Estimation of $R = P[Y < X]$ for Three Parameter Generalized Rayleigh Distribution

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Abstract

Surles and Padgett [15] introduced two-parameter Burr Type X distribution, which can be described as a generalized Rayleigh distribution. In this paper we consider the estimation of the stress-strength parameter $R = P[Y < X]$, when X and Y are both three-parameter generalized Rayleigh distribution with the same scale and locations parameters but different shape parameters. It is assumed that they are independently distributed. It is observed that the maximum likelihood estimators (MLEs) do not exist, and we propose a modified maximum likelihood estimator of R. We obtain the asymptotic distribution of the modified maximum likelihood estimator of R and it can be used to construct the asymptotic confidence interval of R. We also propose the Bayes estimate of R and the construction of the associated credible interval based on importance sampling technique. Analysis of two real data sets, (i) simulated and (ii) real, have been performed for illustrative purposes.

Key Words and Phrases: Stress-Strength model; maximum likelihood estimator; Bayes estimator; bootstrap confidence intervals; credible intervals; asymptotic distributions.

Short Running Title: Estimation of $P[Y < X]$.

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1 Introduction

The two-parameter generalized Rayleigh distribution, introduced by Surles and Padgett [15], has the following cumulative distribution function (CDF) for $x > 0$;

$$F(x;\alpha,\lambda) = \left(1 - e^{-\lambda x^2}\right)^{\alpha}; \qquad x > 0, \tag{1}$$

and the corresponding probability density function (PDF) becomes

$$f(x;\alpha,\lambda) = 2\alpha\lambda x e^{-\lambda x^2}\left(1 - e^{-\lambda x^2}\right)^{\alpha-1}; \qquad x > 0. \tag{2}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. From now on, a two-parameter GR distribution with shape parameter α and scale parameter λ will be denoted by $\mathrm{GR}(\alpha,\lambda)$.

A three-parameter GR distribution can be obtained from a two-parameter GR distribution by introducing the location or threshold parameter μ. Therefore, for $\alpha > 0$, $\lambda > 0$ and $-\infty < \mu < \infty$, a three-parameter GR distribution has the CDF

$$F(x;\alpha,\lambda,\mu) = \left(1 - e^{-\lambda(x-\mu)^2}\right)^{\alpha}; \qquad x > \mu, \tag{3}$$

and it has the corresponding PDF as

$$f(x;\alpha,\lambda,\mu) = 2\alpha\lambda(x-\mu)e^{-\lambda(x-\mu)^2}(1 - e^{-\lambda(x-\mu)^2})^{\alpha-1}; \qquad x > \mu. \tag{4}$$

From now on, a three-parameter GR distribution with CDF (3) will be denoted by $\mathrm{GR}(\alpha,\lambda,\mu)$.

In this paper, we consider the problem of estimating the stress-strength parameter $R = P(Y < X)$, when X and Y are independent and $X \sim \mathrm{GR}(\beta,\lambda,\mu)$, $Y \sim \mathrm{GR}(\alpha,\lambda,\mu)$. Here the notation '$\sim$' means 'follows' or 'has the distribution'. The estimation of the stress-strength parameter R is quite common in the statistical literature, and R is well known as the stress-strength parameter. For example, if X is the strength of a system which is subjected to a stress Y, then R is a measure of system performance, and it arises quite naturally in the mechanical reliability of a system. A book length treatment on this topic can be found in Kotz et al. [5]. For some of the recent references the readers are referred to Kundu and Gupta [6], Kundu and Raqab [9]. A recent review article on the Bayesian inference on the stress-strength parameter can be obtained in Ventura and Racugno [16].

Although, extensive treatment for estimating the stress-strength parameter for different models in presence of shape and scale parameters are available, not much attention has been paid when the unknown location (threshold) parameter is also present, mainly due to its analytical difficulty. It is observed that when the location parameter is also present, the maximum likelihood estimators of the unknown parameters do not exist and the three-parameter GR distribution is no longer a regular family, due to the fact that the support depends on the parameter. In this case, we propose to use modified MLEs of the unknown parameters following suggestion of Smith [13] and provide the joint asymptotic distribution of the modified MLEs of the shape and scale parameters. It has been used to obtain the asymptotic distribution of the modified MLE of R.

Further we develop the Bayesian estimate of R and the associated credible interval based on the gamma priors on the shape and the common scale parameters, and uniform prior on the common location parameter. Although the Bayes estimate and the associated credible interval cannot be obtained in explicit form, the importance sampling technique can be used quite conveniently to compute the Bayes estimate and the associated credible interval. The implementation of the above procedure is quite simple moreover the credible interval obtained using the Bayesian method is valid even for small sample also, as it is not based on the asymptotic result. Two data sets: (i) simulated and (ii) real have been analyzed for illustrative purposes to see the effectiveness of the proposed methods.

The rest of the paper is organized as follows. In Section 2, we provide the modified MLE of R. The asymptotic distributions of the modified MLEs are provided in Section 3. The biased corrected estimator is presented in Section 4. In Section 5, we present the Bayesian inference on the stress-strength parameter. The analysis of two data sets are presented in Section 6, and finally we conclude the paper in Section 7.

2 Modified Maximum Likelihood Estimators

Suppose, $Y \sim \text{GR}(\alpha, \lambda, \mu)$ and $X \sim \text{GR}(\beta, \lambda, \mu)$, and they are independently distributed. Therefore,

$$R = P(Y < X) = \frac{\beta}{\alpha + \beta}. \tag{5}$$

To compute the MLE of R, we need to compute MLEs of α and β. Moreover, to compute the MLE of R, we need to compute the MLEs of λ and μ also. Suppose, we have $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ independent random samples from $\text{GR}(\beta, \lambda, \mu)$ and $\text{GR}(\alpha, \lambda, \mu)$, respectively. Based on the random sample, we want to estimate R.

If we denote the ordered samples from X and Y as $\{x_{(1)} < \cdots < x_{(n)}\}$ and $\{y_{(1)} < \cdots < y_{(m)}\}$ respectively, then the likelihood function becomes

$$\begin{aligned} L(\alpha, \beta, \lambda, \mu) \quad &\propto \quad \lambda^{m+n} \alpha^m \beta^n \prod_{i=1}^{n} (x_{(i)} - \mu) \prod_{j=1}^{m} (y_{(j)} - \mu) e^{-\lambda \{\sum_{i=1}^{n} (x_{(i)} - \mu)^2 + \sum_{j=1}^{m} (y_{(j)} - \mu)^2\}} \times \\ & \prod_{i=1}^{n} \left(1 - e^{-\lambda (x_{(i)} - \mu)^2}\right)^{\beta - 1} \prod_{j=1}^{m} \left(1 - e^{-\lambda (y_{(j)} - \mu)^2}\right)^{\alpha - 1} \times 1_{z > \mu}(\mu), \end{aligned} \tag{6}$$

where $z = \min\{x_{(1)}, y_{(1)}\}$, and $1_{z>\mu}(\mu)$ is an indicator function takes value 1 or 0, if $z > \mu$ or $z \leq \mu$, accordingly. It is immediate that for $\alpha < \frac{1}{2}$ and $\beta < \frac{1}{2}$, if $\mu \uparrow z$, then $l(\alpha, \beta, \lambda, \mu) \to \infty$.

