Bayesian Inference and Prediction of Order Statistics for a Type-II Censored Weibull Distribution

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Abstract

This paper describes the Bayesian inference and prediction of the two-parameter Weibull distribution when the data are Type-II censored data. The aim of this paper is two fold. First we consider the Bayesian inference of the unknown parameters under different loss functions. The Bayes estimates cannot be obtained in closed form. We use Gibbs sampling procedure to draw Markov Chain Monte Carlo (MCMC) samples and it has been used to compute the Bayes estimates and also to construct symmetric credible intervals. Further we consider the Bayes prediction of the future order statistics based on the observed sample. We consider the posterior predictive density of the future observations and also construct a predictive interval with a given coverage probability. Monte Carlo simulations are performed to compare different methods and one data analysis is performed for illustration purposes.

Key Words and Phrases: Bayes estimates; Asymptotic distribution; Type-II censoring; Markov Chain Monte Carlo; Predictive density.

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1 Introduction

Weibull distribution is one of the most popular distribution in analyzing skewed data. In reliability and lifetesting analysis, often the data are censored. Among the different censoring schemes, Type-I and Type-II censoring schemes are the most used ones in reliability and life testing experiments. In this paper we mainly restrict our attention on Type-II censoring, although all our results are valid for Type-I and other censoring schemes also. In Type-II censoring it is assumed that n items are put on a test. The integer $m < n$ is pre-fixed, and the experiment stops as soon as the m-th failure is observed. In this paper it is further assumed that the lifetimes of the items being tested have a Weibull distribution with the PDF;

$$f(t;\alpha,\lambda) = \begin{cases} \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases} \tag{1}$$

Here $\alpha > 0, \lambda > 0$ are the shape and scale parameters, respectively. The Weibull distribution with the shape and scale parameters as α and λ will be denoted by WE(α,λ).

Estimation and prediction problems arise quite naturally in many real life situations. In the Bayesian inference, the performance of the estimator or predictor depends on the prior distribution and also on the loss function used. If the shape parameter α is known, the natural choice of the prior on the scale parameter λ is the conjugate gamma prior. If the shape parameter is also unknown, the continuous conjugate priors do not exist, although there exits a continuous-discrete joint prior distribution, see Soland [18]. The continuous component of this distribution is related to the scale parameter, and the discrete one is related to the shape parameter. This method has been widely criticized in the literature because of its difficulty in applications to real life problems, see Kaminsky and Krivtsov [10]. Another approach is to use the same conjugate prior on λ, and use independent uniform prior on α. Some authors use independent uniform priors on both α and λ, see for example Smith and Naylor [15] or Dellaportas and Wright [6] in this respect. Clearly they have their own limitations. We have used more flexible priors and it will be explained in details in the next section.

The aim of this paper is mainly two fold. First we compute the Bayes estimates of α and λ under different loss functions and compare their performances using extensive computer simulations. It may be mentioned that, although, in the frequentist set up the comparison of the different estimators and various confidence intervals of the Weibull parameters have been obtained by Hossain and Zimmer [8] and Jeng and Meeker [9], but no such comparison has been done in the Bayesian context.

Our second aim of this paper is to consider the prediction of future order statistics based on type-II censored observations. Prediction of future order statistics comes up quite naturally in several real life situations. Extensive work on prediction problem can be found in the literature. Smith [16, 17] investigated the asymptotic property of the predictive inference of Bayes and frequentist procedures for a class of parametric family under smooth loss functions. Al-Hussaini [1] also considered the Bayesian prediction problem for a large class of lifetime distributions. A numerical approach to Bayesian prediction for the two-parameter Weibull distribution was considered by Dellaportas and Wright [6]. They provided the numerical Bayesian predictor of the future observation, under the assumptions that the shape parameter has a uniform prior over a finite interval and the error is squared error. Raqab [13] considered a modification of the maximum likelihood predictor of the future sample for normal distribution. An excellent review of development on prediction problems till late 90s can be found in Kaminsky and Nelson [11]. Recently Ren, Sun and Dey [14] compared different Bayesian and frequentist predictors for exponential distribution. Basak, Basak and Balakrishnan [3] considered the maximum likelihood predictors for different lifetime distributions, when the data are progressively censored.

