On Parameter Estimation of Two-Dimensional Polynomial Phase Signal Model

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Abstract

Two-dimensional (2-D) polynomial phase signals occur in different areas of image processing. When the degree of the polynomial is two it is called chirp signal. In this paper, we consider the least squares estimators of the different unknown parameters of the 2-D polynomial phase signal model in the presence of stationary noise, and derive their properties. The proposed least squares estimators are strongly consistent and we obtained the asymptotic distribution of the least squares estimators. It is observed that asymptotically the least squares estimators are normally distributed. We perform some simulation experiments to observe the behavior of the least squares estimators, and it is observed that the performances are quite satisfactory.

Key Words and Phrases: Polynomial phase signals; least squares estimators; strong consistency; asymptotic distribution; linear processes.

AMS Subject Classification: 62F10, 62F12

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1 Introduction

One dimensional polynomial phase signal models have received considerable attention in the statistical signal processing literature. One dimensional polynomial phase signal model has been used quite successfully in various areas of science and engineering, specifically in sonar, radar communications etc., see for example Barbarossa and Petrone (1997), Barbarossa et al. (1998) and Wu et al. (2008). In Wu et al. (2008), the authors considered a specific case when the degree of polynomial is three, due to its applications in seismology. When the degree of polynomial is two, the polynomial phase signal model is known as chirp model, and it also has received considerable attention in the literature because of its wide scale applicability in the sonar array processing. See for example Djuric and Kay (1990), Gini et al. (2000), Kundu and Nandi (2008), and the references cited therein.

Two-dimensional (2-D) polynomial phase signal model also has received significant amount of attention as it has been used in modeling and analyzing magnetic resonance imaging (MRI), optical imaging and different texture imaging. See for example Francos and Friedlander (1998, 1999), Hedley and Rosenfeld (1992), Peleg and Porat(1991), Cao, Wang and Wang (2006), Zhang and Liu(2006) and Zhang, Wang and Cao (2008). Friedlander and Francos (1996) used 2-D polynomial phase signal model to analyze finger print type data, and Djurovic et al. (2010) considered a specific 2-D cubic phase signal model due to its applications in modeling Synthetic Aperture Radar (SAR) data and in particular Interferometric SAR data.

Surprisingly, although extensive work has been done on estimating the parameters of different 2-D polynomial phase signal models by different methods, no where the least squares estimators (LSEs) of the 2-D polynomial phase signal have been considered, nor their properties have been discussed. The reason might be although that the least squares estimators are the most natural estimators, deriving the properties of the least squares estimators may not be very simple. In the literature many different estimators have been proposed and they were compared with the Cramer-Rao lower bound. But note that unless it is established that the asymptotic variances of the maximum likelihood estimators attend the corresponding Cramer-Rao lower bound, this comparison may not be meaningful. It may be mentioned that when the error random variables $X(m,n)$s are i.i.d. Gaussian random variables, then the LSEs become the maximum likelihood estimators (MLEs) also.

In this paper we consider the most general 2-D polynomial (of degree r) phase signal model which has the following form,

$$y(m,n) = A^0 \cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j) m^j n^{p-j}\right) + B^0 \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p} \alpha^0(j,p-j) m^j n^{p-j}\right) + X(m,n); \qquad m = 1,\cdots,M; \quad n = 1,\cdots,N, \tag{1}$$

here $X(m,n)$ is stationary error, A^0 and B^0 are non zero amplitudes and for $j = 0,\cdots,p,\ p = 1,\cdots,r,\ \alpha^0(j,p-j)$'s are distinct frequency rates of order $(j,p-j)$ respectively, and lie strictly between 0 and π, $\alpha^0(0,1), \alpha^0(1,0)$ are called frequencies. The explicit assumptions on the errors $X(m,n)$ will be provided later.

The main aim of this paper is to provide the properties of the least squares estimators of the unknown parameters of the model (1). It is expected that due to the complicated nature of the model, deriving the exact distribution of the least squares estimators may not be possible, and therefore we mainly rely on the asymptotic results. It may be mentioned that the properties of 1-D chirp signal model have been discussed by Kundu and Nandi (2008). They established the strong consistency and asymptotic normality properties of the least squares estimators. Unfortunately, their results cannot be used directly to establish the asymptotic properties of the least squares estimators of the 2-D polynomial phase signal model (1).

It can be observed that this model also does not satisfy the sufficient conditions of Jennrich (1969) and Wu (1981) for the least squares estimators to be consistent. Therefore, the results of Jennrich (1969) or Wu (1981) cannot be used directly to establish the asymptotic properties of the least squares estimators of the model (1). In this paper we establish the strong consistency and asymptotic normality properties of the least squares estimators of the unknown parameters of the model (1). It is observed that the least squares estimators of $\alpha^0(j, p-j)$ for $j = 1, \cdots, p,\ p = 1, \cdots, r$ have the convergence rates $O_p(M^{-j-1/2}N^{-(p-j)-1/2})$. Moreover, the least squares estimators of A^0 and B^0 have the convergence rate $(MN)^{-1/2}$. Therefore, it is clear that the convergence rates of the least squares estimators of the linear parameters are much slower than the convergence rates of the least squares estimators of the corresponding non-linear parameters. It can be easily observed that the convergence rate of the estimators of $\alpha^0(j, p-j)$ for $j = 1, \cdots, p,\ p = 1, \cdots, r$ is much faster than $(MN)^{-1/2}$, which is the usual convergence rate of an estimator for a general non-linear model. Moreover, when $X(m, n)$'s are i.i.d. random variables, then the asymptotic variances of the MLEs, which are same as the LSEs, attend the Cramer-Rao lower bound.

We perform some simulation experiments to study the effectiveness of the least squares estimators for different sample sizes, for different models, for different error structures and for different error random variables. We have considered independent as well correlated error random variables. Further it is assumed that the error random variables might be Gaussian or Laplace distributions, and we have considered the polynomial phase with degree two and three. In all the cases considered, it is observed that the performances of the least squares estimators are quite satisfactory.

The rest of the paper is organized as follows. In Section 2, we provide the necessary assumptions, preliminary results and the methodology for the least squares estimators. Strong consistency and asymptotic results are established in Section 3. Discussions on extensive simulation results and the analysis of a data set are presented in Section 4. Finally we conclude the paper in Section 5. All the proofs and all the numerical results based on extensive simulations are provided in the Supplementary Section.

2 Model Assumptions, Preliminary Results and Methodology

2.1 Model Assumptions

Assumption 1: The error $X(m,n)$ has the following form;

$$X(m,n)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}a(j,k)\varepsilon(m-j,n-k) \tag{2}$$

with

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|a(j,k)|<\infty. \tag{3}$$

Here $\varepsilon(m,n)$ is a double array sequences of independent and identically distributed ($i.i.d.$) random variables with zero mean, finite variance σ^2 and with finite $2r$-th moment.

Assumption 2: Let us denote the true parameters by $\theta^0=(A^0,B^0,\alpha^0(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r)$ and the parameter space by $\Theta=[-K,K]\times[-K,K]\times[0,\pi]^{\otimes\frac{r(r+3)}{2}}$. Here $K>0$ is an arbitrary constant and $[0,\pi]^{\otimes\frac{r(r+3)}{2}}$ denotes $\dfrac{r(r+3)}{2}$ fold of $[0,\pi]$. It is assumed that θ^0 is an interior point of Θ.

2.2 Preliminary Results

We need the following results for further development.

Proposition 1. *Suppose* $(\alpha^0(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r)\ \in(0,\pi)^{\otimes\frac{r(r+3)}{2}}$. *Then except for countable number of points* $\alpha^0(j,p-j)$, *for* $s,t=0,1,2,\cdots$,

$$\lim_{\min\{M,N\}\to\infty}\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j}\right)=0 \tag{4}$$

$$\lim_{\min\{M,N\}\to\infty}\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j}\right)=0. \tag{5}$$

$$\lim_{\min\{M,N\}\to\infty} \frac{1}{M^{(s+1)}N^{(t+1)}} \sum_{n=1}^{N}\sum_{m=1}^{M} m^s n^t \cos^2\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}\right)$$
$$= \frac{1}{2(s+1)(t+1)} \tag{6}$$
$$\lim_{\min\{M,N\}\to\infty} \frac{1}{M^{(s+1)}N^{(t+1)}} \sum_{n=1}^{N}\sum_{m=1}^{M} m^s n^t \sin^2\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}\right)$$
$$= \frac{1}{2(s+1)(t+1)}. \tag{7}$$

Proof: See in the Supplementary Section.