One way to overcome this problem is to first estimate μ by its natural estimator $\widetilde{\mu} = z$, and obtain the modified data set as $\{x_1 - \widetilde{\mu}, \cdots, x_n - \widetilde{\mu}\}$, $\{y_1 - \widetilde{\mu}, \cdots, y_m - \widetilde{\mu}\}$. Obtain the estimates of α, β, λ (say modified MLEs) from the $m + n - 1$ observations, by removing the

'0' observation, see Smith [13]. Once the modified MLEs of α, β and λ are obtained, say $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\lambda}$ respectively, the modified MLE of R can be easily obtained as

$$\widetilde{R} = \frac{\widetilde{\beta}}{\widetilde{\alpha} + \widetilde{\beta}}. \tag{7}$$

The modified log-likelihood function based on $(m+n-1)$ observations for $x_{(1)} < y_{(1)}$ without the additive constant is given by

$$\begin{aligned} l(\alpha, \beta, \lambda, \widetilde{\mu}) \;\; &= \;\; (m+n-1)\ln\lambda + m\ln\alpha + (n-1)\ln\beta - \lambda\left(\sum_{i=2}^{n}\ln(x_{(i)} - \widetilde{\mu}) + \sum_{j=1}^{m}\ln(y_{(j)} - \widetilde{\mu})\right) \\ &\quad + (\beta-1)\sum_{i=2}^{n}\ln\left(1 - e^{-\lambda(x_{(i)} - \widetilde{\mu})^2}\right) + (\alpha-1)\sum_{j=1}^{m}\ln\left(1 - e^{-\lambda(y_{(j)} - \widetilde{\mu})^2}\right) \end{aligned} \tag{8}$$

and for $x_{(1)} > y_{(1)}$

$$\begin{aligned} l(\alpha, \beta, \lambda, \widetilde{\mu}) \;\; &= \;\; (m+n-1)\ln\lambda + (m-1)\ln\alpha + n\ln\beta - \lambda\left(\sum_{i=1}^{n}\ln(x_{(i)} - \widetilde{\mu}) + \sum_{j=2}^{m}\ln(y_{(j)} - \widetilde{\mu})\right) \\ &\quad + (\beta-1)\sum_{i=1}^{n}\ln\left(1 - e^{-\lambda(x_{(i)} - \widetilde{\mu})^2}\right) + (\alpha-1)\sum_{j=2}^{m}\ln\left(1 - e^{-\lambda(y_{(j)} - \widetilde{\mu})^2}\right). \end{aligned} \tag{9}$$

Therefore, the modified MLEs of α, β and λ can be obtained by maximizing the modified log-likelihood function with respect to α, β and λ. It easily follows that for fixed λ, the modified MLEs of α and β are

$$\widehat{\alpha}(\lambda) = -\frac{m}{\sum_{j=1}^{m}\ln(1 - e^{-\lambda(y_{(j)} - \widehat{\mu})^2})} \qquad \widehat{\beta}(\lambda) = -\frac{n-1}{\sum_{i=2}^{n}\ln(1 - e^{-\lambda(x_{(i)} - \widehat{\mu})^2})} \qquad \text{if} \quad x_{(1)} < y_{(1)} \tag{10}$$

and

$$\widehat{\alpha}(\lambda) = -\frac{m-1}{\sum_{j=2}^{m}\ln(1 - e^{-\lambda(y_{(j)} - \widehat{\mu})^2})} \qquad \widehat{\beta}(\lambda) = -\frac{n}{\sum_{i=1}^{n}\ln(1 - e^{-\lambda(x_{(i)} - \widehat{\mu})^2})} \qquad \text{if} \quad x_{(1)} > y_{(1)}. \tag{11}$$

The modified MLE of λ can be obtained by maximizing the profile log-likelihood function of λ, namely $l(\widehat{\alpha}(\lambda), \widehat{\beta}(\lambda), \lambda, \widetilde{\mu})$ with respect to λ. Since it is a one-dimensional optimization problem it is quite easy to solve. Because of the complicated nature of the profile log-likelihood function, we could not prove theoretically, but it is observed in the analysis of both the data sets that the profile log-likelihood function of λ is unimodal. Once the modified MLE of λ, say $\widetilde{\lambda}$ is obtained, the modified MLEs of α and β can be obtained as $\widetilde{\alpha} = \widetilde{\alpha}(\widetilde{\lambda})$ and $\widetilde{\beta} = \widetilde{\beta}(\widetilde{\lambda})$ respectively. Once the modified MLEs of α, β are obtained, the modified MLE of R can be obtained as in (7). Finally, observe that using Theorem 2, the asymptotic distribution of $\widetilde{R}$ can be easily obtained.

3 Asymptotic Distribution

In this section we derive the asymptotic distribution of the modified MLEs of the unknown parameters, and this can be used to obtain the asymptotic distribution of $\widehat{R}$. The asymptotic distribution of $\widehat{R}$ can be used to construct asymptotic confidence interval also. We have the following results.

Theorem 1: (a) The marginal distribution of $\widetilde{\mu} = \min\{X_{(1)}, Y_{(1)}\}$ is given by

$$P(\widetilde{\mu} \le t) = 1 - \left(1 - \left(1 - e^{-\lambda(t-\mu)^2}\right)^{\beta}\right)^{n} \times \left(1 - \left(1 - e^{-\lambda(t-\mu)^2}\right)^{\alpha}\right)^{m}$$

(b) If $m/n \to p > 0$, as $m, n \to \infty$, then

for $\alpha < \beta$

$$m^{\frac{1}{2\alpha}} (\widetilde{\mu} - \mu) \stackrel{d}{=} \lambda^{-1/2} Z^{1/2\alpha}$$

and for $\beta < \alpha$,

$$n^{\frac{1}{2\beta}} (\widetilde{\mu} - \mu) \stackrel{d}{=} \lambda^{-1/2} Z^{1/2\beta},$$

here Z is a standard exponential random variable, and $\stackrel{d}{=}$ means equal in distribution.

Proof: (a) Trivial.

(b) Let us first consider the case when $\alpha < \beta$.

$$P\left[m^{\frac{1}{2\alpha}} (\widetilde{\mu} - \mu) \le t\right] \quad = \quad P\left[\widetilde{\mu} \le m^{-\frac{1}{2\alpha}} t + \mu\right]$$

$$
\begin{aligned}
&= 1-\left(1-\left(1-e^{-\lambda m^{-1/\alpha}t^2}\right)^\beta\right)^{m\times\frac{n}{m}}\times\left(1-\left(1-e^{-\lambda m^{-1/\alpha}t^2}\right)^\alpha\right)^m\\
&\rightarrow 1-e^{-\lambda^\alpha t^{2\alpha}}.
\end{aligned}
$$

The other case follows along the same manner. ∎

THEOREM 2: The asymptotic distribution of $(\widetilde{\alpha},\widetilde{\beta},\widetilde{\lambda})$, as $\min\{m,n\}\rightarrow\infty$, and $m/n\rightarrow p$, is as follows;

$$
\left(\sqrt{m}(\widetilde{\alpha}-\alpha),\sqrt{m}(\widetilde{\beta}-\beta),\sqrt{m}(\widetilde{\lambda}-\lambda)\right)\xrightarrow{d}N_3(\mathbf{0},\mathbf{A}^{-1}(\alpha,\beta,\lambda)).
$$