In this paper, we consider the estimation of the posterior predictive density of a future order statistic, based on the current type-II censored data, and also construct predictive intervals of the future order statistic. The main difference of our work with the existing work is that we have considered different loss functions which has not been considered before, and moreover our proposed priors are also quite general in nature. We have used the MCMC samples to compute the predictive density and also to compute the corresponding credible interval.

The rest of the paper is organized as follows. In section 2, we provide the model assumptions, different loss functions, and the prior distributions. Computations of the Bayes estimates and construction of the corresponding credible intervals are provided in section 3.

Numerical comparisons of the different Bayes estimates and MLEs are provided in section 4. Bayes predictions are provided in section 5. One illustrative example has been presented in section 6, and finally we conclude the paper in section 7.

2 Models, Loss Functions and Priors

Our problem is to estimate and to construct credible interval of some function of α and λ, say $g(\alpha, \lambda) = \theta$. For estimating θ, the following loss functions are considered.

Quadratic Loss Function: $L_1(\theta, \delta) = (\theta - \delta)^2$.

Linear Loss Function: $L_2(\theta, \delta) = K_1(\theta - \delta)$ if $\theta > \delta$, and it is $K_2(\delta - \theta)$ if $\theta \leq \delta$. Here K_1 and K_2 are constants. Under the assumption that the loss function is symmetric, *i.e.* $K_1 = K_2$, it is equivalent to $L_2(\theta, \delta) = |\theta - \delta|$.

LINEX Loss Function: $L_3(\theta, \delta) = \left(\frac{\delta}{\theta}\right)^{a^*} - a^* \ln\left(\frac{\delta}{\theta}\right) - 1$ for $a^* \neq 0$.

Entropy Loss Function: $L_4(\theta, \delta) = \frac{\delta}{\theta} - \ln\left(\frac{\delta}{\theta}\right) - 1$. Note that the entropy loss function is a special case of the LINEX loss function, and can be obtained by substituting $a^* = 1$.

When α is known, it is assumed that $\lambda > 0$ has a Gamma(a, b) prior with the PDF $\pi_1(\lambda|a, b) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}$ if $\lambda > 0$, and 0 otherwise. Here the hyper parameters $a > 0$, $b > 0$, and $\Gamma(a)$ is the gamma function. When both parameters are unknown, following the approach of Berger and Sun [4] or Kundu [12], it is assumed that λ has the same prior namely $\pi_1(\lambda|a, b)$, the prior on α, $\pi_2(\cdot)$, is independent of $\pi_1(\lambda)$, and $\pi_2(\alpha)$ is log-concave on the support $(0, \infty)$.

3 Bayes Estimates and Credible Intervals

3.1 Shape Parameter Known

Based on the Type-II censored sample $t_1 < \cdots < t_m$, the posterior density function of λ is well known to be Gamma$(a + m, b + \sum_{i=1}^m t_i^\alpha + (n - m)t_m^\alpha)$. Therefore, the Bayes estimate of

λ under the squared error loss function (L_1) is the posterior mean and that is

$$\widehat{\lambda}_1 = \frac{a+m}{b+\sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}}. \tag{2}$$

Jeffrey's prior is a limiting case of the gamma prior $\pi_1(\lambda|a,b)$, and under Jeffrey's prior ($a = b = 0$), the Bayes estimate under the loss function L_1 is same as the maximum likelihood estimator of λ. The Bayes estimate of λ with respect to the loss function L_2 is the median of the posterior density function. In this case we do not have explicit expressions of the median. But using the result of Haldane [7], the median of gamma distribution can be approximated. The Bayes estimate of λ with respect to the loss function L_2 becomes

$$\widehat{\lambda}_2 = \frac{m+c_1}{b+\sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}} + o(m^{-3}), \tag{3}$$

here $c_1 = a - \dfrac{1}{3} + \dfrac{8}{405m}$. See Ren, Sun and Dey [14] also in this respect.

