ij}^{M-1,M})) = \boldsymbol{\Sigma_{M,M-1}^T}, \qquad \sigma_{ij}^{M-1,M} = \frac{1}{2}\sum_{j_1=1}^{P_1} \dots \sum_{j_{M-2}=1}^{P_{M-2}} |A_{j_1\dots j_{M-2}ij}|^2 \end{aligned}$$

$$\boldsymbol{\Sigma_{j,M+1}^T} = \boldsymbol{\Sigma_{M+1,j}} = -Im(\boldsymbol{C_{j,M+1}^T}), \qquad \boldsymbol{\Sigma_{j,M+2}^T} = \boldsymbol{\Sigma_{M+2,j}} = Re(\boldsymbol{C_{j,M+2}^T}),$$

$$\boldsymbol{\Sigma_{M+1,M+1}} = \boldsymbol{\Sigma_{M+2,M+2}} = 2\mathbf{I}_{P_1\dots P_M}, \qquad \boldsymbol{\Sigma_{M+1,M+2}} = \boldsymbol{\Sigma_{M+2,M+1}} = \mathbf{0}$$

$$\begin{aligned} \boldsymbol{\Sigma^{11}} &= 6 \quad diag\{||A_{1\dots}||^{-2}, \dots, ||A_{P_1\dots}||^{-2}\} \\ \vdots &= \qquad\qquad \vdots \\ \boldsymbol{\Sigma^{MM}} &= 6 \quad diag\{||A_{\dots 1}||^{-2}, \dots, ||A_{\dots P_M}||^{-2}\} \end{aligned}$$

$$\boldsymbol{\Sigma^{i,M+1}} = \left(\boldsymbol{\Sigma_{M+1,i}^{-1}}\right)^T = \frac{1}{2} Im\left(\boldsymbol{\Sigma_{ii}^{-1} C_{i,M+1}}\right)$$
$$\boldsymbol{\Sigma^{i,M+2}} = \left(\boldsymbol{\Sigma_{M+2,i}^{-1}}\right)^T = -\frac{1}{2} Re\left(\boldsymbol{\Sigma_{ii}^{-1} C_{i,M+2}}\right)$$

for $i = 1, \dots, M$.

$$\boldsymbol{\Sigma^{M+1,M+1}} = \frac{1}{4}\left(Im(\boldsymbol{C_{M+1,1}})\boldsymbol{\Sigma^{11}}\boldsymbol{Im(C_{1,M+1}}) + \dots + Im(\boldsymbol{C_{M+1,M}})\boldsymbol{\Sigma^{MM}}\boldsymbol{Im(C_{M,M+1}})\right)$$

$$
\begin{aligned}
& + \quad \frac{1}{2}\mathbf{I} \\
\boldsymbol{\Sigma}^{\boldsymbol{M+2,M+2}} &= \frac{1}{4}\Big(Re(\boldsymbol{C_{M+1,1}})\boldsymbol{\Sigma}^{\mathbf{11}}\boldsymbol{Re}(\boldsymbol{C_{1,M+1}}) + \ldots + Re(\boldsymbol{C_{M+1,M}})\boldsymbol{\Sigma}^{\boldsymbol{MM}}\boldsymbol{Re}(\boldsymbol{C_{M,M+1}})\Big) \\
& + \quad \frac{1}{2}\mathbf{I} \\
\boldsymbol{\Sigma}^{\boldsymbol{M+1,M+2}} &= -\frac{1}{4}\Big(Im(\boldsymbol{C_{M+1,1}})\boldsymbol{\Sigma}^{\mathbf{11}}\boldsymbol{Re}(\boldsymbol{C_{1,M+2}}) + \ldots + Im(\boldsymbol{C_{M+1,M}})\boldsymbol{\Sigma}^{\boldsymbol{MM}}\boldsymbol{Re}(\boldsymbol{C_{M,M+2}})\Big) \\
&= \boldsymbol{\Sigma}^{\boldsymbol{(M+2,M+1)T}}
\end{aligned}
$$

and the remaining $\boldsymbol{\Sigma}^{\boldsymbol{ij}}$ matrices are zero matrices. Now we can state the general result as follows.