Here

$$
\mathbf{A}(\alpha,\beta,\lambda)=\begin{bmatrix} a_{11} & 0 & a_{13}\\ 0 & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}
$$

where

$$
a_{11}=\frac{1}{\alpha^2},\quad a_{13}=a_{31}=\frac{2c}{\lambda},\quad a_{22}=\frac{1}{\beta^2},\quad a_{23}=a_{32}=\frac{2d}{\sqrt{p}\lambda},\quad a_{33}=-g(\alpha)-\frac{1}{p}g(\beta),
$$

$$
\begin{aligned}
c &= \left[\frac{1}{\alpha-1}(\psi(\alpha)-\psi(1))-\frac{1}{\alpha}\right]\quad\text{if}\quad\alpha\neq 1\\
&= \sum_{i=0}^{\infty}\frac{1}{(i+2)^2}\quad\text{if}\quad\alpha=1,\\
d &= \left[\frac{1}{\beta-1}(\psi(\beta)-\psi(1))-\frac{1}{\beta}\right]\quad\text{if}\quad\beta\neq 1\\
&= \sum_{i=0}^{\infty}\frac{1}{(i+2)^2}\quad\text{if}\quad\beta=1,
\end{aligned}
$$

$$
\begin{aligned}
g(\mu) &= -\frac{2}{\lambda^2}\left[1+\left(\psi(\mu+1)-\psi(1)-\frac{1}{\mu}\right)+2((\psi(2)-\psi(\mu+1))^2+(\psi'(2)-\psi'(\mu+1)))\right.\\
&\quad\left.+\frac{2}{\mu-2}((\psi(3)-\psi(\mu+1))^2+(\psi'(3)-\psi'(\mu+1)))\right]\quad\text{if}\quad\mu\neq 1\text{ or }2,\\
&= -\frac{2}{\lambda^2}\left[1+\psi(2)-\psi(1)\right]\quad\text{if}\quad\mu=1\\
&= -\frac{2}{\lambda^2}\left[\frac{3}{2}+\psi(3)-\psi(2)+2((\psi(2)-\psi(3))^2+(\psi'(2)-\psi'(3))+4\sum_{i=0}^{\infty}\frac{2}{(3+i)^3}\right]\\
&\qquad\text{if}\quad\mu=2.
\end{aligned}
$$

Proof: To prove Theorem 2, we mainly use the results of Smith [13], and the fact that the PDF of the three-parameter GR distribution (4) is of the form

$$2\alpha c(x-\mu)^{2\alpha-1}, \quad \text{as} \quad x \downarrow \mu. \tag{12}$$

We use the following notation. If U_n is a random variable, then $U_n = O_p(1)$, means as $n \to \infty$, U_n is bounded in probability, and if $U_n = o_p(1)$, means U_n converges to zero in probability. We will denote $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$ as the MLEs of α, β and λ respectively when μ is known. When μ is known, using Theorem 1 of Raqab and Kundu [12] it follows that

$$\left(\sqrt{m}(\widehat{\alpha}-\alpha), \sqrt{m}(\widehat{\beta}-\beta), \sqrt{m}(\widehat{\lambda}-\lambda)\right) \xrightarrow{d} N_3(\mathbf{0}, \mathbf{A}^{-1}(\alpha,\beta,\lambda)). \tag{13}$$

Case I: $0 < \alpha, \beta \le 1$

From Theorem 4 of Smith [13] and due to the fact (12), it follows that

$$\widehat{\alpha}-\widetilde{\alpha} = \begin{cases} o_p(m^{-1/\alpha}) & \text{if} \quad \frac{1}{2} < \alpha \le 1 \\ o_p(\ln m/m) & \text{if} \quad 0 < \alpha \le \frac{1}{2}, \end{cases} \qquad \widehat{\beta}-\widetilde{\beta} = \begin{cases} o_p(m^{-1/\beta}) & \text{if} \quad \frac{1}{2} < \beta \le 1 \\ o_p(\ln m/m) & \text{if} \quad 0 < \beta \le \frac{1}{2} \end{cases} \tag{14}$$

and if $\alpha < \beta$, then

$$\widehat{\lambda}-\widetilde{\lambda} = \begin{cases} o_p(m^{-1/\alpha}) & \text{if} \quad \frac{1}{2} < \alpha \le 1 \\ o_p(\ln m/m) & \text{if} \quad 0 < \alpha \le \frac{1}{2} \end{cases} \tag{15}$$

otherwise

$$\widehat{\lambda}-\widetilde{\lambda} = \begin{cases} o_p(m^{-1/\beta}) & \text{if} \quad \frac{1}{2} < \beta \le 1 \\ o_p(\ln m/m) & \text{if} \quad 0 < \beta \le \frac{1}{2} \end{cases}. \tag{16}$$

Since

$$\begin{aligned} \sqrt{m}(\widetilde{\alpha}-\alpha), \sqrt{m}(\widetilde{\beta}-\beta), \sqrt{m}(\widetilde{\lambda}-\lambda) &= \sqrt{m}(\widehat{\alpha}-\alpha), \sqrt{m}(\widehat{\beta}-\beta), \sqrt{m}(\widehat{\lambda}-\lambda) \\ &+ \sqrt{m}(\widetilde{\alpha}-\widehat{\alpha}), \sqrt{m}(\widetilde{\beta}-\widehat{\beta}), \sqrt{m}(\widetilde{\lambda}-\widehat{\lambda}). \end{aligned} \tag{17}$$

Since for $0 < \alpha, \beta \le 1$,

$$\sqrt{m}(\widetilde{\alpha}-\widehat{\alpha}) = o_p(1), \quad \sqrt{m}(\widetilde{\beta}-\widehat{\beta}) = o_p(1) \quad \text{and} \quad \sqrt{m}(\widetilde{\lambda}-\widehat{\lambda}) = o_p(1), \tag{18}$$

the result follows.

CASE 2: $0 < \alpha \leq 1$ and $\beta > 1$

In this case, $\widehat{\beta} - \widetilde{\beta} = o_p(m^{-1/2})$, and

$$\widehat{\alpha} - \widetilde{\alpha} = \begin{cases} o_p(m^{-1/\alpha}) & \text{if } \frac{1}{2} < \alpha \leq 1 \\ o_p(\ln m/m) & \text{if } 0 < \alpha \leq \frac{1}{2}, \end{cases} \qquad \widehat{\lambda} - \widetilde{\lambda} = \begin{cases} o_p(m^{-1/\alpha}) & \text{if } \frac{1}{2} < \alpha \leq 1 \\ o_p(\ln m/m) & \text{if } 0 < \alpha \leq \frac{1}{2} \end{cases}. \tag{19}$$

Therefore, (18) is satisfied, and the result follows immediately.

CASE 3: $0 < \beta \leq 1$ and $\alpha > 1$

The proof can be obtained along the same line as in Case 2, by interchanging the role of α and β.