Lemma 1. *If $X(m,n)$ satisfies the Assumptions 1 & 2, then as $\min\{M,N\}\to\infty$,*

$$\sup_{\alpha(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r} \left| \frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M} X(m,n) e^{i\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j}\right)} \right| \to 0 \ a.s.$$

Proof: See in the Supplementary Section.

Lemma 2. *If $X(m,n)$ satisfies Assumptions 1 & 2, then as $\min\{M,N\}\to\infty$, and for $s,t=0,1,\cdots$,*

$$\sup_{\alpha(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r} \left| \frac{1}{M^{s+1}N^{t+1}}\sum_{n=1}^{N}\sum_{m=1}^{M} m^s n^t X(m,n) e^{i\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^j n^{p-j}\right)} \right| \to 0 \ a.s.$$

Proof: See in the Supplementary Section.

2.3 Methodology

We will use the following notations;

$$\phi = \begin{bmatrix} A \\ B \end{bmatrix}$$

and

$$W(\alpha)=\begin{bmatrix}
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)\right)\\
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)2^j\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)2^j\right)\\
\vdots & \vdots\\
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)M^j\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)M^j\right)\\
\vdots & \vdots\\
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)N^{p-j}\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)N^{p-j}\right)\\
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)2^jN^{p-j}\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)2^jN^{p-j}\right)\\
\vdots & \vdots\\
\cos\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)M^jN^{p-j}\right) & \sin\left(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)M^jN^{p-j}\right)
\end{bmatrix}$$

and Y is the $MN \times 1$ data vector defined as follows:

$$Y = (y(1,1),\cdots,y(M,1),\cdots,y(1,N),\cdots,y(M,N))^T. \tag{8}$$

The least squares estimators of $\theta = (A, B, \alpha(j,p-j),\ j = 0,\cdots,p,\ p = 1,\cdots,r)$, can be obtained by minimizing

$$Q(\theta) = (Y - W(\alpha)\phi)^T(Y - W(\alpha)\phi) \tag{9}$$

with respect to θ. Now using the separable regression technique of Richards (1961), it can be seen that for fixed $(\alpha(j,p-j),\ j = 0,\cdots,p,\ p = 1,\cdots,r)$, the minimization of $Q(\theta)$ with respect to A and B can be obtained as

$$\widehat{\phi}(\alpha) = \begin{bmatrix}\widehat{A}(\alpha)\\ \widehat{B}(\alpha)\end{bmatrix} = (W(\alpha)^TW(\alpha))^{-1}W(\alpha)^TY.$$

Therefore, the minimization of $Q(\theta)$ can be obtained by minimizing

$$R(\alpha) = Y^T(I - P(\alpha))Y$$

with respect to $(\alpha(j,p-j),\ j = 0,\cdots,p,\ p = 1,\cdots,r)$, where

$$P(\alpha) = W(\alpha)(W(\alpha)^TW(\alpha))^{-1}W(\alpha)^T$$

is the projection matrix on the column space of $W(\alpha)$.

If $(\widehat{\alpha}(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r)$ minimizes $R(\alpha)$, the least squares estimates of A and B can be obtained as

$$\widehat{A}=\widehat{A}(\widehat{\alpha}) \quad \text{and} \quad \widehat{B}=\widehat{B}(\widehat{\alpha}).$$

We will use $\widehat{\theta}=(\widehat{A},\widehat{B},\widehat{\alpha}(j,p-j),\ j=0,\cdots,p,\ p=1,\cdots,r)$. Note that by using the separable regression technique, the least squares estimators of the unknown parameters of the model (1) can be obtained by solving a $r(r-3)/2$ dimensional optimization problem, rather than a $2+r(r-3)/2$ dimensional optimization problem.

3 Asymptotic Properties of the Least Squares Estimators

3.1 Consistency of the Least Squares Estimators

In this section we provide the consistency results of the estimators.

Theorem 1. *If the Assumptions 1 & 2 are satisfied then $\widehat{\theta}$, the least squares estimators of θ^0, is a strongly consistent estimator of θ^0.*

Proof: See in the Supplementary Section.

The following result might be useful for error analysis of the model, or it may have some independent interests also.

Lemma 3. *If the Assumptions 1 & 2 are satisfied, then for $j=0,\cdots,p,\ p=1,\cdots,r$*

$$M^jN^{p-j}(\widehat{\alpha}(j,p-j)-\alpha^0(j,p-j))\to 0 \ a.s.$$

Proof: See in the Supplementary Section.

Using Lemma 3, we immediately obtain

$$\begin{aligned}\widehat{A} &= A^0 + o(1) \ \ a.s., \quad \widehat{B} = B^0 + o(1) \ \ a.s., \\ \widehat{\alpha}(j, p-j) &= \alpha^0(j, p-j) + o(M^j N^{p-j}) \ \ a.s.,\end{aligned}$$

So we get,

$$\begin{aligned}\widehat{y}(m,n) &= \widehat{A}\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\widehat{\alpha}(j,p-j)m^j n^{p-j}) + \widehat{B}\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\widehat{\alpha}(j,p-j)m^j n^{p-j}) \\ &= A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) + B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) \\ &\quad +o(1) \ \ a.s.\end{aligned}$$

which gives,

$$y(m,n) - \widehat{y}(m,n) = X(m,n) + o(1) \quad a.s. \tag{10}$$

Therefore, clearly (10) can be used for checking the error assumptions.

3.2 Asymptotic normality of the estimators

In this section we will provide the asymptotic normal distribution for the estimators.

Theorem 2. *If the Assumptions 1 & 2 are satisfied, then* $(\widehat{\theta}-\theta^0)D^{-1} \rightarrow N_d(0, 2c\sigma^2\Sigma^{-1})$ *where the matrix* D *is a* $(2+\frac{r(r+3)}{2}) \times (2+\frac{r(r+3)}{2})$ *diagonal matrix as follows:*

$$D = diag\left(M^{-\frac{1}{2}}N^{-\frac{1}{2}}, M^{-\frac{1}{2}}N^{-\frac{1}{2}}, M^{-j-\frac{1}{2}}N^{-(p-j)-\frac{1}{2}}, \ j=0,\cdots,p, \ p=1,\cdots,r\right),$$

$$\Sigma = \begin{bmatrix} 1 & 0 & V_1 \\ 0 & 1 & V_2 \\ V_1^T & V_2^T & M \end{bmatrix} \tag{11}$$

where $V_1 = (\frac{B^0}{(j+1)(p-j+1)}, \ j=0,\cdots,p, \ p=1,\cdots,r)$, $V_2 = (-\frac{A^0}{(j+1)(p-j+1)}, \ j = 0,\cdots,p, \ p=1,\cdots,r)$, *are vectors of order* $1 \times \frac{r(r+3)}{2}$,

$$M = (\frac{{A^0}^2+{B^0}^2}{(j+k+1)(p+q-j-k+1)}, j=0,\cdots,p, \ p=1,\cdots,r, \ k=0,\cdots,q, \ q=1,\cdots,r),$$

a matrix of order $\frac{r(r+3)}{2} \times \frac{r(r+3)}{2}$ *and* $c = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a(j,k)^2$. *Further,* $N_d(0, 2c\sigma^2\Sigma^{-1})$ *denotes a d-variate normal distribution with the mean vector* 0, *and the dispersion matrix* $2c\sigma^2\Sigma^{-1}$, *where* $d = 2+\frac{r(r+3)}{2}$.

Proof: See in the Supplementary Section.

Comments: Note that when $X(m,n)$'s are i.i.d. Gaussian random variables, then the maximum likelihood estimator of θ is the same as the least squares estimator. Hence due to Theorem 2, it follows that $(\widehat{\theta} - \theta^0)D^{-1} \rightarrow N_d(0, 2\sigma^2\Sigma^{-1})$. Now if $l(\theta)$ denotes the log-likelihood function in this case, then from the expressions of the elements of $Q''(\theta)$, see the Proof of Theorem 2, it follows that

$$E\left[D\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta'}D\right]_{\theta=\theta^0} \rightarrow \frac{1}{2\sigma^2}\Sigma.$$

Hence, it follows that the asymptotic variance of $\widehat{\theta}$ with proper normalization attains the Cramer-Rao lower bound.