Theorem 5: Under the assumptions 4, 5 and 6 as $N_{(M)} \to \infty$, the least squares estimators of $\boldsymbol{\theta}$ of the model (1.1) is strongly consistent.

Proof of Theorem 5: The detailed proof for $M = 2$ when $N_{(1)} \to \infty$ is provided in Kundu and Mitra (1996, Theorem 1). Note that in Kundu and Mitra (1996) it is assumed that the fourth order moments are finite and $N_{(1)} \to \infty$ where as here Lemma 1 is proved using the finiteness of the second order moments and as $N_{(M)} \to \infty$. For general M the idea of the proof is exactly the same and it follows using Lemma 1 and Lemma 2.

If we denote the diagonal matrix $\mathbf{D}$ as

$$
\mathbf{D} = diag\{\underbrace{N_1 N^{\frac{1}{2}}, \ldots, N_1 N^{\frac{1}{2}}}_{P_1}, \ldots, \underbrace{N_M N^{\frac{1}{2}}, \ldots, N_M N^{\frac{1}{2}}}_{P_M}, \underbrace{N^{\frac{1}{2}}, \ldots, N^{\frac{1}{2}}}_{2P_1 \ldots P_M}\}
$$

with $N = N_1 \ldots N_M$. Then we have the following result.

Theorem 6: Under the assumptions 4, 5 and 6 as $N_{(1)} \to \infty$, the asymptotic distribution of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\boldsymbol{D}$ is asymptotically $(P_1 + \ldots + P_M + 2P_1 \ldots P_M)$ variate normal distribution with mean zero and dispersion matrix $\sigma^2\ \boldsymbol{\Sigma}^{\mathbf{-1}}$, where $\boldsymbol{\Sigma}^{\mathbf{-1}}$ is as defined in this section before.

Proof of theorem 6: The main idea of the proof is quite simple. It involves the routine computation of the first and second derivative of $Q(\boldsymbol{\theta})$, where

$$Q(\boldsymbol{\theta}) = \sum_{n_1=1}^{N_1} \cdots \sum_{n_M=1}^{N_M} \left| y(\mathbf{n}) - \sum_{j_1=1}^{P_1} \cdots \sum_{j_M=1}^{P_M} A_{j_1 \ldots j_M} e^{i(n_1\omega_{1,j_1} + \ldots + n_M\omega_{M,j_M})} \right|^2$$

Then expanding the first derivative of $Q(\boldsymbol{\theta})$ in terms of Taylor series and use the proper normalizing constant to obtain the limiting normal distribution. The crucial point is to obtain the matrix $\boldsymbol{\Sigma}$ and then the matrix $\boldsymbol{\Sigma^{-1}}$.

Comments: Note that the results of Bai *et al.* (1987), Rao *et al.* (1994), Rao and Zhao (1993) and Kundu and Mitra (1996, 1999) follow from theorems 5 and 6. Theorems 5 and 6 generalize all the existing one or two dimensional results to their most general form.

4. CONSISTENCY AND ASYMPTOTIC NORMALITY OF $\hat{\sigma}^2$:

In this section we state the consistency and the asymptotic normality results of $\hat{\sigma}^2$, an estimator of σ^2, obtained as follows;

$$\hat{\sigma}^2 = \frac{1}{N} Q(\hat{\boldsymbol{\theta}}). \tag{4.1}$$

Here $Q(\boldsymbol{\theta})$ is same as defined in (3.1). We have the following results.

Theorem 7: If $\hat{\boldsymbol{\theta}}$ is the LSE of $\boldsymbol{\theta^0}$ of the model (1.1) and the error random variables $e(\mathbf{n})$'s satisfy assumption 4, then as $N_{(M)} \to \infty$, $\hat{\sigma}^2$ is a strongly consistent estimator of σ^2.