Using Lemma 5 of Ren, Sun and Dey [14], the Bayes estimate of λ under the loss function L_3 can be seen as

$$\widehat{\lambda}_3 = \frac{m+c_2}{b+\sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}} + o(m^{-3}), \tag{4}$$

here $c_2 = a - \dfrac{a^*+1}{2} - \dfrac{(a^*-1)}{24m}$. Because, the posterior distribution of λ follows gamma, a credible interval of λ can be easily obtained. Moreover, if $a+m$ is a positive integer, then the chi-square table values can be used for constructing credible intervals.

3.2 Shape Parameter Unknown

Based on the prior distributions $\pi_1(\lambda|a,b)$ and $\pi_2(\alpha)$, the posterior distribution of α and λ given the *data* is

$$\pi(\alpha,\lambda|data) = \frac{l(data|\alpha,\lambda)\pi_1(\lambda|a,b)\pi_2(\alpha)}{\int_0^{\infty}\int_0^{\infty} l(data|\alpha,\lambda)\pi_1(\lambda|a,b)\pi_2(\alpha)d\alpha d\lambda}. \tag{5}$$

It is clear that even if we have specific form of $\pi_2(\alpha)$, the Bayes estimate with respect to different loss functions may not be obtained in explicit forms. We propose to use Gibbs sampling technique to generate samples from the joint posterior distribution function, and use them to compute Bayes estimates and also to construct credible intervals. The following results can be used to generate samples from the joint posterior density function. Note that the conditional PDF of λ given α and the *data* is Gamma$(a+m, b+\sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha})$. Moreover, we have the following result, whose proof can be obtained from the result of Kundu [12].

THEOREM 1: The conditional PDF of α given the data is given by

$$g(\alpha|data) \propto \pi_2(\alpha)\alpha^m \prod_{i=1}^{m} t_i^{\alpha-1} \times \frac{1}{(b+\sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha})^{a+m}}, \tag{6}$$

and it is log-concave.

4 Numerical Experiments

In this section we present some simulation results to examine the behavior of the different Bayes estimators for different Type-II censoring schemes. We have taken the following Type-II censoring schemes: Scheme 1: $n = 25$, $m = 10$; Scheme 2: $n = 25$, $m = 15$; Scheme 3: $n = 25$, $m = 20$; Scheme 4: $n = 50$, $m = 10$; Scheme 5: $n = 50$, $m = 20$; Scheme 6: $n = 50$, $m = 40$. For all the censoring schemes, we have used $\alpha = 2.0$, $\lambda = 1.0$. To compute the Bayes estimates, it is assumed that $\pi_2(\alpha)$ has gamma density function with the shape and scale parameters as c and d respectively.

First we consider the non-informative prior for both α and λ, *i.e.* $a = b = c = d = 0$. In this case the prior becomes improper and it is Jeffrey's prior. It should be mentioned that even if prior of α is not log-concave, but the posterior is still log-concave. We call this prior as Prior 0. We have taken one informative priors, namely Prior 1: $a = b = 1$, $c = 2$, $d = 1$. Purposely we have taken the prior means the same as the original means.

In case case we have computed the average Bayes estimates, mean squared errors (MSEs), coverage percentages, and the average credible interval lengths based on 1000 data generation replications. For comparison purposes we have computed the MLEs and the confidence intervals based on Fisher information matrix. The results are reported in Tables 1 to 3. In Table 1 we report the average estimates and the mean squared of the MLEs and different Bayes estimators when Prior 0 has been used. In Table 2 we report the average estimates and the mean squared errors of different Bayes estimators when Prior 1 has been used. In

Table 3 we report the average confidence lengths based on MLEs and the credible intervals based on posterior distributions when Prior 0 and Prior 1 were used.

Some of the points are quite clear from this experiments. It is observed that for fixed n (sample size) as m (effective sample size) increases the performances become better in terms of biases and MSEs in all cases considered. But for fixed m as n increases the performances becomes worse in most of the cases. The performances of MLEs and Bayes 1 (Squared Error) in terms of biases and MSEs, are very similar in nature as expected when non-informative priors are used. The performances of the Bayes 1 and Bayes 4 are very similar in nature in all cases. When Prior 0 is used, MLEs and all the Bayes estimators except the Bayes 3 (a^* $= 10$) overestimate the parameters, where as Bayes 3 underestimate the parameters. It may not be very surprising, because when $a^* = 10$, the loss function puts severe penalty for over estimation. The performances of all the Bayes estimators become better for Prior 1 than Prior 0. Although in case of Bayes 3, the biases are more for Prior 1 than Prior 0, we really do not have any proper explanation for this.