4 Simulations and Data Analysis

4.1 Simulations

We perform some simulation experiments for different models, for different sample sizes and for different error variances mainly to see how the least squares estimators perform in practice based on the biases and mean squared errors (MSEs). We have considered the following two models:

Model 1:

$$\begin{aligned} y(m,n) = A^0\cos(\alpha^0 m + \beta^0 m^2 + \gamma^0 n + \delta^0 n^2) + B^0\sin(\alpha^0 m + \beta^0 m^2 + \gamma^0 n + \delta^0 n^2) \\ + X(m,n); \qquad m = 1,\cdots,M; \quad n = 1,\cdots,N. \end{aligned} \tag{12}$$

Here the model parameters are

$$A^0 = 5.0,\ B^0 = 5.0,\ \alpha^0 = 1.0,\ \beta^0 = 0.05,\ \gamma^0 = 1.5, \delta^0 = 0.5. \tag{13}$$

We have taken the following different sample sizes: $50 \times 50, 75 \times 75, 100 \times 100$ and the following two error structures namely;

$$\text{Error-I:}\ \ X(m,n) = \varepsilon(m,n); \tag{14}$$

$$\text{Error-II:}\ \ X(m,n) = \varepsilon(m,n) + 0.5\varepsilon(m-1,n) + 0.33\varepsilon(m,n-1). \tag{15}$$

We have taken two different distributions of $\varepsilon(m,n)$, (a) $\varepsilon(m,n)$s are *i.i.d.* Gaussian random variables with mean 0 and variance σ^2 and (b) $\varepsilon(m,n)$s are *i.i.d.* Laplace random variables with mean 0 and variance σ^2. We have considered different σ^2, namely 0.05 and 0.5, in our simulation experiments.

Model 2:

$$\begin{aligned} y(m,n) =& A^0 \cos(\alpha^0 m + \beta^0 m^2 + \eta^0 m^3 + \gamma^0 n + \delta^0 n^2 + \xi^0 n^3) + \\ & B^0 \sin(\alpha^0 m + \beta^0 m^2 + +\eta^0 m^3 + \gamma^0 n + \delta^0 n^2 + \xi^0 n^3) + X(m,n); \\ & m = 1, \cdots, M; \quad n = 1, \cdots, N. \end{aligned} \tag{16}$$

Here the model parameters are

$$A^0 = 2.0,\ B^0 = 2.0,\ \alpha^0 = 1.0,\ \beta^0 = 0.05,\ \eta^0 = 0.01,\ \gamma^0 = 1.0, \delta^0 = 0.05,\ \xi^0 = 0.01. \tag{17}$$

In this case also we have taken the following different sample sizes: $50 \times 50, 75 \times 75, 100 \times 100$ and the error structures Error-I only as defined in (15). It is further assumed that $\varepsilon(m,n)$s are *i.i.d.* Gaussian random variables with mean 0 and variance $\sigma^2 = 0.5$

We have used the random number generator RAN2 of Press et al. (1992) for generating the uniform random numbers. In each case the least squares estimators of the unknown parameters are obtained by using the *Downhill Simplex Algorithm*, see for example Press et al. (1992), whereas, the initial guesses are obtained by using grid search method using grid size of 0.01 around the true parameter values.

In each case we computed the least squares estimators, and obtained the average estimates, mean squared errors and variances over 1000 replications. We report the true parameter values (PARA), the average estimates (MEAN), the associated mean squared errors (MSE), variances (VAR). For comparison purposes we report the asymptotic variances (ASYV) also obtained using Theorem 2. For Model 1, in case of Gaussian errors, the results are reported in Tables 1 - 4, and in case of Laplace errors the results are reported in Tables 5 - 8. For Model 2, the results are reported in Table 9. All the tables are provided in the Supplementary Section.

Some of the points are quite clear from these tables. First of all, it is observed that as the error variances decrease the performance of the estimators in terms of MSEs improve. Also if the sample size increases the variances and the mean squared errors decrease in all the cases considered, as expected. The simulation results show that the least squares estimates are quite close to the true parameter values. For both the error structures it is observed that the mean squared errors of the least squares estimators match quite well with the corresponding asymptotic variances.

Comparing the two different error random variables it is observed that MSEs for the LSEs of the model parameters are slightly lower when the error variances follow Gaussian distribution than when the error variances follow Laplace distribution. But the LSEs behave quite well even when the error variances are Laplace distribution. Even for Model 2 the LSEs of the unknown parameters behave quite satisfactorily compared to the asymptotic variances of the corresponding estimators of the unknown parameters. It seems that the asymptotic results work quite well even for moderate sample sizes for different cases considered here.

4.2 Data Analysis

For illustrative purposes, mainly to show how the proposed method can be implemented in practice, we have analyzed two simulated data sets obtained from the model (1). We have used the following parameter values:

$$A^0 = 5.0, \quad B^0 = 1.0, \quad \alpha^0 = 1.55, \quad \beta^0 = 0.05, \quad \gamma^0 = 1.25, \quad \delta^0 = 0.075.$$

$X(n)$s are as follows,

$$X(m,n) = \epsilon(m,n) + 0.5\epsilon(m-1,n) + 0.33\epsilon(m,n-1) + 0.2\epsilon(m-1,n-1)$$

where $\epsilon's$ are assumed to be $i.i.d.$ Gaussian random variables with mean 0 and variance $\sigma^2 = 2.5$. We have also plotted one generated data set $\{y(m,n); m = 1,\ldots,100, n =$

$1, \ldots, 100$, in Figure 1. Figure 1 represents the 2-D image plot of a simulated noise corrupted $y(m,n)$, whose gray level at (m,n) is proportional to the value of $y(m,n)$. The problem is to extract the true texture, see Figure 2, from the contaminated one.

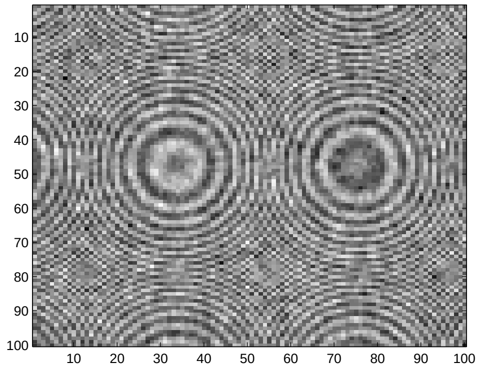

Figure 1: Noisy signal

We use the least squares technique and estimate the unknown parameters, and they are as follow:

$\widehat{A} = 5.003434,\ \widehat{B} = 0.965267,\ \widehat{\alpha} = 1.549228,\ \widehat{\beta} = 0.050006,\ \widehat{\gamma} = 1.250852, \widehat{\delta} = 0.074991.$

The estimated $y(m,n)$, namely $\widehat{y}(m,n)$ as in (10) are plotted in Figure 3. It is clear that the original and the estimated plots match quite well.

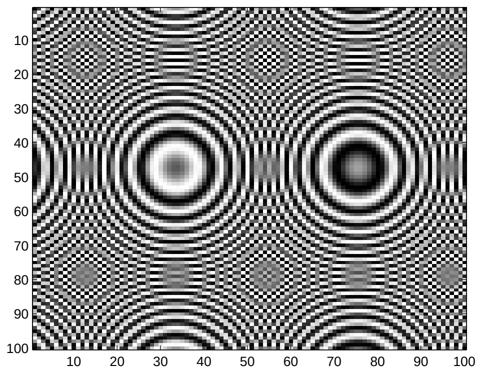

Figure 2: True signal

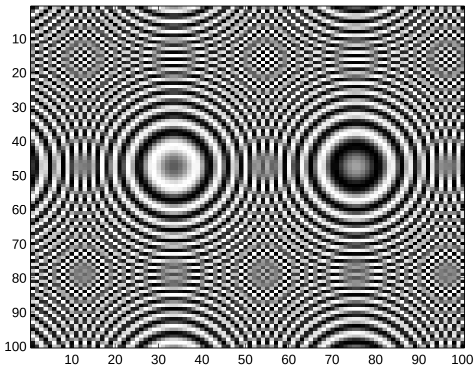

Figure 3: Estimated signal

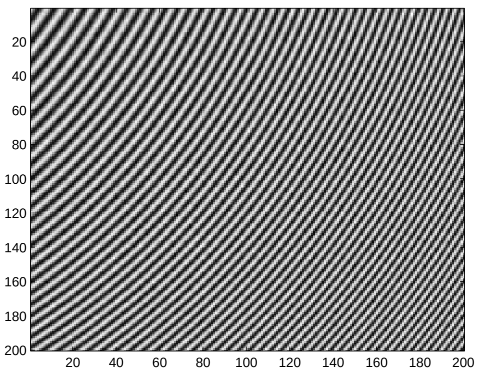

Figure 4: True image

Next we have generated another data with similar set of values as in Friedlander and Francos (1996) suitable for our model.

$$A^0 = 1.0, \quad B^0 = 1.0, \quad \alpha^0 = 0.45, \quad \beta^0 = 0.0015, \quad \gamma^0 = 0.82, \quad \delta^0 = 0.0022.$$

and $\sigma^2 = 0.005$. Then we have simulated the data and used our method as before to get the estimated image. They two images, true and estimated, also looks very similar. The estimates of the corresponding parameters are as follows $\widehat{A} = 0.305698$, $\widehat{B} = -0.054401$, $\widehat{\alpha} = 0.586525$, $\widehat{\beta} = 0.000995$, $\widehat{\gamma} = 0.969553$, $\widehat{\delta} = 0.001654$.