Proof of Theorem 7: Using lemma 1, the proof can be obtained similarly as the one dimensional case (see Kundu and Mitra; 1999), so it is omitted.

Theorem 8: If $\hat{\sigma}^2$ is the estimator of σ^2 as defined in (4.1) and the error random variables $e(\mathbf{n})$'s satisfy assumption 4. Furthermore if

$$E[(Re\{e(\mathbf{n})\})^4] < \infty \quad \text{and} \quad E[(Im\{e(\mathbf{n})\})^4] < \infty$$

then

$$\sqrt{N}(\hat{\sigma}^2 - \sigma^2) \to N(0, \sigma^*) \quad \text{as} \quad N_{(1)} \to \infty,$$

where $\sigma^* = E[(Re\{e(\mathbf{n})\})^4] + E[(Im\{e(\mathbf{n})\})^4] - \frac{\sigma^4}{2}$.

Proof of Theorem 8: This proof is also quite similar to the proof for the one dimensional case, therefore it is not provided here. The detailed proof can be obtained on request from the author.

5. CONCLUSIONS:

In this paper we discuss the asymptotic properties of the least squares estimators of the multidimensional exponential model. It is observed that several particular cases mainly one or two dimensional results can be obtained as particular cases. We consider two special cases of the general multidimensional model and obtain the asymptotic distribution. It may be mentioned that other than the least squares estimators, it may be possible to consider the approximate least squares estimators as defined by Hannan (1971) or Walker (1971). The approximate least squares estimators of the frequencies can be obtained by maximizing the periodogram function. It can be proved along the same line as the one dimensional case that the least squares estimators and the approximate least squares estimators are asymptotically equivalent. Therefore, the asymptotic distribution of the ALSE's will be same as that of the LSE's in all the cases considered. Note that in this paper we assume that the errors are independent and identically distributed. Suppose that we have the following error structure $X(\mathbf{n})$ instead of $e(\mathbf{n})$ in (1.1),

$$X(\mathbf{n}) = \sum_{i_1=1}^{Q_1} \dots \sum_{i_M=1}^{Q_M} a_{i_1 \dots i_M} e(n_1 - i_1, \dots, n_M - i_M) \tag{5.1}$$

here $a_{i_1 \dots i_M}$ are unknown constants and $Q_1, \dots, Q_M$ are known integers. The error structure (5.1) indicates that the errors are of the moving average type in M dimensions. The results can be extended even when the errors are of the type (5.1), see for example Nandi and Kundu (1999) for the two dimensional result in this context.

Now we would like to mention some open problems. First of all the estimation of $P_1, \ldots, P_M$ is very important but difficult problem. Even in the case of two dimensions, the problem is not yet resolved (see Rao *et al.* (1996)) satisfactorily. May be some model selection criteria like AIC or BIC or their modifications can be used to resolve this problem. Numerically obtaining the LSE's of the one or two dimensional problems are well known to be very difficult problems. It will be important to find out some good estimation schemes for obtaining the LSE's of the unknown parameters for higher dimensional model. Another important problem is to study the properties of the LSE's when the errors are from a stationary distribution not necessarily from a finite order moving average type. If the errors are from a stationary random field then it will be important to derive the properties of the LSE's. More work is needed in these directions.