Comparing the confidence intervals and credible intervals, it is clear that for small effective sample sizes the confidence intervals often do not have the nominal coverage percentages, but in most of the cases the credible intervals maintain the coverage percentages, even for small effective sample sizes. When Prior 1 is used, the average length of the credible intervals become smaller as expected.

5 Prediction of Future Order Statistics

In this section we consider the Bayes prediction of future order statistics, based on the current type-II censored sample. Note that this problem has received considerable attention recently, see for example the recent article by Balakrishnan, Beutner and Cramer [2] and the references cited therein. The problem can be formulated as follows. Let $T_1 < \cdots < T_{(m)}$ be the observed sample known as informative sample and $T_{(m+1)} < \cdots < T_{(n)}$ be the unobserved future order statistics from the same sample, which is yet to observed. The prediction problem involves the prediction of the future order statistics $T_{(m+k)}$; for $0 < k \leq n - m$.

For predicting the future order statistics $T_{(m+k)}$ first we obtain the posterior predictive density of $T_{(m+k)}$ for $0 < k \leq n-m$, given the first m ordered observations. The posterior predictive density of $T_{(m+k)}$ is given by

$$\pi_{T_{(m+k)}}(y|data) = \int_0^\infty \int_0^\infty f_{T_{(m+k)}}(y|\alpha,\lambda,data)\pi(\alpha,\lambda|data)d\alpha d\lambda, \quad y > t_{(m)}. \tag{7}$$

Here $f_{T_{(m+k)}}(\cdot|\alpha,\lambda,data)$ is the conditional density of $T_{(m+k)}$ given $t_{(1)} < \cdots < t_{(m)}$, See for example Chen, Shao and Ibrahim [5]. Using the Markov property of the conditional order statistics

$$\begin{aligned} f_{T_{(m+k)}}(y|\alpha,\lambda,data) &= f_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data), \\ &= \frac{(n-m)!}{(k-1)!(n-k-m)!}\alpha\lambda y^{\alpha-1}e^{-\lambda(n-k-m+1)y^\alpha}\left(e^{-\lambda t_{(m)}^\alpha} - e^{-\lambda y^\alpha}\right)^{k-1} e^{\lambda(n-m)t_{(m)}^\alpha} \end{aligned} \tag{8}$$

for $y > t_{(m)}$. Therefore, the predictive density of $T_{(m+k)}$ at any point '$y > t_{(m)}$' is then

$$\begin{aligned} f^*_{T_{(m+k)}}(y|data) &= E_{Posterior}\left[f_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data)\right] \\ &= \int_0^\infty \int_0^\infty f_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data)\pi(\alpha,\lambda|data)d\alpha d\lambda. \end{aligned} \tag{9}$$

Similarly, the predictive survival function at any point '$y > t_{(m)}$' is then

$$\begin{aligned} S^*_{T_{(m+k)}}(y|data) &= E_{Posterior}\left[S_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data)\right] \\ &= \int_0^\infty \int_0^\infty S_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data)\pi(\alpha,\lambda|data)d\alpha d\lambda. \end{aligned} \tag{10}$$

Here $S_{T_{(m+k)}|T_{(m)}=t_{(m)}}(y|\alpha,\lambda,data) = \int_y^\infty f_{T_{(m+k)}|T_{(m)}=t_{(m)}}(u|\alpha,\lambda,data)du$, and it can be easily obtained from (8). A simulation based consistent estimator of $f^*_{T_{(m+k)}}(y|data)$ and $S^*_{T_{(m+k)}}(y|data)$ can be obtained by using the Gibbs sampling procedure as described before.