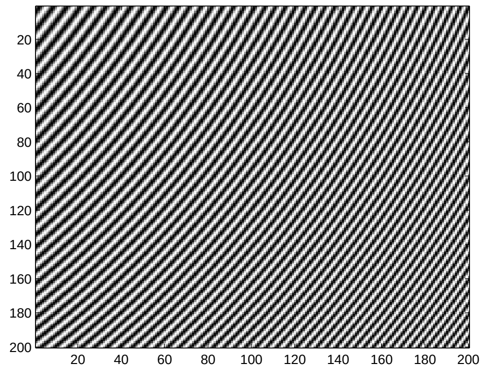

Figure 5: Estimated image

5 Conclusion

In this paper we consider the 2-D polynomial phase signal model and study the properties of the least squares estimators of the unknown parameters. We have proved that the least squares estimators are strongly consistent and they are asymptotically normally distributed. It is observed that the least squares estimators can be obtained as a $\frac{r(r+3)}{2}$ dimensional optimization problem. Our simulation results suggest that the asymptotic properties of the least squares estimators can be used quite effectively even for moderate sample sizes.

There are several open issues and generalizations which are of interests for future work. For example, it is observed that the least squares estimators can be obtained using a $\frac{r(r+3)}{2}$ dimensional optimization problem. It will be interesting to develop some numerically efficient algorithm to find a solution of this optimization problem. Moreover, although the least squares estimators are quite efficient, it is well known that they may not be very robust. Developing robust parameter estimation in this case will be of interest. More work is needed along that direction.

Supplementary

In this section we have provided all the tables based on the simulation experiments discussed in Section 4. In case of Model 1, for Gaussian errors the results are presented in Tables 1 - 4, and in case of Laplace errors the results are reported in Tables 5 - 8. For Model 2, the results are reported in Table 9. All the proofs also are presented in this section.

Table 1: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.05, Error-I and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998749	5.000799	0.999997	0.050000	1.500020	0.500000
MSE	(0.12381E-02)	(0.10095E-02)	(0.11243E-06)	(0.39859E-10)	(0.38522E-06)	(0.11305E-09)
VAR	(0.12365E-02)	(0.10089E-02)	(0.11242E-06)	(0.39853E-10)	(0.38490E-06)	(0.11297E-09)
ASYV	(0.72000E-03)	(0.72000E-03)	(0.61440E-07)	(0.23040E-10)	(0.61440E-07)	(0.23040E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.001370	4.998416	1.000027	0.050000	1.499981	0.500000
MSE	(0.43412E-03)	(0.43844E-03)	(0.37391E-07)	(0.63734E-11)	(0.37549E-07)	(0.63756E-11)
VAR	(0.43225E-03)	(0.43594E-03)	(0.36688E-07)	(0.62422E-11)	(0.37172E-07)	(0.63756E-11)
ASYV	(0.32000E-03)	(0.32000E-03)	(0.12136E-07)	(0.20227E-11)	(0.12136E-07)	(0.20221E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983871	5.015692	1.000094	0.049999	1.500076	0.500000
MSE	(0.35075E-03)	(0.32362E-03)	(0.15354E-07)	(0.14174E-11)	(0.75770E-08)	(0.75615E-12)
VAR	(0.90565E-04)	(0.77356E-04)	(0.64723E-08)	(0.62758E-12)	(0.17133E-08)	(0.32169E-12)
ASYV	(0.24000E-03)	(0.24000E-03)	(0.68285E-08)	(0.85333E-12)	(0.68285E-08)	(0.85333E-12)

Table 2: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.5, Error-I and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.993152	5.003687	1.000072	0.049999	1.500018	0.499999
MSE	(0.14862E-01)	(0.89375E-02)	(0.11333E-05)	(0.42206E-09)	(0.56846E-05)	(0.21873E-08)
VAR	(0.14815E-01)	(0.89239E-02)	(0.11280E-05)	(0.42062E-09)	(0.56842E-05)	(0.21871E-08)
ASYV	(0.72000E-02)	(0.72000E-02)	(0.61440E-06)	(0.23040E-09)	(0.61440E-06)	(0.23040E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998179	5.000843	1.000020	0.050000	1.500002	0.500000
MSE	(0.30544E-02)	(0.30886E-02)	(0.27571E-06)	(0.46163E-10)	(0.27355E-06)	(0.47246E-10)
VAR	(0.30511E-02)	(0.30879E-02)	(0.27533E-06)	(0.46128E-10)	(0.27354E-06)	(0.47246E-10)
ASYV	(0.32000E-02)	(0.32000E-02)	(0.12136E-06)	(0.20227E-10)	(0.12136E-06)	(0.20227E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.984965	5.013754	1.000111	0.049999	1.500057	0.500000
MSE	(0.69261E-03)	(0.71473E-03)	(0.52219E-07)	(0.55155E-11)	(0.56498E-08)	(0.96203E-12)
VAR	(0.46656E-03)	(0.52554E-03)	(0.39950E-07)	(0.42982E-11)	(0.24600E-08)	(0.65270E-12)
ASYV	(0.24000E-02)	(0.24000E-02)	(0.68285E-07)	(0.85333E-11)	(0.68285E-07)	(0.85333E-11)

Table 3: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.05, Error-II and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998425	5.001181	1.000016	0.050000	1.500002	0.500000
MSE	(0.68604E-03)	(0.68611E-03)	(0.11592E-06)	(0.44304E-10)	(0.11251E-06)	(0.40000E-10)
VAR	(0.68357E-03)	(0.68472E-03)	(0.11565E-06)	(0.44250E-10)	(0.11250E-06)	(0.40000E-10)
ASYV	(0.97840E-03)	(0.97840E-03)	(0.83490E-07)	(0.31309E-10)	(0.83490E-07)	(0.31309E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000978	4.998699	1.000024	0.050000	1.499987	0.500000
MSE	(0.48896E-03)	(0.47950E-03)	(0.44114E-07)	(0.76008E-11)	(0.45481E-07)	(0.78724E-11)
VAR	(0.48799E-03)	(0.47780E-03)	(0.43524E-07)	(0.74974E-11)	(0.45366E-07)	(0.78669E-11)
ASYV	(0.43484E-03)	(0.43484E-03)	(0.16492E-07)	(0.27486E-11)	(0.16492E-07)	(0.27486E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983525	5.016331	1.000103	0.049999	1.500071	0.500000
MSE	(0.43866E-03)	(0.42898E-03)	(0.24280E-07)	(0.21339E-11)	(0.67342E-08)	(0.68071E-12)
VAR	(0.16725E-03)	(0.16233E-03)	(0.13803E-07)	(0.12060E-11)	(0.16579E-08)	(0.31423E-12)
ASYV	(0.32613E-03)	(0.32613E-03)	(0.92768E-08)	(0.11596E-11)	(0.92768E-08)	(0.11596E-11)

Table 4: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.5, Error-II and $\varepsilon(m,n)$'s are Gaussian random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.995960	5.002028	1.000068	0.049999	1.499953	0.500001
MSE	(0.69557E-02)	(0.69637E-02)	(0.14144E-05)	(0.53330E-09)	(0.11924E-05)	(0.43918E-09)
VAR	(0.69394E-02)	(0.69596E-02)	(0.14098E-05)	(0.53240E-09)	(0.11900E-05)	(0.43844E-09)
ASYV	(0.97840E-02)	(0.97840E-02)	(0.83490E-06)	(0.31309E-09)	(0.83490E-06)	(0.31309E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.998765	4.999673	1.000021	0.050000	1.499990	0.500000
MSE	(0.44162E-02)	(0.44612E-02)	(0.39796E-06)	(0.61326E-10)	(0.31483E-06)	(0.53257E-10)
VAR	(0.44147E-02)	(0.44611E-02)	(0.39755E-06)	(0.61311E-10)	(0.31474E-06)	(0.53251E-10)
ASYV	(0.43484E-02)	(0.43484E-02)	(0.16492E-06)	(0.27486E-10)	(0.16492E-06)	(0.27486E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.981299	5.018357	1.000159	0.049998	1.500048	0.500000
MSE	(0.86866E-03)	(0.10029E-02)	(0.80940E-07)	(0.82032E-11)	(0.45432E-08)	(0.93503E-12)
VAR	(0.51900E-03)	(0.66586E-03)	(0.55460E-07)	(0.57826E-11)	(0.22498E-08)	(0.76692E-12)
ASYV	(0.32613E-02)	(0.32613E-02)	(0.92768E-07)	(0.11596E-10)	(0.92768E-07)	(0.11596E-10)