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Appendix A

First we prove the following lemma:

Lemma 0: Let $\{e(\mathbf{n})\}$ be a *i.i.d.* sequence of M-dimensional random variables with mean zero and finite variance, then

$$\lim_{N_{(1)}\to\infty} \sup_{\alpha_1,\ldots,\alpha_M} \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(\mathbf{n}) cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \to 0 \qquad (A1.1)$$

Proof of Lemma 0: Note that proving (A1.1) is equivalent to prove

$$\lim_{\substack{N_1\to\infty\\ \vdots \\ N_M\to\infty}} \sup_{\alpha_1,\ldots,\alpha_M} \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(\mathbf{n}) cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \to 0 \quad a.s.$$

Consider the following random variables

$$\begin{aligned} Z(\mathbf{n}) &= e(\mathbf{n}) \text{ if } e(\mathbf{n}) \le (n_1 \ldots n_M)^{\frac{3}{4}} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

First we will show that $Z(\mathbf{n})$ and $e(\mathbf{n})$ are equivalent sequences. Consider

$$\sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} P\{e(\mathbf{n}) \ne Z(\mathbf{n})\} = \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} P\{|e(\mathbf{n})| \ge (n_1\ldots n_M)^{\frac{3}{4}}\}.$$

Now observe that there are at most $2^k k^{M-1}$ combinations of $(n_1,\ldots,n_M)$'s such that $n_1\ldots n_M < 2^k$. Therefore, we have

$$\begin{aligned} & \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} P\{|e(\mathbf{n})| \ge (n_1\ldots n_M)^{\frac{3}{4}}\} \\ \le\ & \sum_{k=1}^{\infty} \sum_{2^{k-1}\le r<2^k} P\{|e(\mathbf{n})| \ge r^{\frac{3}{4}}\} \text{ here } [r = n_1\ldots n_M] \\ \le\ & \sum_{k=1}^{\infty} 2^k k^{M-1} P\{|e(1,\ldots,1)| \ge 2^{(k-1)\frac{3}{4}}\} \\ \le\ & C\sum_{k=1}^{\infty} \frac{2^k k^{M-1}}{2^{(k-1)\frac{3}{2}}} \le C\sum_{k=1}^{\infty} \frac{k^{M-1}}{2^{\frac{k}{2}}} < \infty. \end{aligned}$$

Here C is a constant and it may take different values at different places. Therefore, $e(\mathbf{n})$ and $Z(\mathbf{n})$ are equivalent sequences. So,

$$P\{e(\mathbf{n}) \ne Z(\mathbf{n})\ i.o.\} = 0.$$

Here $i.o.$ means infinitely often. Let

$$U(\mathbf{n}) = Z(\mathbf{n}) - E(Z(\mathbf{n})),$$

then

$$\sup_{\alpha_1,\ldots,\alpha_M} \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} E(Z(\mathbf{n}))cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \le \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} |E(Z(\mathbf{n}))|.$$

Since $E(Z(\mathbf{n})) \to 0$ as $N_{(1)} \to \infty$, therefore,

$$\frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} |E(Z(\mathbf{n}))| \to 0.$$

as $N_{(1)} \to \infty$. So, it is enough to prove that

$$\sup_{\alpha_1,\ldots,\alpha_M} \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} U(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \to 0$$

as $N_{(1)} \to \infty$. Now for any fixed $\epsilon > 0$, $-\pi < \alpha, \beta < \pi$ and $0 < h \le \frac{1}{2N^{\frac{3}{4}}}$, we have

$$P\left\{ \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} U(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \ge \epsilon \right\}$$

$$\le 2e^{-hN\epsilon} \prod_{n_1=1}^{N_1} \ldots \prod_{n_M=1}^{N_M} Ee^{h(U(\mathbf{n}))\times cos(n_1\alpha_1)\ldots cos(n_m\alpha_M)}$$

Since $|hU(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M)| \le \frac{1}{2}$, using $e^x < 1 + x + x^2$ for $|x| \le \frac{1}{2}$, we have

$$2e^{-hN\epsilon} \prod_{n_1=1}^{N_1} \ldots \prod_{n_M=1}^{N_M} Ee^{h(U(\mathbf{n}))\times cos(n_1\alpha_1)\ldots cos(n_m\alpha_M)}$$

$$\le 2e^{-hN\epsilon}(1 + h^2\sigma^2)^N.$$

Choose $h = \frac{1}{2N^{\frac{3}{4}}}$, therefore for large $N_1, \ldots, N_M$,

$$P\left\{ \left| \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} U(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M) \right| \ge \epsilon \right\} \le Ce^{-N^{\frac{1}{4}}\frac{\epsilon}{2}}.$$