CASE 4: $\alpha, \beta > 1$

The proof of this part follows along the same line as in Mudholkar et al. [11] and Kundu and Raqab [9]. In this proof only we denote the true values of α, β, λ and μ as α^0, β^0, λ^0 and μ^0 respectively. We use $\gamma = (\alpha, \beta, \lambda)$, $\gamma^0 = (\alpha^0, \beta^0, \lambda^0)$, and $\widetilde{\gamma} = (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\lambda})$. Let us denote, $l = l(\alpha, \beta, \lambda, \widetilde{\mu})$, the modified log-likelihood function centered around $\widetilde{\mu}$ as defined (8) and (9). We further use $G(\gamma, \widetilde{\mu}) = \dfrac{\partial l}{\partial \gamma}$, the 3×1 derivative vector, and $H(\gamma, \widetilde{\mu}) = \dfrac{\partial^2 l}{\partial \gamma^2}$ be the 3×3 Hessian matrix. We have $G(\widetilde{\gamma}, \widetilde{\mu}) = 0$. Since conditioning on $\widetilde{\mu}$, $\widetilde{\gamma}$ is a $\sqrt{m}$ consistent estimator of γ^0, expanding $G(\widetilde{\gamma}, \widetilde{\mu})$ around γ^0, gives us

$$G(\gamma^0, \widetilde{\mu}) + H(\gamma^0, \widetilde{\mu})(\widetilde{\gamma} - \gamma^0) + O_p(1) = 0. \tag{20}$$

Assuming that m is large enough, so that $\widetilde{\mu}$ can be replaced by μ^0 in $H(\gamma^0, \widetilde{\mu})$, we get

$$G(\gamma^0, \widetilde{\mu}) + H(\gamma^0, \mu^0)(\widetilde{\gamma} - \gamma^0) + O_p(1) = 0. \tag{21}$$

Therefore,

$$\begin{aligned} \sqrt{m}(\widetilde{\gamma} - \gamma^0) &= -\sqrt{m} H^{-1}(\gamma^0, \mu^0) G(\gamma^0, \widetilde{\mu}) + \left[\frac{1}{m} H(\gamma^0, \mu^0)\right]^{-1} \frac{O_p(1)}{\sqrt{m}} \\ &= -\sqrt{m} H^{-1}(\gamma^0, \mu^0) G(\gamma^0, \widetilde{\mu}) + o_p(1). \end{aligned} \tag{22}$$

Note that (22) follows due to the fact that $\frac{1}{m} H(\gamma^0, \mu^0)$ converges to a positive definite matrix. Now expanding $G(\gamma^0, \widetilde{\mu})$ around μ^0, (Theorem 4, of [11]) we have

$$-\sqrt{m} H^{-1}(\gamma^0, \mu^0) G(\gamma^0, \mu^0) - \sqrt{m}(\widetilde{\mu} - \mu^0) H^{-1}(\gamma^0, \mu^0) \frac{d}{d\mu} G(\gamma^0, \mu)\Big|_{\mu^0} + o_p(1). \qquad (23)$$

Since for $\alpha, \beta > 1$ and $\frac{1}{\sqrt{m}} G(\gamma^0, \mu^0)$ converges to a multivariate normal distribution with mean zero and finite variance covariance matrix, the second term on the right hand side of (23) is $o_p(1)$. Now the result follows by observing the facts

$$\frac{1}{\sqrt{m}} G(\gamma^0, \mu^0) \longrightarrow N_3(\mathbf{0}, \mathbf{A}(\alpha^0, \beta^0, \lambda^0)), \quad \text{and} \quad \frac{1}{m} H(\gamma^0, \mu^0) \longrightarrow -\mathbf{A}(\alpha^0, \beta^0, \lambda^0).$$

Theorem 3: As $m \to \infty$ and $n \to \infty$ so that $m/n \to p$, then

$$\sqrt{m}(\widehat{R} - \widehat{R}) \to N(0, B),$$

where

$$B = \frac{1}{u(\alpha + \beta)^4} \left[\beta^2(a_{22}a_{33} - a_{23}^2) - 2\alpha\beta\sqrt{p} a_{23}a_{31} + \alpha^2 p(a_{11}a_{33} - a_{13}^2)\right],$$

and

$$u = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}.$$

Proof: It follows from Theorem 2 and using δ method. ∎

Remark: Note that the normalizing constants $\sqrt{m}$ and $\sqrt{n}$ in Theorems 1 and 2, can be interchanged.

4 Biased Corrected Estimator

It is clear that when μ is estimated by $\min\{x_{(1)}, y_{(1)}\}$, it is a biased estimator. To overcome that Hall and Wang [4] proposed a biased corrected estimator of μ. The main idea of their method can be described as follows. Suppose $z_1, \cdots, z_n$ is a random sample from a distribution function with the PDF of the following form for $z > \mu$;

$$f(z) \sim (z-\mu)^{\delta-1} g(z|\mu,\eta) \quad \text{as} \quad z \downarrow \mu, \tag{24}$$

and zero otherwise. Here $\delta > 0$, μ is a single location parameter, η is a vector of parameters other than μ, and $g(x|\mu,\eta)$ converges to a strictly positive constant as $x \downarrow \mu$. First estimate η for fixed μ as $\bar{\eta}(\mu)$ by maximizing

$$L_1(\mu,\eta) = \prod_{i=1}^{n} g(z_i|\mu,\eta), \tag{25}$$

with respect to η. Then obtain an estimate of μ, say $\bar{\mu}$, by maximizing

$$L_2(\mu) = \frac{(z_{(1)}-\mu)}{(z_{(2)}-\mu)} \times \left\{ \prod_{i=1}^{n} (z_i-\mu)^{\delta-1} \right\} \left\{ \prod_{i=1}^{n} g(z_i|\mu,\bar{\eta}(\mu)) \right\}, \tag{26}$$

with respect to μ. Here $z_{(1)}$ and $z_{(2)}$ are the first and second order statistics of the sample $\{z_1, \ldots, z_n\}$. Finally obtain an estimate of η by $\bar{\eta}(\bar{\mu})$. It has been shown by Hall and Wang [4] that although the asymptotic distribution of $\bar{\mu}$ is not the same as the asymptotic distribution of $z_{(1)}$, the asymptotic distribution of $\bar{\eta}(\bar{\mu})$ is same as the corresponding asymptotic distribution of the modified MLE of η as proposed by Smith [13].

Motivating by this approach, and since the three-parameter GR distribution has the PDF (4) which is of the form (24) with $\delta = 2\alpha$, $\eta = (\alpha, \mu)$ and $g(z|\mu,\eta) = \alpha\lambda^{\alpha} exp(-\lambda(z-\mu)^2)$, we propose the following estimator of $\alpha, \beta, \lambda, \mu$ of our problem, which can be used to obtain an estimator of R.

Consider

$$L_1(\alpha,\beta,\lambda,\mu) = \frac{(x_{(1)}-\mu)(y_{(1)}-\mu)}{(x_{(2)}-\mu)(y_{(2)}-\mu)} \times L(\alpha,\beta,\lambda,\mu), \tag{27}$$

here $L(\alpha,\beta,\lambda,\mu)$ is same as defined in (6). For fixed μ, maximize $L_1(\alpha,\beta,\lambda,\mu)$ with respect to α, β and λ say $\bar{\alpha}(\mu)$, $\bar{\beta}(\mu)$ and $\bar{\lambda}(\mu)$ respectively. Maximize

$$L_1(\bar{\alpha}(\mu), \bar{\beta}(\mu), \bar{\lambda}(\mu)) \tag{28}$$

with respect to μ, to obtain an estimate of μ, say $\bar{\mu}$. Finally obtain an estimate of α, β and λ as $\bar{\alpha}(\bar{\mu})$, $\bar{\beta}(\bar{\mu})$ and $\bar{\lambda}(\bar{\mu})$ respectively. Although the asymptotic distributions of $\bar{\mu}$ and $\widetilde{\mu}$ are different, due to Hall and Wang [4], the asymptotic distribution of $(\bar{\alpha}(\bar{\mu}), \bar{\beta}(\bar{\mu}), \bar{\lambda}(\bar{\mu}))$ is same as the asymptotic distribution of $(\widetilde{\alpha}(\widetilde{\mu}), \widetilde{\beta}(\widetilde{\mu}), \widetilde{\lambda}(\widetilde{\mu}))$ as derived in Section 3. Hence the asymptotic distributions of the modified MLE of R based on $\bar{\mu}$ and $\widetilde{\mu}$ are same.