Another important aspect of prediction is to construct a two-sided predictive interval for $T_{(m+k)}$. A symmetric $100\gamma\%$ predictive interval of $T_{(m+k)}$ can be obtained by solving the non-linear equations (11) and (12) simultaneously for the lower bound, L and upper bound, U:

$$\frac{1+\gamma}{2} = P(T_{(m+k)} > L|Data) \quad \Rightarrow \quad S^*_{T_{(m+k)}|Data}(L) = \frac{1+\gamma}{2} \tag{11}$$

and

$$\frac{1-\gamma}{2} = P(T_{(m+k)} > U|Data) \quad \Rightarrow \quad S^*_{T_{(m+k)}|Data}(U) = \frac{1-\gamma}{2}. \tag{12}$$

We need to apply a suitable numerical method as they cannot be solved analytically.

6 Illustrative Example

For illustrative purposes, we have analyzed one real data which has been recently considered by Balakrishnan, Beutner and Cramer [2]. It represents the lifetimes (in hours) of appliance cord put under a specific test. The sample consists of the smallest nine observations of twelve appliance cords as follows: 57.5, 77.8, 88.0, 98.4, 102.1, 105.3, 139.3, 143.9, 148.0. The problem is to predict the unobserved order statistics and the associated credible intervals.

Since the empirical hazard function shows an increasing trend, Weibull distribution can be used for analyzing this data set. We have divided all the points by 100. First we compute the MLEs of α and λ and they are 4.3490 and 0.5112 respectively. The 95% confidence intervals of α and λ based on the Fisher information matrix become (1.8476, 6.8504) and (0.0354, 0.9871) respectively. To compute the Bayes estimates we need to assume specific form of $\pi_2(\alpha)$ and it is assumed that it is also gamma with shape and scale parameters as c and d respectively. Since we do not have any prior information on a, b, c, and d, we assume the non-informative priors, *i.e.* $a = b = c = d = 0$. It should be mentioned that when $c = d = 0$, then $\pi_2(\alpha)$ is not log-concave, but it may be easily observed that the posterior density function of α as given in (6) is still log-concave. The posterior density function (6) in this case can be approximated very well with a two-parameter gamma density function with the shape and scale parameters as 13.7217 and 3.3782 respectively. The posterior density function and the approximate posterior density function are plotted in Figure 1, clearly they are very close. We have generated 10000 MCMC samples of (α, λ). It is observed that the marginal posterior density function of λ can be very well approximated by a two-parameter gamma distribution with the shape and scale parameters as 5.1560 and 9.3009 respectively.

Now we compute the Bayes estimates with respect to different loss functions, namely squared error (Sq. err), absolute error (Abs. err) and LINEX loss functions. For LINEX loss function, we have taken different a^* values and all the results are presented in Table 4. All the estimates are quite close to each other and they are quite different than the MLEs. In fact, it is observed that based on the Kolmogorov-Smirnov distance measures the Bayes estimates provide a much better fit than the MLEs. We obtain the 95% credible intervals of α and λ and they are (1.8132, 6.3137) and (0.1881, 1.1354) respectively. Now we consider the prediction of the 10-th, 11-th and 12-th order statistics, which are missing. The 95%

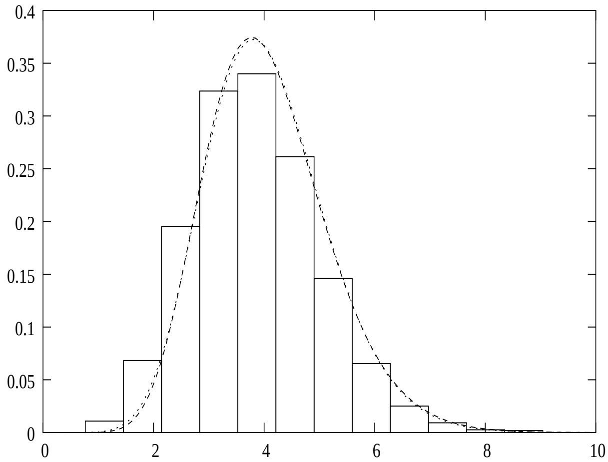

Figure 1: Posterior density function, approximate posterior density function and the generated MCMC samples of α.

predictive interval of the 10-th, 11-th and 12-th order statistics are (148.0, 191.9), (148.0, 233.7) and (148.0, 302.2) respectively.