Let $K = N^2$, choose K points $\beta_1 = (\alpha_{11}, \ldots, \alpha_{M1})$, ..., $\beta_K = (\alpha_{1K}, \ldots, \alpha_{MK})$ in $(-\pi, \pi) \times \ldots \times (-\pi, \pi)$ such that for each $\beta = (\beta_1, \ldots, \beta_M) \in (-\pi, \pi) \times \ldots \times (-\pi, \pi)$, there exists a point β_j satisfying

$$|\beta_{1j} - \beta_1| + \ldots + \beta_{Mj} - \beta_M| \le \frac{2\pi}{N^2}.$$

Note that

$$\left|\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}U(\mathbf{n})\{cos(n_1\alpha_1)\ldots cos(n_M\alpha_M)-cos(n_1\alpha_{1j})\ldots cos(n_M\alpha_{Mj})\}\right|$$

$$\leq C\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}\frac{N^{\frac{3}{4}}}{N^2}(n_1+\ldots+n_M)\to 0\ a.s.$$

as $N_{(1)}\to\infty$. Therefore, for large $N_1,\ldots,N_M$, we have

$$P\left\{\left|\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}U(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M)\right|\geq 2\epsilon\right\}$$

$$\leq P\left[\max_{j\leq N^2}\left|\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}U(\mathbf{n})cos(n_1\alpha_{1j})\ldots cos(n_M\alpha_{Mj})\right|\geq\epsilon\right\}$$

$$\leq CN^2e^{-N^{\frac{1}{4}}\frac{\epsilon}{2}}$$

Since $\sum_{t=1}^{\infty}t^2e^{-t^{\frac{1}{4}}}<\infty$ therefore as $N_{(1)}\to\infty$,

$$\sup_{\alpha_1\ldots\alpha_M}\left|\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}U(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M)\right|\to 0\ a.s.$$

Proof of Lemma 1: Note that $N_{(M)}$ goes ∞ if and only if k of the N_i go to ∞ and $M-k$ of the N_i are bounded for $k=1,\ldots,M$. For $k=M$, lemme 1 reduces to lemma 0. We prove the result for $k=1,\ldots,M-1$. Consider any fixed k, and with out any loss of generality we can assume that $N_1,\ldots,N_k$ go to ∞ and $N_{k+1},\ldots,N_M$ are bounded. Therefore, lemma 1 will be proved if we prove the following, for any bounded $N_{k+1},\ldots,N_M$,

$$\lim_{\substack{N_1\to\infty\\ \vdots\\ N_k\to\infty}}\sup_{\alpha_1,\ldots,\alpha_M}\left|\frac{1}{N}\sum_{n_1=1}^{N_1}\ldots\sum_{n_M=1}^{N_M}e(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_M\alpha_M)\right|\to 0\quad a.s. \qquad (A1.2)$$

Since

$$\lim_{\substack{N_1\to\infty\\ \vdots\\ N_k\to\infty}}\sup_{\alpha_1,\ldots,\alpha_k}\left|\frac{1}{N_1\ldots N_k}\sum_{n_1=1}^{N_1}\ldots\sum_{n_k=1}^{N_k}e(\mathbf{n})cos(n_1\alpha_1)\ldots cos(n_k\alpha_k)\right|\to 0\quad a.s. \qquad (A1.3)$$

(follows from lemma 0 by putting n = k), therefore, (A1.2) follows simply by taking limit inside the summation and repeatedly using (A1.3).