Table 5: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.05, Error-I and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.006754	5.001654	1.002316	0.049912	1.500532	0.499913
MSE	(0.13214E-02)	(0.10765E-02)	(0.11765E-06)	(0.40145E-10)	(0.39675E-06)	(0.12267E-09)
VAR	(0.13112E-02)	(0.10498E-02)	(0.11111E-06)	(0.39264E-10)	(0.39541E-06)	(0.12001E-09)
ASYV	(0.72000E-03)	(0.72000E-03)	(0.61440E-07)	(0.23040E-10)	(0.61440E-07)	(0.23040E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.003412	5.000132	0.999911	0.049992	1.500016	0.499981
MSE	(0.44563E-03)	(0.44983E-03)	(0.37998E-07)	(0.64876E-11)	(0.37999E-07)	(0.64236E-11)
VAR	(0.44123E-03)	(0.43889E-03)	(0.37076E-07)	(0.63991E-11)	(0.37571E-07)	(0.63998E-11)
ASYV	(0.32000E-03)	(0.32000E-03)	(0.12136E-07)	(0.20227E-11)	(0.12136E-07)	(0.20221E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.983871	5.015692	1.000094	0.049999	1.500076	0.500000
MSE	(0.35657E-03)	(0.32463E-03)	(0.15588E-07)	(0.14768E-11)	(0.75991E-08)	(0.75899E-12)
VAR	(0.90786E-04)	(0.77651E-04)	(0.64915E-08)	(0.62887E-12)	(0.17387E-08)	(0.32342E-12)
ASYV	(0.24000E-03)	(0.24000E-03)	(0.68285E-08)	(0.85333E-12)	(0.68285E-08)	(0.85333E-12)

Table 6: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.5, Error-I and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.996754	4.998767	1.000564	0.050031	1.500784	0.499943
MSE	(0.15123E-01)	(0.94325E-02)	(0.12875E-05)	(0.436574-09)	(0.57865E-05)	(0.22998E-08)
VAR	(0.14993E-01)	(0.91234E-02)	(0.11887E-05)	(0.43018E-09)	(0.57003E-05)	(0.21997E-08)
ASYV	(0.72000E-02)	(0.72000E-02)	(0.61440E-06)	(0.23040E-09)	(0.61440E-06)	(0.23040E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000153	5.001065	1.000776	0.049965	1.500232	0.500116
MSE	(0.32287E-02)	(0.327781-02)	(0.28671E-06)	(0.47861E-10)	(0.28112E-06)	(0.47998E-10)
VAR	(0.31943E-02)	(0.31671E-02)	(0.28001E-06)	(0.46995E-10)	(0.27967E-06)	(0.47848E-10)
ASYV	(0.32000E-02)	(0.32000E-02)	(0.12136E-06)	(0.20227E-10)	(0.12136E-06)	(0.20227E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000119	5.011234	1.000675	0.050014	1.500165	0.500087
MSE	(0.69867E-03)	(0.71675E-03)	(0.52568E-07)	(0.55376E-11)	(0.56621E-08)	(0.96701E-12)
VAR	(0.47212E-03)	(0.53019E-03)	(0.41671E-07)	(0.43789E-11)	(0.25105E-08)	(0.67562E-12)
ASYV	(0.24000E-02)	(0.24000E-02)	(0.68285E-07)	(0.85333E-11)	(0.68285E-07)	(0.85333E-11)

Table 7: The MEAN, MSE, VAR and ASYV of the least squares estimators when σ^2 = 0.05, Error-II and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.991154	4.976781	1.004516	0.0499856	1.500176	0.500232
MSE	(0.71657E-03)	(0.71245E-03)	(0.13561E-06)	(0.46678E-10)	(0.13391E-06)	(0.42584E-10)
VAR	(0.70651E-03)	(0.70067E-03)	(0.128971-06)	(0.455423-10)	(0.12891E-06)	(0.41675E-10)
ASYV	(0.97840E-03)	(0.97840E-03)	(0.83490E-07)	(0.31309E-10)	(0.83490E-07)	(0.31309E-10)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.999651	4.995543	1.000667	0.050112	1.500143	0.499945
MSE	(0.50187E-03)	(0.500957-03)	(0.45875E-07)	(0.78098E-11)	(0.471541-07)	(0.80009E-11)
VAR	(0.49089E-03)	(0.48761E-03)	(0.43998E-07)	(0.75671E-11)	(0.460091-07)	(0.78998E-11)
ASYV	(0.43484E-03)	(0.43484E-03)	(0.16492E-07)	(0.27486E-11)	(0.16492E-07)	(0.27486E-11)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000561	5.001761	1.000451	0.049965	1.500110	0.499993
MSE	(0.44541E-03)	(0.438716-03)	(0.24678E-07)	(0.21667E-11)	(0.67981E-08)	(0.68121E-12)
VAR	(0.172345-03)	(0.16876E-03)	(0.14365E-07)	(0.14671E-11)	(0.18867E-08)	(0.33451E-12)
ASYV	(0.32613E-03)	(0.32613E-03)	(0.92768E-08)	(0.11596E-11)	(0.92768E-08)	(0.11596E-11)

Table 8: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-II and $\varepsilon(m,n)$'s are Laplace random variables (Model 1)

M=N=50						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	5.000541	5.006571	1.000675	0.050071	1.499786	0.500112
MSE	(0.71453E-02)	(0.71981E-02)	(0.166547-05)	(0.55543E-09)	(0.13675E-05)	(0.45871E-09)
VAR	(0.69878E-02)	(0.700876-02)	(0.157643-05)	(0.54018E-09)	(0.12287E-05)	(0.45001E-09)
ASYV	(0.97840E-02)	(0.97840E-02)	(0.83490E-06)	(0.31309E-09)	(0.83490E-06)	(0.31309E-09)
M=N=75						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.994311	5.000113	1.000254	0.050176	1.500087	0.500267
MSE	(0.46654E-02)	(0.46098E-02)	(0.41076E-06)	(0.63245E-10)	(0.33528E-06)	(0.55267E-10)
VAR	(0.45186E-02)	(0.459981-02)	(0.40185E-06)	(0.62376E-10)	(0.32675E-06)	(0.54987E-10)
ASYV	(0.43484E-02)	(0.43484E-02)	(0.16492E-06)	(0.27486E-10)	(0.16492E-06)	(0.27486E-10)
M=N=100						
PARA	5.00	5.00	1.00	0.05	1.50	0.50
MEAN	4.982453	5.012546	1.000441	0.050014	1.500067	0.500132
MSE	(0.88876E-03)	(0.11768E-02)	(0.82001E-07)	(0.83176E-11)	(0.46651E-08)	(0.94093E-12)
VAR	(0.52176E-03)	(0.66987E-03)	(0.55860E-07)	(0.58019E-11)	(0.22998E-08)	(0.77016E-12)
ASYV	(0.32613E-02)	(0.32613E-02)	(0.92768E-07)	(0.11596E-10)	(0.92768E-07)	(0.11596E-10)

Table 9: The MEAN, MSE, VAR and ASYV of the least squares estimators when $\sigma^2 = 0.5$, Error-I and $\varepsilon(m,n)$'s are Laplace random variables (Model 2)