Now we provide how to implement the above estimation procedure. Note that $l_1(\alpha, \beta, \lambda, \mu)$ $= \ln L_1(\alpha, \beta, \lambda, \mu)$ can be written as

$$\begin{aligned} l_1(\alpha, \beta, \lambda, \mu) &= (m+n)\ln\lambda + m\ln\alpha + n\ln\beta + \sum_{i=3}^{n}\ln(x_{(i)}-\mu) + 2\ln(x_{(1)}-\mu) \\ &\quad + \sum_{j=3}^{m}\ln(y_{(j)}-\mu) + 2\ln(y_{(1)}-\mu) - \lambda(\sum_{i=1}^{n}(x_i-\mu)^2 + \sum_{j=1}^{m}(y_j-\mu)^2) \\ &\quad + (\beta-1)\sum_{i=1}^{n}\ln\left(1-e^{-\lambda(x_i-\mu)^2}\right) + (\alpha-1)\sum_{j=1}^{m}\ln\left(1-e^{-\lambda(y_j-\mu)^2}\right). \end{aligned} \tag{29}$$

For fixed λ and μ, the maximization of (29) with respect to α and β can be obtained at

$$\bar{\alpha}(\lambda, \mu) = -\frac{m}{\sum_{j=1}^{m}\ln\left(1-e^{-\lambda(y_j-\mu)^2}\right)} \tag{30}$$

$$\bar{\beta}(\lambda, \mu) = -\frac{n}{\sum_{i=1}^{n}\ln\left(1-e^{-\lambda(x_i-\mu)^2}\right)} \tag{31}$$

respectively. Now for fixed μ, $\bar{\lambda}(\mu)$ can be obtained by maximizing

$$\begin{aligned} l_2(\lambda, \mu) &= (m+n)\ln\lambda + m\ln(\bar{\alpha}(\mu, \lambda)) + n\ln(\bar{\beta}(\lambda, \mu)) - \lambda(\sum_{i=1}^{n}(x_i-\mu)^2 + \sum_{j=1}^{m}(y_j-\mu)^2) \\ &\quad - \sum_{i=1}^{n}\ln\left(1-e^{-\lambda(x_i-\mu)^2}\right) - \sum_{j=1}^{m}\ln\left(1-e^{-\lambda(y_j-\mu)^2}\right) + \sum_{i=3}^{n}\ln(x_{(i)}-\mu) \\ &\quad + 2\ln(x_{(1)}-\mu) + \sum_{j=3}^{m}\ln(y_{(j)}-\mu) + 2\ln(y_{(j)}-\mu) \end{aligned} \tag{32}$$

with respect to λ for fixed μ. Clearly, $\bar{\lambda}(\mu)$ cannot be obtained in explicit form, it has to be obtained numerically. Once $\bar{\lambda}(\mu)$ is obtained, the estimate of μ can be obtained by maximizing $l_2(\bar{\lambda}(\mu), \mu)$ with respect to μ. Finally the estimates of α, β and λ can be obtained as $\bar{\alpha}(\bar{\lambda}(\bar{\mu}), \bar{\mu})$, $\bar{\beta}(\bar{\lambda}(\bar{\mu}), \bar{\mu})$ and $\bar{\lambda}(\bar{\mu})$ respectively.

5 Bayesian Inference

So far we have discussed the frequentist inference on R. It is observed (see next section) that the estimates of α, β and λ depend quite significantly on the estimate of μ. Moreover, the asymptotic distribution of the maximum likelihood estimator R is also quite involved. It seems that the Bayesian inference is a natural choice in this case, since the Bayes estimate and the associated credible of R can be obtained in a quite straight forward manner.

We will be using the following notation. A random variable U is said to have a gamma distribution with the shape and scale parameters as $a > 0$ and $b > 0$, respectively, if the PDF of U is

$$f_U(u; a, b) = \begin{cases} \frac{b^a}{\Gamma(a)} u^{a-1} e^{-bu} & \text{if} \quad u \geq 0 \\ 0 & \text{if} \quad u < 0 \end{cases} \tag{33}$$

and it will be denoted by $\text{GA}(a, b)$. A random variable V is uniform over (c, d) if it has the PDF

$$f_V(v) = \begin{cases} 1 & \text{if} \qquad c < v < d \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

We will be using the following prior distributions on α, β, λ and μ.

$$\alpha \sim \text{GA}(a_1, b_1), \quad \beta \sim \text{GA}(a_2, b_2), \quad \lambda \sim \text{GA}(a_0, b_0), \quad \mu \sim U(c, d), \tag{35}$$

and they are assumed to be independently distributed.

Based on the above prior the posterior density function of α, β, λ and μ given the data is

$$\begin{aligned} \pi(\alpha, \beta, \lambda, \mu | data) \;\; = \;\; & K \; g_\lambda(\lambda; m + n + a_0, T_1(\mu) + T_2(\mu) + b_0) \times \\ & g_\alpha(\alpha; m + a_1, b_1 + D_1(\lambda, \mu)) \times g_\beta(\beta; n + a_2, b_2 + D_2(\lambda, \mu)) \times \\ & g(\mu) \times h(\lambda, \mu), \end{aligned} \tag{36}$$

here K is the normalizing constant, $g_\lambda(\lambda; m + n + a_0, T_1(\mu) + T_2(\mu) + b_0)$ is the PDF of $\text{GA}(m + n + a_0, T_1(\mu) + T_2(\mu) + b_0)$. Similarly, $g_\alpha(\alpha; m + a_1, b_1 + D_1(\lambda, \mu)$ and $g_\beta(\beta; n +$

$a_2, b_2 + D_2(\lambda, \mu)$ are the PDFs of GA$(m + a_1, b_1 + D_1(\lambda, \mu))$ and GA$(n + a_2, b_2 + D_2(\lambda, \mu)$, respectively, where

$$D_1(\lambda, \mu) = -\sum_{i=1}^{n} \ln(1 - e^{-\lambda(x_i-\mu)^2}) \quad \text{and} \quad D_2(\lambda, \mu) = -\sum_{j=1}^{m} \ln(1 - e^{-\lambda(y_j-\mu)^2}),$$

$$T_1(\mu) = \sum_{i=1}^{n}(x_i - \mu)^2 \quad \text{and} \quad T_2(\mu) = \sum_{j=1}^{m}(y_j - \mu)^2.$$