7 Conclusions

In this paper we have considered the Bayes estimates and prediction of a two-parameter Weibull distribution when the data are Type-II censored. We have compared the MLEs and different Bayes estimators obtained using different loss functions by computer simulations in terms of the average biases and MSEs for different censoring schemes. We have also compared the confidence intervals obtained using asymptotic distribution of the MLEs and credible intervals obtained from the posterior distribution function. It is observed that if we have informative priors, Bayesian inference has a clear advantage over the frequentist one. It may be mentioned although we have provided the results mainly for Type-II censored sample, but our method can be applied for other censoring mechanism also, namely Type-I, hybrid or progressive censoring also. More work is needed along these direction.

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Table 1: MLEs and Bayes estimates with respect to different loss functions when when Prior 0 is used. True $\alpha = 2.0$, $\lambda = 1.0$.

Scheme		MLE	Sq. err (Bayes 1)	Abs. err (Bayes 2)	$a^* = 10$ (Bayes 3)	$a^* = 0.1$ (Bayes 4)	$a^* = 1.0$ (Bayes 5)
1	α	2.4016	2.3799	2.3496	1.9865	2.3301	2.2898
(25,10)		(0.9167)	(0.8720)	(0.8524)	(0.6831)	(0.8400)	(0.8156)
	λ	1.4135	1.4858	1.3817	0.9682	1.4312	1.3051
		(1.4213)	(1.4707)	(1.2126)	(0.5888)	(1.4422)	(1.0590)
2	α	2.2919	2.2293	2.2113	1.9772	2.1997	2.1756
(25,15)		(0.7184)	(0.6402)	(0.6297)	(0.5357)	(0.6233)	(0.6103)
	λ	1.0955	1.1349	1.1158	0.9479	1.1102	1.0904
		(0.4534)	(0.4895)	(0.4625)	(0.3398)	(0.4618)	(0.4407)
3	α	2.2242	2.1520	2.1399	1.9785	2.1325	2.1166
(25,20)		(0.5501)	(0.4907)	(0.4847)	(0.4296)	(0.4810)	(0.4736)
	λ	1.0421	1.0437	1.0355	0.9302	1.0305	1.0196
		(0.2439)	(0.2737)	(0.2696)	(0.2475)	(0.2679)	(0.2637)
4	α	2.5101	2.4096	2.3771	1.9956	2.3568	2.3139
(50,10)		(0.9897)	(0.9139)	(0.8927)	(0.7100)	(0.8796)	(0.8532)
	λ	2.2143	2.6848	2.1391	1.0797	2.1566	1.8427
		(1.9113)	(2.0174)	(2.0216)	(1.2365)	(2.0244)	(1.9771)
5	α	2.2287	2.1671	2.1528	1.9631	2.1437	2.1248
(50,20)		(0.5832)	(0.5127)	(0.5051)	(0.4414)	(0.5006)	(0.4915)
	λ	0.9983	1.1607	1.1340	0.9571	1.1324	1.1099
		(0.4462)	(0.5141)	(0.4764)	(0.3278)	(0.4781)	(0.4510)
6	α	2.1474	2.0792	2.0734	1.9913	2.0699	2.0622
(50,40)		(0.3368)	(0.3108)	(0.3086)	(0.2879)	(0.3073)	(0.3045)
	λ	1.0234	1.0259	1.0220	0.9675	1.0195	1.0144
		(0.1681)	(0.1754)	(0.1741)	(0.1666)	(0.1735)	(0.1721)

Note: Here corresponds to each scheme, the first entry represents the average estimate and the corresponding MSE is reported within bracket below.

Table 2: Bayes estimates with respect to different loss functions when Prior 1 is used. True $\alpha = 2.0$, $\lambda = 1.0$.