Appendix B

$$\mathbf{C^T_{1,M+1}} = \begin{bmatrix} A_{1\ldots1} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{1P_2\ldots P_M} & 0 & \ldots & 0 \\ 0 & A_{21\ldots1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & A_{2P_2\ldots P_M} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \ldots & \vdots & A_{P_11\ldots1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & A_{P_1\ldots P_M} \end{bmatrix}$$

$$
\mathbf{C_{2,M+1}^T} = \begin{bmatrix}
A_{11\ldots1} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
A_{11P_3\ldots P_M} & 0 & \ldots & \ldots \\
0 & A_{121\ldots1} & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & A_{12P_3\ldots P_M} & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A_{1P_21\ldots1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A_{1P_2P_3\ldots P_M} \\
A_{21\ldots1} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
A_{21P_3\ldots P_M} & 0 & \ldots & 0 \\
0 & A_{221\ldots1} & \ldots & 0 \\
0 & \vdots & \vdots & 0 \\
0 & A_{22P_3\ldots P_M} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A_{2P_21\ldots1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A_{2P_2P_3\ldots P_M} \\
\vdots & \vdots & \vdots & \vdots \\
A_{P_11\ldots1} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
A_{P_11P_3\ldots P_M} & 0 & \ldots & 0 \\
0 & A_{P_121\ldots1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & A_{P_12P_3\ldots P_M} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \ldots & A_{P_1P_21\ldots1} \\
0 & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A_{P_1P_2\ldots P_M}
\end{bmatrix}
$$

$$
\mathbf{C_{M,M+1}}^T = \begin{bmatrix}
A_{11\ldots1} & 0 & 0 & \ldots & 0 \\
0 & A_{11\ldots12} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{11\ldots1P_M} \\
A_{11\ldots21} & 0 & 0 & \ldots & 0 \\
0 & A_{11\ldots22} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{11\ldots2P_M} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_{P_1\ldots P_{M-1}1} & 0 & 0 & \ldots & 0 \\
0 & A_{P_1\ldots P_{M-1}2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{P_1\ldots P_M}
\end{bmatrix}
$$

Appendix C

Proof of Lemma 2: In this proof only we write $\hat{\boldsymbol{\theta}}_{N_1 \ldots N_M}$ as the least squares estimator of $\boldsymbol{\theta}^0$ and $Q(\boldsymbol{\theta})$ as defined in (2.2) as $Q_{N_1 \ldots N_M}(\boldsymbol{\theta})$. Suppose $\hat{\boldsymbol{\theta}}_{N_1 \ldots N_M}$ is not consistent, therefore either

Case I: For all subsequences $(N_{1k}, \ldots, N_{Mk})$ of $(N_1, \ldots, N_M)$, $|\hat{\boldsymbol{\theta}}_{N_1 \ldots N_M}| \to \infty$ with a positive probability or

Case II: There exists a $\delta > 0$ and a $T < \infty$ and a subsequence $(N_{1k}, \ldots, N_{Mk})$ of $(N_1, \ldots, N_M)$, such that $\hat{\boldsymbol{\theta}}_{N_{1k}, \ldots, N_{Mk}} \in S_{\delta, T}$,

for all k = 1,2, ... Now note that

$$\frac{1}{N_{1k} \ldots N_{Mk}} Q_{N_{1k}, \ldots, N_{Mk}}(\hat{\boldsymbol{\theta}}_{N_{1k}, \ldots, N_{Mk}}) \leq \frac{1}{N_{1k} \ldots N_{Mk}} Q_{N_{1k}, \ldots, N_{Mk}}(\boldsymbol{\theta}^0) \quad a.s. \qquad (A3.1)$$

as $\hat{\boldsymbol{\theta}}_{N_{1k}, \ldots, N_{Mk}}$ is the least squares estimator of $\boldsymbol{\theta}^0$, when $N_1 = N_{1k}, \ldots, N_M = N_{Mk}$ from the strong law of large numbers and using the similar argument as of the proof of lemma 1, it easily follows that

$$\lim_{N_{(M)} \to \infty} \frac{1}{N} Q_{N_1, \ldots, N_M}(\boldsymbol{\theta}^0) \to \sigma^2 \quad a.s.$$