M=N=50								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	1.987911	2.012565	1.000834	0.050032	0.010001	1.000878	0.050028	0.010003
MSE	(0.37671E-01)	(0.35645E-01)	(0.25645E-05)	(0.75678E-09)	(0.41178E-12)	(0.25538E-05)	(0.76017E-09)	(0.43176E-12)
VAR	(0.29879E-01)	(0.276756-01)	(0.21786E-05)	(0.74564E-09)	(0.39876E-12)	(0.22176E-05)	(0.74018E-09)	(0.40173E-12)
ASYV	(0.21543E-01)	(0.21543E-01)	(0.18232E-05)	(0.68124E-09)	(0.13672E-12)	(0.18232E-05)	(0.68124E-09)	(0.13672E-12)
M=N=75								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	1.995671	1.994532	1.000675	0.050010	0.010000	1.000112	0.500112	0.010000
MSE	(0.16523E-01)	(0.166587-01)	(0.62367E-06)	(0.82567E-09)	(0.12765E-13)	(0.61786E-06)	(0.81987E-09)	(0.13451E-13)
VAR	(0.13668E-01)	(0.140171-01)	(0.58764E-06)	(0.81198E-09)	(0.111453-13)	(0.599765-06)	(0.81076E-09)	(0.12017E-13)
ASYV	(0.94789E-02)	(0.94789E-02)	(0.34578E-06)	(0.61235E-10)	(0.90123E-14)	(0.34587E-06)	(0.61235E-10)	(0.90123E-14)
M=N=100								
PARA	2.00	2.00	1.00	0.05	0.01	1.0	0.05	0.01
MEAN	2.000897	2.000765	1.000675	0.050006	0.010000	1.499786	0.500112	0.010000
MSE	(0.99865E-02)	(0.98769E-02)	(0.29876E-05)	(0.35672E-11)	(0.15467E-16)	(0.30156E-06)	(0.34561E-11)	(0.15569E-16)
VAR	(0.90167E-02)	(0.914536-02)	(0.26778E-05)	(0.31987E-11)	(0.14221E-16)	(0.27801E-06)	(0.31675E-11)	(0.14451E-16)
ASYV	(0.70167E-02)	(0.70167E-02)	(0.22167E-06)	(0.23156E-11)	(0.10176E-16)	(0.22167E-06)	(0.23156E-11)	(0.10176E-16)

Proof of Proposition-1 The proofs can be obtained using the results of Vinogradov (1954) for estimating Weyl (1916)'s sum for one and multi-dimensions. Suppose for a given $k > 0$, $f(n) = \alpha_1 n + \alpha_2 n^2 + \ldots \alpha_n n^k$ for $n = 1, 2, \cdots$, where $\alpha_1, \ldots, \alpha_k$ are real numbers, and for a positive integer N, $S = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Then except for countable number of points $\alpha_1, \ldots, \alpha_k$, $S = O(N^{1-\rho})$, for some $\rho > 0$. Hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k)} = 0.$$

Therefore, we immediately have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \cos(\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k) = 0 \text{ and } \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sin(\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k) = 0.$$

Similarly, if we denote $f(m, n) = \sum_{p=1}^{r} \sum_{j=0}^{p} \alpha^0(j, p-j) m^j n^{p-j}$ and $S = \sum_{m=1}^{M} \sum_{n=1}^{N} e^{2\pi i f(m,n)}$, then except for countable number of points $\{\alpha^0(j, p-j), j = 0, 1, \ldots, p, p = 1, \ldots, r\}$, $S = O(N^{1-\rho_1} M^{1-\rho_2})$ for some $\rho_1 > 0$ and $\rho_2 > 0$. Hence (4) and (5) follow as

$$\lim_{\min\{M,N\}\to\infty} \frac{1}{NM} \sum_{m=1}^{M} \sum_{n=1}^{N} e^{2\pi i f(m,n)} = 0.$$

For s, t positive integers, we also have

$$\lim_{\min\{M,N\}\to\infty} \frac{1}{N^{t+1} M^{s+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^s n^t e^{2\pi i f(m,n)} = 0. \tag{18}$$

Now for s, t positive integers, using (18) and

$$\lim_{M\to\infty} \frac{1}{M^{s+1}} \sum_{m=1}^{M} m^s = \frac{1}{s+1} \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{M} n^t = \frac{1}{t+1},$$

(6), (7) follow.

Proof of Lemma-1 First we will prove for $r = 2$ to observe how the proof works and then provide the proof for general r. We will use the following results to prove it.

Result 1: For fixed $j(k) = 1, 2, \cdots, M - 1(N - 1)$, and for fixed l, t such that

$|j-l|<M,\ |k-t|<N$, using Holder's inequality we have

$$
\begin{aligned}
& E\left|\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j}\varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right| \\
\leq & \left[E\left(\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j}\varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right)^2\right]^{\frac{1}{2}} \\
= & \left[E\sum_{n=k+1}^{N-t+k}\sum_{m=j+1}^{M-l+j}\left(\varepsilon(m,n)\varepsilon(m-j,n-k)\varepsilon(m+l,n+t)\varepsilon(m-j+l,n-k+t)\right)^2\right]^{\frac{1}{2}} \\
= & O(MN)^{\frac{1}{2}}
\end{aligned}
$$

The equality at the third step of the above expression holds as contribution over cross product terms is zero. ∎

Result 2: If we denote $q(\phi,m,n)=t_1\alpha m+t_2\beta n$, i.e. $q(\phi,m,n)$ is a linear function of m and n, for some t_1,t_2, where $\phi=(\alpha,\beta)$ then again using Holder's inequality, we have

$$
\begin{aligned}
& E\sup_{\phi}\left|\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right| \\
\leq & \left[E\left(\sup_{\phi}\left|\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right|\right)^2\right]^{\frac{1}{2}} \\
= & \left[E\sup_{\phi}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{iq(\phi,m,n)}\right)\right. \\
& \left.\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m-j,n-k)e^{-iq(\phi,m,n)}\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \Bigg[E\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m,n)^2\varepsilon(m-j,n-k)^2 \\
+\ & 2E\left|\sum_{n=1}^{N-1}\sum_{m=1}^{M}\varepsilon(m,n)\varepsilon(m,n+1)\varepsilon(m-j,n-k)\varepsilon(m-j,n-k+1)\right| \\
+\ & 2E\left|\sum_{n=1}^{N}\sum_{m=1}^{M-1}\varepsilon(m,n)\varepsilon(m+1,n)\varepsilon(m-j,n-k)\varepsilon(m-j+1,n-k)\right| \\
+\ & \cdots + 2E\left|\sum_{m=1}^{M}\varepsilon(m,1)\varepsilon(m,N)\varepsilon(m-j,1-k)\varepsilon(m-j,N-k)\right| \\
& \quad +2E\left|\sum_{n=1}^{N}\varepsilon(1,n)\varepsilon(M,n)\varepsilon(1-j,n-k)\varepsilon(M-j,n-k)\right| \\
+\ & 2E\left|\sum_{n=1}^{N-1}\sum_{m=1}^{M-1}\varepsilon(m,n)\varepsilon(m+1,n+1)\varepsilon(m-j,n-k)\varepsilon(m-j+1,n-k+1)\right| \\
+\ & \cdots + 2E\left|\varepsilon(M,1)\varepsilon(M,N)\varepsilon(M-j,1-k)\varepsilon(M-j,N-k)\right| \\
+\ & 2E\left|\varepsilon(1,N)\varepsilon(M,N)\varepsilon(1-j,N-k)\varepsilon(M-j,N-k)\right|\Bigg]^{\frac{1}{2}} \\
=\ & O(MN + MN.(MN)^{\frac{1}{2}})^{\frac{1}{2}} = O((MN)^{\frac{3}{4}}).
\end{aligned}
$$

∎

If $q(\xi,m,n) = \alpha m + \beta m^2 + \gamma n + \delta n^2 + \nu mn$, i.e. $q(\xi,m,n)$ is a quadratic function of m and n, where $\xi = (\alpha,\beta,\gamma,\delta,\nu)$ and let $m-j = m', n-k = n'$. Then

$$
\begin{aligned}
& E\sup_{\xi}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}X(m,n)e^{iq(\xi,m,n)}\right| \\
& = E\sup_{\xi}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}a(j,k)\varepsilon(m-j,n-k)e^{iq(\xi,m,n)}\right| \\
& = E\sup_{\xi}\left|\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}a(j,k)\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m',n')e^{iq(\xi,m,n)}\right| \\
& \leq E\sup_{\xi}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left|a(j,k)\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m',n')e^{iq(\xi,m,n)}\right| \\
& = E\sup_{\xi}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|a(j,k)|\frac{1}{MN}\left|\sum_{n=1}^{N}\sum_{m=1}^{M}\varepsilon(m',n')e^{iq(\xi,m,n)}\right|
\end{aligned}
$$

$$\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \left[E \sup_{\xi} \frac{1}{MN} \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n') e^{iq(\xi,m,n)} \right| \right]$$

$$\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \left[E \sup_{\xi} \frac{1}{MN} \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n') e^{iq(\xi,m,n)} \right|^2 \right]^{\frac{1}{2}}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)|$$