Further,

$$h(\lambda, \mu) = e^{D_1(\lambda,\mu)} \times e^{D_2(\lambda,\mu)} \tag{37}$$

and

$$g(\mu) = \prod_{i=1}^{n}(x_i - \mu) \times \prod_{j=1}^{m}(y_j - \mu) \times 1_{c<\mu<z}(\mu) \tag{38}$$

here $1_{c<\mu<z}(\mu)$ is an indicator function which takes value one, if $c < \mu < z$, and zero otherwise. Therefore, the Bayes estimate of $R = R(\alpha, \beta)$, say $\widehat{R}_B$, with respect to squared error loss function can be obtained as the posterior mean of R, *i.e.*

$$\begin{aligned}\widehat{R}_B &= \int_{\alpha=0}^{\alpha=\infty}\int_{\beta=0}^{\beta=\infty}\int_{\lambda=0}^{\lambda=\infty}\int_{\mu=c}^{z} R(\alpha,\beta)\pi(\alpha,\beta,\lambda,\mu|data)d\mu d\lambda d\beta d\alpha \\ &= \frac{\int_{\alpha=0}^{\alpha=\infty}\int_{\beta=0}^{\beta=\infty}\int_{\lambda=0}^{\lambda=\infty}\int_{\mu=c}^{z} R(\alpha,\beta)\pi_0(\alpha,\beta,\lambda,\mu|data)d\mu d\lambda d\beta d\alpha}{\int_{\alpha=0}^{\alpha=\infty}\int_{\beta=0}^{\beta=\infty}\int_{\lambda=0}^{\lambda=\infty}\int_{\mu=c}^{z} \pi_0(\alpha,\beta,\lambda,\mu|data)d\mu d\lambda d\beta d\alpha},\end{aligned} \tag{39}$$

where $\pi_0(\alpha, \beta, \lambda, \mu|data) = \pi(\alpha, \beta, \lambda, \mu|data)/K$. Clearly, $\widehat{R}_B$ cannot be obtained in explicit form, it needs to be calculated numerically, or Lindley [10] type approximation may be used. Unfortunately in both the cases it is not possible to obtain the credible interval. Due to this reason we use Monte Carlo technique as suggested by Chen and Shao [2] to compute $\widehat{R}_B$ and also to construct the associated credible interval. We need the following result for further development.

THEOREM 4: The $g(\mu)$ as defined in (38) is log-concave for $c < \mu < z$.

PROOF: For $c < \mu < z$,

$$\ln g(\mu) = \sum_{i=1}^{n} \ln(x_i - \mu) + \sum_{j=1}^{m} \ln(y_j - \mu).$$

Therefore,

$$\frac{d^2}{d\mu^2}\ln g(\mu) = -\sum_{i=1}^{n}\frac{1}{(x_i-\mu)^2} - \sum_{j=1}^{m}\frac{1}{(y_j-\mu)^2} < 0.$$

Hence the result follows. ∎

Note that due to log-concavity of $g(\mu)$, the method proposed by Devroye [3] can be used to generate samples from the PDF proportional to $g(\mu)$. We propose the following algorithm to compute the Bayes estimate and also to compute the HPD credible interval of R, using the idea of Chen and Shao [2], see also Kundu and Pradhan [8] in this respect.

ALGORITHM:

STEP 1: Generate μ_1 from a PDF proportional to $g(\mu)$ using the method proposed by Devroye [3], also generate λ_1, α_1 and β_1 from $\text{GA}(m+n+a_0, T_1(\mu)+T_2(\mu)+b_0)$, $\text{GA}(m+a_1, b_1+D_1(\lambda,\mu))$ and $\text{GA}(n+a_2, b_2+D_2(\lambda,\mu))$ respectively.

STEP 2: Repeat this procedure to obtain $(\alpha_1,\beta_1,\lambda_1,\mu_1),\cdots,(\alpha_N,\beta_N,\lambda_N,\mu_N)$.

STEP 2: The approximate value of $\widehat{R}_B$ can be obtained as

$$\frac{\sum_{i=1}^{N} R(\alpha_i,\beta_i)h(\lambda_i,\mu_i)}{\sum_{i=1}^{N} h(\lambda_i,\mu_i)}$$

STEP 3: From the generated samples compute

$$w_i = \frac{h(\lambda_i,\mu_i)}{\sum_{i=1}^{N} h(\lambda_i,\mu_i)}; \quad i=1,\cdots,N,$$

and rearrange $\{(R_1,w_1),\cdots,(R_N,w_N)\}$ as $\{(R_{(1)},w_{[1]}),\cdots,(R_{(N)},w_{[N]})\}$, where $R_{(1)} < \cdots < R_{(N)}$. Note that although $R_{(i)}$'s are ordered, $w_{[i]}$'s are not ordered, they are just associated with $R_{(i)}$'s.

STEP 4: Suppose for $0 < p < 1$, $P(R(\alpha,\beta) \le r_p|data) = p$, then a simulation consistent estimate of r_p can be obtained as $\widehat{r}_p = R_{(N_p)}$, where N_p is an integer satisfying

$$\sum_{i=1}^{N_p} w_{[i]} \le p < \sum_{i=1}^{N_p+1} w_{[i]},$$

Table 1: Ordered Data Set X.

0.00049	0.01633	0.01649	0.01811	0.02211
0.03675	0.04221	0.04911	0.05535	0.13239
0.20175	0.22942	0.27841	0.35904	0.43645
0.50868	0.89570	0.91850	1.06802	1.36165

Table 2: Ordered Data Set Y.

0.02508	0.03053	0.06461	0.07875	0.08541
0.17233	0.22555	0.24749	0.24995	0.31585
0.37454	0.45032	0.45662	0.55637	0.65419
0.73808	1.16973	1.22977	1.82091	1.91542

see Chen and Shao [2]. Now, using the above procedure, a 100(1-γ)% credible interval of R can be obtained as $(\widehat{r}_\delta, \widehat{r}_{\delta+1-\gamma})$, for $\delta = w_{[1]}, w_{[1]} + w_{[2]}, \cdots, \sum_{i=1}^{N_{1-\gamma}} w_{[i]}$. Therefore, a 100(1-$\gamma$)% HPD credible interval becomes $(\widehat{r}_{\delta^*}, \widehat{r}_{\delta^*+1-\gamma})$, where δ^* is such that

$$\widehat{r}_{\delta^*+1-\gamma} - \widehat{r}_{\delta^*} \leq \widehat{r}_{\delta+1-\gamma} - \widehat{r}_\delta \quad \text{for all} \quad \delta.$$

6 Data Analysis

In this section we analyze two data sets. One is a set of simulated data and the other is a real data set.

6.1 Simulated Data Set:

We have generated samples from X and Y, where $X \sim$ GR(0.3,1.0,0.0), $Y \sim$ GR(0.2,1.0,0.0) and $m = n = 20$. The data set X and data set Y are presented in Table 1 and Table 2 respectively. In this case the true value of $R = 0.6$.

First we obtain the modified maximum likelihood estimator of R. The modified MLE of μ is $\widetilde{\mu} = \min\{x_{(1)}, y_{(1)}\} = 0.00049$. We provide the profile log-likelihood function of λ in Figure 1, which is an unimodal function, and obtain the estimate of λ as $\widetilde{\lambda} = 1.9299$. Using $\widetilde{\lambda}$, obtain $\widetilde{\alpha} = 0.6247$ and $\widetilde{\beta} = 0.9992$. Based on $\widetilde{\alpha}$ and $\widetilde{\beta}$, $\widetilde{R} = 0.6153$, and a 95% confidence interval is (0.4915, 0.7391). Interestingly, although, the estimates of α and β are quite bad, the estimate of R is quite good.