Scheme		Sq. err (Bayes 1)	Abs. err (Bayes 2)	$a^* = 10$ (Bayes 3)	$a^* = 0.1$ (Bayes 4)	$a^* = 1.0$ (Bayes 5)
1	α	1.7437	1.7236	1.4784	1.7108	1.6841
(25,10)		(0.4442)	(0.4531)	(0.6075)	(0.4591)	(0.4726)
	λ	0.9775	0.9446	0.7299	0.9397	0.9091
		(0.3181)	(0.2997)	(0.3553)	(0.3017)	(0.2934)
2	α	1.8513	1.8371	1.6495	1.8278	1.8086
(25,15)		(0.3924)	(0.3956)	(0.4770)	(0.3978)	(0.4031)
	λ	1.0404	1.0257	0.8807	1.0202	1.0038
		(0.2930)	(0.2817)	(0.2521)	(0.2810)	(0.2724)
3	α	1.9125	1.9023	1.7602	1.8955	1.8816
(25,20)		(0.3473)	(0.3481)	(0.3909)	(0.3490)	(0.3511)
	λ	1.0323	1.0251	0.9218	1.0202	1.0096
		(0.2396)	(0.2363)	(0.2232)	(0.2347)	(0.2313)
4	α	1.7150	1.6964	1.4688	1.6848	1.6602
(50,10)		(0.4300)	(0.4407)	(0.6048)	(0.4479)	(0.4634)
	λ	0.8907	0.8367	0.5923	0.8366	0.7955
		(0.3380)	(0.3313)	(0.4441)	(0.3324)	(0.3365)
5	α	1.8716	1.8601	1.7062	1.8528	1.8375
(50,20)		(0.3478)	(0.3505)	(0.4175)	(0.3525)	(0.3570)
	λ	0.9973	0.9800	0.8458	0.9776	0.9617
		(0.2818)	(0.2709)	(0.2599)	(0.2718)	(0.2650)
6	α	1.9459	1.9405	1.8640	1.9372	1.9301
(50,40)		(0.2574)	(0.2578)	(0.2764)	(0.2582)	(0.2591)
	λ	1.0211	1.0172	0.9632	1.0148	1.0097
		(0.1638)	(0.1626)	(0.1573)	(0.1620)	(0.1608)

Note: Here corresponds to each scheme, the first entry represents the average estimate and the corresponding MSE is reported within bracket below.

Table 3: Confidence/ Credible intervals based on MLEs and posterior density function and the associated coverage percentages. True $\alpha = 2.0$, $\lambda = 1.0$.

Scheme		MLE	Prior 0	Prior 1
1	α	2.3786	2.5528	1.9179
(25,10)		(0.88)	(0.92)	(0.95)
	λ	1.8716	1.9173	1.6697
		(0.94)	(0.93)	(0.96)
2	α	1.9678	2.0566	1.6723
(25,15)		(0.91)	(0.93)	(0.95)
	λ	1.3245	1.3695	1.1798
		(0.96)	(0.93)	(0.96)
3	α	1.2565	1.6372	1.4445
(25,20)		(0.91)	(0.93)	(0.96)
	λ	0.9355	0.9429	0.9292
		(0.95)	(0.93)	(0.96)
4	α	2.2145	2.8586	1.8211
(50,10)		(0.82)	(0.92)	(0.95)
	λ	1.9767	2.5561	2.1157
		(0.94)	(0.92)	(0.96)
5	α	1.2767	1.7977	1.4996
(50,20)		(0.91)	(0.94)	(0.96)
	λ	1.2341	1.5339	1.1649
		(0.95)	(0.95)	(0.95)
6	α	0.7729	1.1143	1.0402
(50,40)		(0.90)	(0.95)	(0.95)
	λ	0.6420	0.6445	0.6403
		(0.96)	(0.95)	(0.96)

Note: Here corresponds to each scheme, the first entry represents the average length of the confidence/ credible interval and the corresponding coverage percentage is presented below within bracket.

Table 4: Bayes estimates with respect to different loss functions

	Sq. err	Abs. err	$a^* = 3$	$a^* = 2.0$	$a^* = 1$	$a^* = 0.5$	$a^* = 0.1$
α	3.8081	3.7804	3.5135	3.5845	3.6583	3.6957	3.7257
λ	0.5333	0.5106	0.4479	0.4669	0.4880	0.4991	0.5082