where $\sigma^2 = \mathrm{Var}(e(\mathbf{n}))$. In both the cases under the definition of $Q_{N_{1k}, \ldots, N_{Mk}}(\boldsymbol{\theta})$ (see (2.2)) and because of the assumption of Lemma 2, we get that there exists a $M_0 > 0$ such that for all $k \geq M_0$,

$$\frac{1}{N_{1k} \ldots N_{Mk}} Q_{N_{1k}, \ldots, N_{Mk}}(\hat{\boldsymbol{\theta}}_{N_{1k}, \ldots, N_{Mk}}) > \frac{1}{N_{1k} \ldots N_{Mk}} Q_{N_{1k}, \ldots, N_{Mk}}(\boldsymbol{\theta}^0),$$

with a positive probability. This contradicts (A3.1). It proves the lemma.

Proof of Theorem 1: Note that

$$\frac{1}{N} Q(\boldsymbol{\theta}) = \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(\boldsymbol{n})^2 +$$

$$\frac{1}{N}\sum_{n_1=1}^{N_1}\dots\sum_{n_M=1}^{N_M}\left(\sum_{k=1}^{P}A_k^0 cos(n_1\omega_{1k}^0+\ldots+n_M\omega_{Mk}^0)-\sum_{k=1}^{P}A_k cos(n_1\omega_{1k}+\ldots+n_M\omega_{Mk})\right)^2$$
$$\frac{2}{N}\sum_{n_1=1}^{N_1}\dots\sum_{n_M=1}^{N_M}e(\mathbf{n})\left(\sum_{k=1}^{P}A_k^0 cos(n_1\omega_{1k}^0+\ldots+n_M\omega_{Mk}^0)-\sum_{k=1}^{P}A_k cos(n_1\omega_{1k}+\ldots+n_M\omega_{Mk})\right). \tag{A3.2}$$

Now observe that as $N_{(M)} \rightarrow \infty$, the first term on the right hand side of (A3.2) converges to σ^2 $a.s.$ because of the strong law of large number and the third term converges to zero because of Lemma 1. Therefore due to Lemma 2, to prove the consistency of the least squares estimator it is enough to prove

$$\liminf_{N_{(M)}\rightarrow\infty}\inf_{\boldsymbol{\theta}\in S_{\delta,T}} g(\boldsymbol{\theta},\boldsymbol{\theta}^0) > 0 \quad a.s., \tag{A3.3}$$

where $g(\boldsymbol{\theta},\boldsymbol{\theta}^0) =$

$$\frac{1}{N}\sum_{n_1=1}^{N_1}\dots\sum_{n_M=1}^{N_M}\left(\sum_{k=1}^{P}A_k^0 cos(n_1\omega_{1k}^0+\ldots+n_M\omega_{Mk}^0)-\sum_{k=1}^{P}A_k cos(n_1\omega_{1k}+\ldots+n_M\omega_{Mk})\right)^2.$$

Consider the following sets

$$\begin{aligned}
A_{1\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |A_1 - A_1^0| \geq \delta, |\boldsymbol{\theta}| \leq T\} \\
&\vdots \\
A_{M\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |A_M - A_M^0| \geq \delta, |\boldsymbol{\theta}| \leq T\} \\
\omega_{11\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |\omega_{11} - \omega_{11}^0| \geq \delta, |\boldsymbol{\theta}| \leq T\} \\
&\vdots \\
\omega_{1M\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |\omega_{1M} - \omega_{1M}^0| \geq \delta, |\boldsymbol{\theta}| \leq T\} \\
&\vdots \\
\omega_{P1\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |\omega_{P1} - \omega_{P1}^0| \geq \delta, |\boldsymbol{\theta}| \leq T\} \\
&\vdots \\
\omega_{PM\delta} &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_M); |\omega_{PM} - \omega_{PM}^0| \geq \delta, |\boldsymbol{\theta}| \leq T\}.
\end{aligned}$$