$$\times \frac{1}{MN} \left[E \sup_{\xi} \left(\sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n') e^{iq(\xi,m,n)} \right) \left(\sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n') e^{-iq(\xi,m,n)} \right) \right]^{\frac{1}{2}}$$

$$\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a(j,k)| \quad \times \frac{1}{MN} \left[E \sum_{n=1}^{N} \sum_{m=1}^{M} \varepsilon(m', n')^2 \right.$$

$$+E \left| \sum_{n=1}^{N-1} \sum_{m=1}^{M} \varepsilon(m', n') \varepsilon(m', n'+1) e^{-i(2\delta n + \nu n)} \right|$$

$$+E \left| \sum_{n=1}^{N} \sum_{m=1}^{M-1} \varepsilon(m', n') \varepsilon(m'+1, n') e^{-i(2\beta m + \nu m)} \right|$$

$$+ \cdots + E \left| \sum_{m=1}^{M} \varepsilon(m', 1-k) \varepsilon(m', N-k) \right|$$

$$+E \left| \sum_{n=1}^{N} \varepsilon(1-j, n') \varepsilon(M-j, n') \right|$$

$$+ \cdots + E \left| \varepsilon(1-j, 1-k) \varepsilon(M-j, N-k) \right|$$

$$\left. +E \left| \varepsilon(1-j, 1-k) \varepsilon(M-j, N-k) \right| \right]^{\frac{1}{2}}$$

$$= \frac{1}{MN} O(MN + MN.(MN)^{\frac{3}{4}})^{\frac{1}{2}} = O((MN)^{-\frac{1}{8}}).$$

Therefore,

$$E \sup_{\xi} \left| \frac{1}{(MN)^9} \sum_{n=1}^{N^9} \sum_{m=1}^{M^9} X(m,n) e^{iq(\xi,m,n)} \right| \leq O((MN)^{-\frac{9}{8}}) \qquad (19)$$

Take

$$Z(M,N) = \sup_{\xi} \left| \frac{1}{(MN)^9} \sum_{n=1}^{N^9} \sum_{m=1}^{M^9} X(m,n) e^{iq(\xi,m,n)} \right|$$

and for $\epsilon \quad > 0$

$$\sum_{N=1}^{\infty} \sum_{M=1}^{\infty} P(Z(M,N) > \epsilon) \leq \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{EZ(M,N)}{\epsilon} \leq \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{O((MN)^{-\frac{9}{8}})}{\epsilon} < \infty$$

Therefore, by Borel Cantelli lemma we have

$$\sup_{\xi}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)}\right| \to \quad 0 \quad a.s.$$

Denote, $\{(J,K): N^9 < K \le (N+1)^9,\ M^9 < J \le (M+1)^9\} \ = S_{JK}$

Define,

$$\begin{aligned}
U(M,N) \quad &= \sup_{\xi}\sup_{S_{JK}}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{JK}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right| \\
&= \sup_{\xi}\sup_{S_{JK}}\left|\frac{1}{(MN)^9}\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{(MN)^9}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right. \\
&\qquad \left. + \frac{1}{(MN)^9}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)} - \frac{1}{JK}\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right| \\
&\le \sup_{\xi}\sup_{S_{JK}}\left[\frac{1}{(MN)^9}\left|\sum_{n=1}^{N^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)} - \sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right|\right. \\
&\qquad \left. + \left(\frac{1}{(MN)^9} - \frac{1}{(M+1)^9(N+1)^9}\right)\left|\sum_{n=1}^{K}\sum_{m=1}^{J}X(m,n)e^{iq(\xi,m,n)}\right|\right] \\
&\le V(M,N) + W(M,N),
\end{aligned}$$

where

$$\begin{aligned}
V(M,N) = \quad & \sup_{\xi}\left|\sum_{n=N^9+1}^{(N+1)^9}\sum_{m=1}^{M^9}X(m,n)e^{iq(\xi,m,n)}\right| + \sup_{\xi}\left|\sum_{n=1}^{N^9}\sum_{m=M^9+1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right| \\
&+ \sup_{\xi}\left|\sum_{n=N^9+1}^{(N+1)^9}\sum_{m=M^9+1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right|,
\end{aligned}$$

$$W(M,N) = \left(\frac{1}{(MN)^9} - \frac{1}{(M+1)^9(N+1)^9}\right)\sup_{\xi}\left|\sum_{n=1}^{(N+1)^9}\sum_{m=1}^{(M+1)^9}X(m,n)e^{iq(\xi,m,n)}\right|.$$

We have

$$E\sup_{\xi}\left|\sum_{n=n_0+1}^{N}\sum_{m=m_0+1}^{M}X(m,n)e^{iq(\xi,m,n)}\right| \le O((M-m_0)(N-n_0))^{\frac{7}{8}} \tag{20}$$

and

$$P(V(M,N) > \epsilon) \le \frac{EV(M,N)}{\epsilon} \le \frac{O(\frac{(M^9N^8)^{\frac{7}{8}} + (M^8N^9)^{\frac{7}{8}}}{(MN)^9})}{\epsilon}. \tag{21}$$

Therefore,

$$\sum_{N=1}^{\infty}\sum_{M=1}^{\infty} P(V(M,N) > \epsilon) \leq \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{O((MN)^{-\frac{9}{8}})}{\epsilon} < \infty.$$

We also have

$$P(W(M,N) > \epsilon) \leq \frac{EW(M,N)}{\epsilon} \leq \frac{O((\frac{1}{M}+\frac{1}{N})(M+1)^{\frac{-9}{8}}(N+1)^{\frac{-9}{8}})}{\epsilon}$$

Therefore,

$$\sum_{N=1}^{\infty}\sum_{M=1}^{\infty} P(W(M,N) > \epsilon) \leq \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{O((MN)^{-\frac{9}{8}})}{\epsilon} < \infty$$

Now by Borel Cantelli lemma we get $U(M,N) \rightarrow \ 0 \ \ a.s.$

Now we will analyze the previous proof. To show almost sure convergence the tool used was Borel Cantelli lemma for which we were required Markov inequality. To get probability bound in Markov inequality, we tried to calculate corresponding expectation as follows:

$$E\sup_{\xi}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M} X(m,n)e^{iq(\xi,m,n)}\right| \tag{22}$$

As $q(\xi,m,n)$ is a quadratic in $m,\ n$ while simplifying (22) we need similar expectation calculations, Result 2 for linear $q(\phi,m,n)$ and Result 1 with zeroth degree polynomial in $m,\ n$. Now in our original case we need similar expectation calculation for rth degree polynomial of m and n. If we would take $q(\alpha,m,n) = \sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}$, i.e. $q(\alpha,m,n)$ is rth degree polynomial of m and n, where $\alpha = (\alpha^0(j,p-j), j = 0,\cdots,p, p = 1,\cdots,r)$ then our object of interest will be $E\sup_{\alpha}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M} X(m,n)e^{iq(\alpha,m,n)}\right|$, for which we would need r many results, like Result 1 and 2, for $r-1, r-2\cdots,1,0$ th degree polynomials. Also note that for quadratic case we need the existence of fourth moment whereas for rth degree polynomial case we need existence of $2r$th moment. Now we can proceed for the proof of Lemma 1. Along same line as before

$$\begin{aligned}&E\sup_{\alpha}\left|\frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M} X(m,n)e^{iq(\alpha,m,n)}\right|\\ &= O((MN)^{-\frac{1}{2^{r+1}}})\end{aligned}$$

In next step, similar way as before, we get

$$\sup_{\alpha} \left| \frac{1}{(MN)^{2^{r+1}+1}} \sum_{n=1}^{N^{2^{r+1}+1}} \sum_{m=1}^{M^{2^{r+1}+1}} X(m,n) e^{iq(\alpha,m,n)} \right| \rightarrow \quad 0 \quad a.s.$$

For rest of the proof replacing subsequences M^9, N^9 by $M^{2^{r+1}+1}, N^{2^{r+1}+1}$, and ξ by α we arrive the final conclusion. ∎.

Proof of Lemma 2: It can be obtained along the same line.

To prove the Theorem-1 we need the following Lemma.