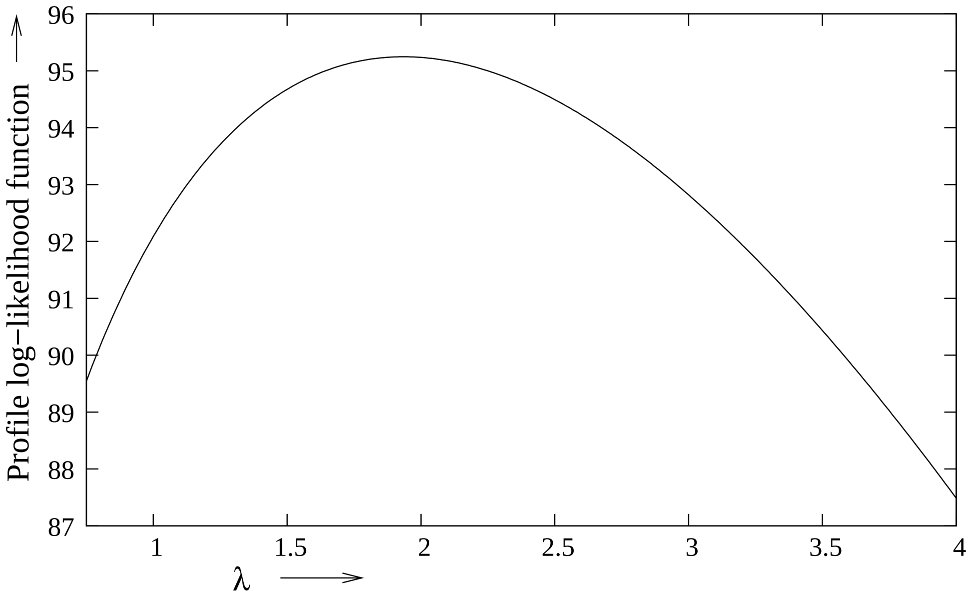

Figure 1: Profile log-likelihood function of λ.

It is observed that the estimates of α and β depend quite heavily on the estimate of μ. Just to see the effect of μ, it is assumed that μ is known and take different values of μ. We report the different values of μ and associated estimates of the different parameters as follows: (i) μ = 0.0, $\widetilde{\lambda} = 1.8699$, $\widetilde{\beta} = 0.9825$, $\widetilde{\alpha} = 0.5307$, $\widetilde{R} = 0.6492$, (ii) $\mu = 0.0001$, $\widetilde{\lambda} = 1.8599$, $\widetilde{\beta} = 0.9785$, $\widetilde{\alpha} = 0.5295$, $\widetilde{R} = 0.6504$, (iii) μ = -0.0001, $\widetilde{\lambda} = 1.8999$, $\widetilde{\beta} = 0.9990$, $\widetilde{\alpha} = 0.5553$, $\widetilde{R} = 0.6428$.

Now we use the bias corrected Hall and Wang's estimates. In Figure 2 we provide the profile modified log-likelihood function of μ. We obtain as estimate of μ, $\bar{\mu} = 0.000175$. Based on $\bar{\mu}$, we obtain $\bar{\lambda} = 0.8700$, $\bar{\alpha} = 0.2086$, $\bar{\beta} = 0.3576$, $\bar{R} = 0.6315$. Clearly, Hall and Wang's estimates provide a better results for λ, α and β, but worse estimate for R. Finally we compute the Bayes estimate of R also using Monte Carlo simulation technique

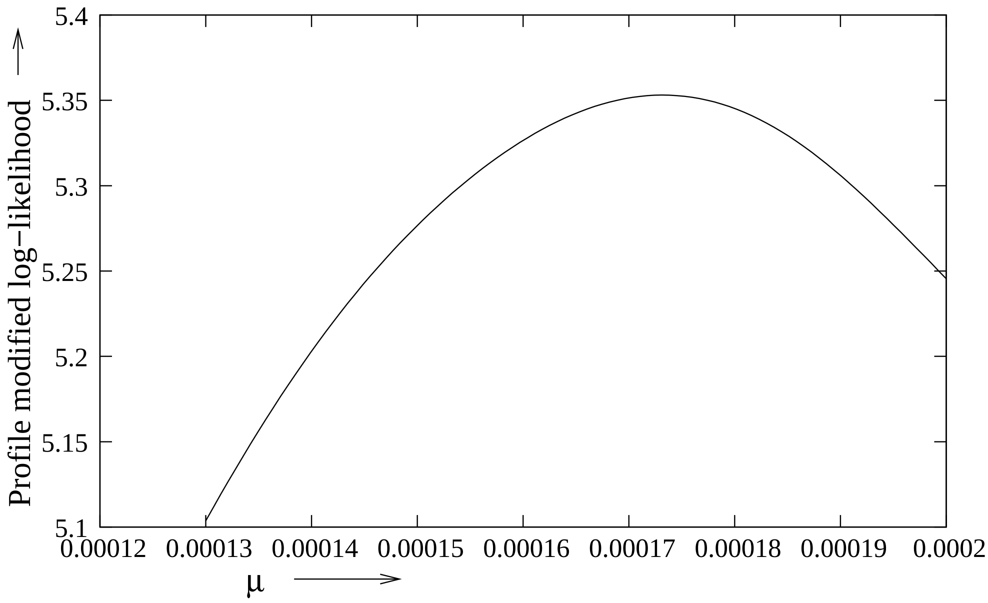

Figure 2: Profile log-likelihood function of μ.

as suggested in the previous section. We have used the following hyper parameters $a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = c = 0$. Based on 10000 Monte Carlo samples ($N = 10000$), we have obtained $\widehat{R}_B = 0.6414$, and the associated 95% HPD credible interval becomes (0.5245, 0.7712).

6.2 Carbon-Fiber Data Sets

In this section we provide a data analysis of a strength data set of Badar and Priest [1]. We have received this data set from Professor R.G. Surles and we are thankful to him due to this. This data set sets were analyzed by Surles and Padgett [14] and by Raqab and Kundu [12] by using two-parameter generalized Rayleigh distribution after subtracting 0.75. In this paper we analyze the data sets with three-parameter Rayleigh distributions.

The data set represents the strength measurement in GPA for single carbon fiber, and impregnated 1000 carbon fibers, and impregnated 1000-carbon fiber tows. These fibers have been tested under different tension with gauge lengths 1,10,20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. We consider single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge lengths. Data Set I and Date Set II have sample sizes 69 and 63 respectively. The data sets are available in Kundu and Gupta [7] and it is not presented here. In this case $\widetilde{\mu} = 1.312$. The profile log-likelihood function is presented in Figure 3, and $\widetilde{\lambda} = 0.4739$. Using the above $\widetilde{\lambda}$, we obtain $\widetilde{\alpha} = 1.5772$, $\widetilde{\beta} = 0.9672$, and $\widetilde{R} = 0.3801$. The 95% confidence interval becomes (0.1248, 0.5049).