$$(A3.4)$$

Since $S_{\delta,T}$ is the union of all the sets defined above in (A3.4), therefore to prove (A3.3), it is enough to prove

$$\liminf_{N_{(M)}\to\infty} \inf_{\boldsymbol{\theta}\in V} g(\boldsymbol{\theta},\boldsymbol{\theta}^0) > 0 \quad a.s. \tag{A3.5}$$

where V is any one of the set defined in (A3.4). We show (A3.5) for $V = A_{1\delta}$,

$$\begin{aligned}\liminf_{N_{(M)}\to\infty} \inf_{\boldsymbol{\theta}\in A_{1\delta}} g(\boldsymbol{\theta},\boldsymbol{\theta}^0) &= (A_1 - A_1^0)^2 \lim_{N_{(M)}\to\infty} \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} cos^2(n_1\omega_{1k}^0 + \ldots + n_M\omega_{Mk}^0) \\ &\geq \delta^2 \lim_{N_{(M)}\to\infty} \frac{1}{N} \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} cos^2(n_1\omega_{1k}^0 + \ldots + n_M\omega_{Mk}^0) > 0\end{aligned}$$

For other V, it follows along the same line. It proves Theorem 1.

The proof that $Q'(\boldsymbol{\theta^0})\mathbf{D}^{-1}$ converges in distribution to a $P(M+2)$ variate normal distribution with mean zero and dispersion matrix $2\sigma^2\ \boldsymbol{\Sigma}$ (page 6).

The different elements of $Q'(\boldsymbol{\theta^0})\mathbf{D}^{-1}$ can be written as follows:

$$\begin{aligned}\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial A_k} &= -2 \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(n) cos(n_1\omega_{1k}^0 + \ldots + n_M\omega_{Mk}^0) \\ \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \omega_{1k}} &= 2 \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(n) A_k^0 n_1 sin(n_1\omega_{1k}^0 + \ldots + n_M\omega_{Mk}^0) \\ \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \omega_{Mk}} &= 2 \sum_{n_1=1}^{N_1} \ldots \sum_{n_M=1}^{N_M} e(n) A_k^0 n_M sin(n_1\omega_{1k}^0 + \ldots + n_M\omega_{Mk}^0)\end{aligned}$$

All the elements of $Q'(\boldsymbol{\theta}^0)$ satisfy the Lindeberg- Feller condition (Chung, K.L.; 1978, A Course in Probability Theory). Therefore, $Q'(\boldsymbol{\theta}^0)$, with proper normalization will converge to a multivariate normal distribution. Now let's look at the asymptotic covariance matrix of $Q'(\boldsymbol{\theta}^0)$. Using the results

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} sin^2(\beta t) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^2} \sum_{t=1}^{n} t sin^2(\beta t) = \frac{1}{4}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} cos^2(\beta t) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n} tcos^2(\beta t) = \frac{1}{4},$$

we have for $i, j = 1, \ldots M$ and for $k = 1, \ldots P$,

$$\frac{1}{N} cov\left(\frac{\partial Q(\boldsymbol{\theta}^0}{\partial A_i}, \frac{\partial Q(\boldsymbol{\theta}^0}{\partial A_j}\right) \to \left\{\begin{array}{ccc} 0 & \text{if} & i \neq j \\ 2\sigma^2 & \text{if} & i = j \end{array}\right\}$$

$$\frac{1}{N_i N_j N} cov\left(\frac{\partial Q(\boldsymbol{\theta}^0}{\partial \omega_{ik}}, \frac{\partial Q(\boldsymbol{\theta}^0}{\partial \omega_{jk}}\right) \to \left\{\begin{array}{ccc} \frac{2}{3}\sigma^2 A_k^{0^2} & \text{if} & i = j \\ \frac{1}{2}\sigma^2 A_k^{0^2} & \text{if} & i \neq j \end{array}\right\}.$$