Lemma 4. *Let $\widehat{\theta}$ be the least squares estimator of θ^0, and consider the set $S_c = \{\theta : \theta \in \Theta; |A - A^0| \geq c, |B - B^0| \geq c, |\alpha(j, p-j) - \alpha^0(j, p-j)| \geq c,\ j = 0, \cdots, p,\ p = 1, \cdots, r\}$. If for any $c > 0$, $\liminf \inf_{\theta \in S_c} \frac{1}{MN}(Q(\theta) - Q(\theta^0)) > \quad 0 \quad a.s.$ then $\widehat{\theta} \rightarrow \theta^0 \quad a.s.$. Here the function $Q(\theta)$ is same as defined in (9).*

Proof of Lemma 4: The proof can be obtained by contradiction, along the lines of lemma 1 of Wu(1981). If $\widehat{\theta^{(N)}}$ does not converges to θ^0 then there exists a subsequence $\{N_k\}_{k=1}^{\infty}$ along which it fails to converge. Let the collection of ω's, on which it fails to converge is Ω_0. Now $\widehat{\theta^{(N_k)}}$ is LSE for θ^0 and so minimize the quantity $Q_1(\theta)$. That implies on Ω_0 (which is subset of whole set of ω) $\frac{1}{N_k}(Q(\widehat{\theta^{(N_k)}}) - Q_1(\theta^0)) < \ 0$. Then on whole set of Ω $\liminf \inf_{\theta \in S_c} \frac{1}{N}(Q(\theta) - Q(\theta^0)) \geq \quad 0 \quad a.s.$ which is a contradiction.

Proof of Theorem 1:

To prove Theorem 1, it is enough to prove that

$$\liminf \inf_{\theta \in S_c} \frac{1}{MN}(Q(\theta) - Q(\theta^0)) > \quad 0 \quad a.s.$$

Observe that

$$\frac{1}{MN}[Q(\theta) - Q(\theta^0)] = f(\theta) + g(\theta),$$

where

$$f(\theta) = \frac{1}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\Bigg[A\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^jn^{p-j}) + B\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^jn^{p-j})$$
$$-A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j}) - B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j})\Bigg]^2$$

$$g(\theta) = \frac{2}{MN}\sum_{n=1}^{N}\sum_{m=1}^{M}\Bigg[A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j}) + B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^jn^{p-j})$$
$$-A\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^jn^{p-j}) - B\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^jn^{p-j})\Bigg]X(m,n)$$

Now $g(\theta)$ is going to zero $a.s.$ because of Lemma-1 . We now observe that

$$S_c \subset S_c^A \cup S_c^B \cup_{p=1}^{r} \cup_{j=0}^{p} S_c^{\alpha(j,p-j)}$$

where $S_c^A = \{\theta; \theta \in \Theta, |A - A^0| \geq c\}$, and the other sets are also similarly defined. It can be shown along the same line as in Kundu (1997) that

$$\liminf \inf_{S_c^{\mathfrak{t}}} f(\theta) > 0,\ a.s. \quad \text{for } \mathfrak{t} \text{ is any one of } A, B, \alpha(j,p-j),\ j = 0,\cdots,p,\ p = 1,\cdots,r$$

and hence the result is proved. ∎

Proof of Lemma 3:

Let us denote $Q'(\theta)$ as the $(2+\frac{r(r+3)}{2})\times 1$ first derivative matrix and $Q''(\theta)$ as the $(2+\frac{r(r+3)}{2})\times(2+\frac{r(r+3)}{2})$ second derivative matrix. Now using multivariate Taylor series expansion of $Q'(\widehat{\theta})$ around θ^0 and we get

$$Q'(\widehat{\theta}) - Q'(\theta^0) = (\widehat{\theta} - \theta^0)Q''(\bar{\theta}) \tag{23}$$

where $\bar{\theta}$ is a point on line joining $\widehat{\theta}$ and θ^0. Since, $Q'(\widehat{\theta}) = 0$, then for the diagonal matrix D, same as defined in Theorem 2, we obtain

$$-Q'(\theta^0)D = (\widehat{\theta} - \theta^0)D^{-1}[DQ''(\bar{\theta})D] \tag{24}$$

which gives,

$$(\widehat{\theta} - \theta^0)D^{-1} = [-Q'(\theta^0)D][DQ''(\bar{\theta})D]^{-1} \tag{25}$$

Dividing by $\sqrt{MN}$ the expression becomes

$$(\widehat{\theta}-\theta^0)(\sqrt{MN}D)^{-1}=[-\frac{1}{\sqrt{MN}}Q'(\theta^0)D][DQ''(\bar{\theta})D]^{-1} \tag{26}$$

Since, $\widehat{\theta}\to\theta^0\ \ a.s.$, $\bar{\theta}\to\theta^0\ \ a.s.$. Therefore,

$$\left[DQ''(\bar{\theta})D\right]^{-1}\to\left[DQ''(\theta^0)D\right]^{-1}$$

Moreover, using Lemma 1 and Proposition 2,

$$\frac{1}{\sqrt{MN}}Q'(\theta^0)D\to 0\ a.s. \tag{27}$$

So,

$$(\widehat{\theta}-\theta^0)(\sqrt{MN}D)^{-1}\to 0\ a.s. \tag{28}$$

Hence using (28) we get for $j=0,\cdots,p,\ p=1,\cdots,r$

$$M^jN^{p-j}(\widehat{\alpha}(j,p-j)-\alpha^0(j,p-j))\to 0\ a.s.$$

∎

Proof of Theorem 2: We recall $q(\alpha,m,n)=\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha(j,p-j)m^jn^{p-j}$. Note that

$$\left[Q'(\theta^0)D\right]^T=\begin{bmatrix} -\frac{2}{\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}\cos(q(\alpha^0,m,n))X(m,n) \\ -\frac{2}{\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}\sin(q(\alpha^0,m,n))X(m,n) \\ \frac{2}{M\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}m[A^0\sin(q(\alpha^0,m,n))-B^0\cos(q(\alpha^0,m,n))]\ X(m,n) \\ \frac{2}{M^2\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}m^2[A^0\sin(q(\alpha^0,m,n))-B^0\cos(q(\alpha^0,m,n))]\ X(m,n) \\ \frac{2}{N\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}n[A^0\sin(q(\alpha^0,m,n))-B^0\cos(q(\alpha^0,m,n))]\ X(m,n) \\ \frac{2}{N^2\sqrt{MN}}\sum_{n=1}^{N}\sum_{m=1}^{M}n^2[A^0\sin(q(\alpha^0,m,n))-B^0\cos(q(\alpha^0,m,n))]\ X(m,n). \end{bmatrix}$$

Now using Central Limit Theorem of linear processes, see Fuller (1996,), page 329, it follows that

$$\left[Q'(\theta^0)D\right]^T\to N_6(0,2c\sigma^2\Sigma). \tag{29}$$

Also,

$$\frac{\partial^2 Q(\theta)}{\partial A^2}\Big|_{\theta^0} = 2\sum_{n=1}^{N}\cos^2(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}),$$

$$\frac{\partial^2 Q(\theta)}{\partial A\partial B}\Big|_{\theta^0} = 2\sum_{n=1}^{N}\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}),$$

$$\frac{\partial^2 Q(\theta)}{\partial B^2}\Big|_{\theta^0} = 2\sum_{n=1}^{N}\sin^2(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}),$$

$$\frac{\partial^2 Q(\theta)}{\partial A\partial\alpha(j,p-j)}\Big|_{\theta^0} = 2\sum_{n=1}^{N}m^j n^{p-j}\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})\times X(n)$$

$$-2\sum_{n=1}^{N}m^j n^{p-j}\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})$$

$$\times[A^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) - B^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})],$$

$$\frac{\partial^2 Q(\theta)}{\partial B\partial\alpha(j,p-j)}\Big|_{\theta^0} = -2\sum_{n=1}^{N}m^j n^{p-j}\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})\times X(n)$$

$$-2\sum_{n=1}^{N}m^j n^{p-j}\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})$$

$$\times[A^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) - B^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})],$$

$$\frac{\partial^2 Q(\theta)}{\partial\alpha(j,p-j)\partial\alpha(k,q-k)}\Big|_{\theta^0} = 2\sum_{n=1}^{N}m^{j+k}n^{p+q-j-k+1}[A^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})$$

$$+B^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})]\times X(n)$$

$$+2\sum_{n=1}^{N}m^{j+k}n^{p+q-j-k+1}[A^0\sin(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j}) - B^0\cos(\sum_{p=1}^{r}\sum_{j=0}^{p}\alpha^0(j,p-j)m^j n^{p-j})]^2,$$

for $k=0,\cdots,q,\ q=1,\cdots,r$ Since

$$\left[DQ''(\theta^0)D\right]\to\Sigma$$

we immediately get

$$(\widehat{\theta}-\theta^0)D^{-1}\to\ N_6(0,2c\sigma^2\Sigma^{-1}).$$

■

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