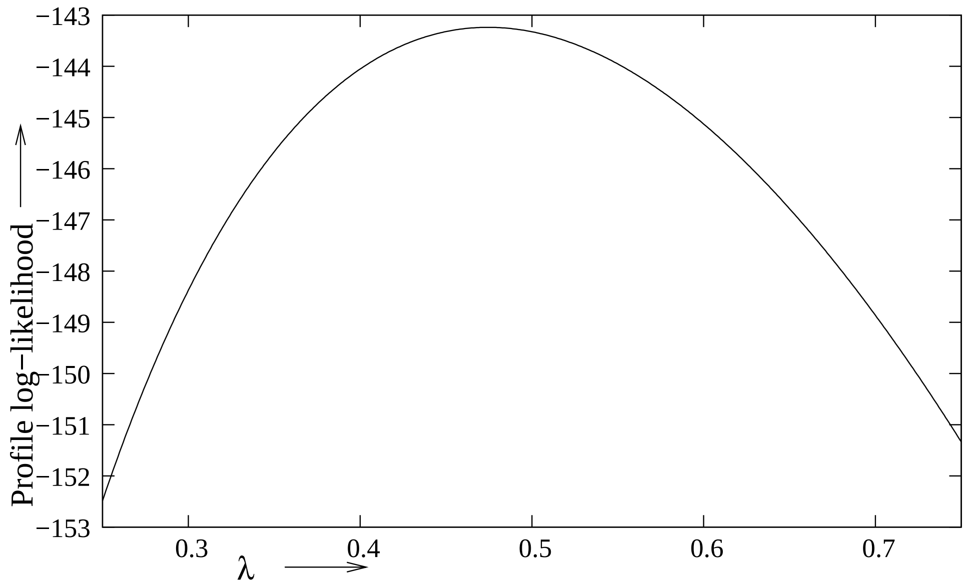

Figure 3: Profile log-likelihood function of λ.

Now we would like to obtain the biased corrected estimates of the unknown parameters. We plot the profile modified log-likelihood function of μ in Figure 4, and it is a monotone function of μ, hence it does not provide any estimate of μ. Finally we compute the Bayes estimate of R using the same hyper-parameters as provided in the previous example, and the Bayes estimate of R becomes 0.3353, and 95% HPD credible interval becomes (0.2179, 0.4586).

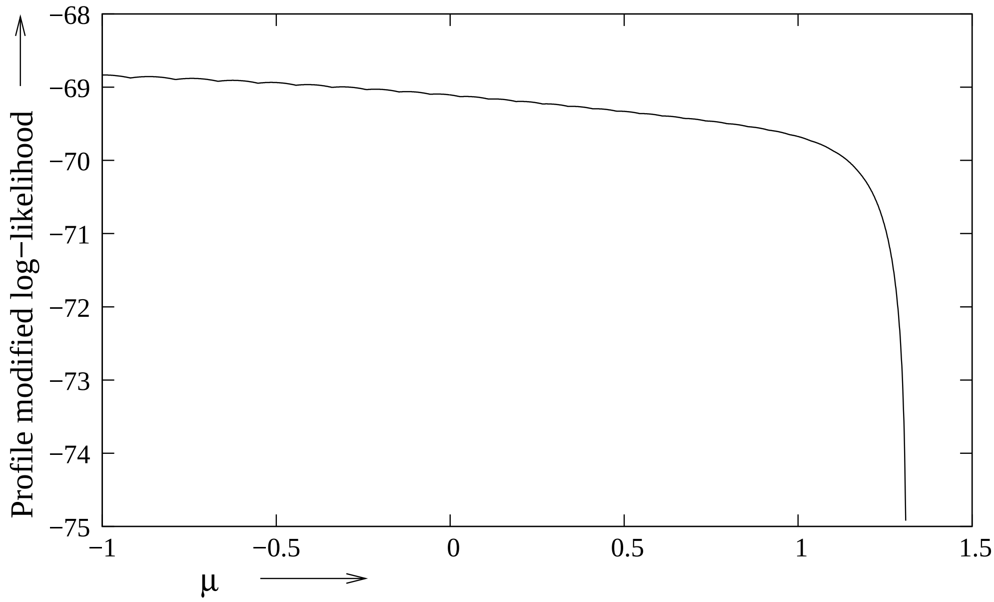

Figure 4: Profile log-likelihood function of μ.

7 Conclusions

In this paper we have consider the inference of the stress strength parameter for two generalized Rayleigh distribution with the same scale and location parameters but different shape parameters. It seems to be a difficult problem, as the model does not satisfy the standard regularity conditions. It is observed that the standard maximum likelihood estimators do not exist, and hence we use the modified maximum likelihood estimators of the scale and shape parameters, and they can be used to compute the modified maximum likelihood estimator of the stress-strength parameter. We have also proposed another biased corrected estimator motivated by the work of Hall and Wang [4]. It seems the method does not work all the times, but when it works, it works quite well. It is observed that the Bayes estimator cannot be obtained in explicit form, but Monte Carlo simulation technique can be used very conveniently to compute Bayes estimate of R and also to construct the associated credible interval. It seems Bayesian inference is a very natural choice in this case. Detailed investigation is needed to compare different methods for various parameter values and for for various sample sizes. More investigation is needed along this line.

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References

[1] Badar, M.G. and Priest, A.M. (1982), "Statistical aspects of fiber and bundle strength in hybrid composites", *Progress in Science and Engineering Composites*, Hayashi, T., Kawata, K. and Umekawa, S. (eds.), ICCM-IV, Tokyo, 1129-1136.

[2] Chen, M.H. and Shao, Q.M. (1999) "Monte Carlo estimation of Bayesian Credible and HPD intervals", *Journal of Computational and Graphical Statistics*, vol. 8, 69-92.

[3] Devroye, L. (1984), "A simple algorithm for generating random variables with a log-concave density", *Computing*, vol. 33, 247 - 257.

[4] Hall, P. and Wang, J.Z. (2005), "Bayesian likelihood methods for estimating the end point of a distribution", *Journal of the Royal Statistical Society*, Ser. B, vol. 67, 717 - 729.

[5] Kotz, S., Lumelskii, Y. and Pensky, M. (2003), *The stress-strength model and its generalizations*, World Scientific Press, Singapore.

[6] Kundu, D. and Gupta, R.D. (2005), "Estimation of $P(Y < X)$ for the generalized exponential distribution", *Metrika*, vol. 61, 291 - 308.

[7] Kundu, D. and Gupta, R.D. (2006), "Estimation of $P(Y < X)$ for the Weibull distribution", *IEEE Transactions on Reliability*, vol. 55, 270 - 280.

[8] Kundu, D. and Pradhan, B. (2009), "Bayesian inference and life testing plans for generalized exponential distribution", *Science in China Series A: Mathematics*, vol. 52, 1373 - 1388.

[9] Kundu, D. and Raqab, M.Z. (2009), "Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution", *Statistics and Probability Letters*, vol. 79, 1839 - 1846.

[10] Lindley, D.V. (1980), "Approximate Bayesian methods", *Trabajos de Estadistica*, vol. 3, 281-288.

[11] Mudholkar, G.S., Srivastava, D.K. and Koria, G.D. (1996), "A generalization of the Weibull distribution with applications to the analysis of survival data", *Journal of the American Statistical Analysis*, vol. 91, 1575 - 1583.

[12] Raqab, M.Z. and Kundu, D. (2005), "Comparison of different estimators of $P(Y < X)$ for a scaled Burr Type-X distribution", *Communications in Statistics - Simulation and Computation*, vol. 34, 465 - 483.

[13] Smith, R.L. (1985), "Maximum likelihood estimation in a class of non-regular cases", *Biometrika*, vol. 72, 67 - 90.

[14] Surles, J.G. and Padgett, W.J. (1998), "Inference for $P(Y < X)$ in the Burr Type X model", *Journal of Applied Statistical Science*, vol. 7, 225-238.

[15] Surles, J.G. and Padgett, W.J. (2001), "Inference for reliability and stress-strength for a scaled Burr Type X distribution", *Lifetime Data Analysis*, vol. 7, 187-200.

[16] Ventura, L. and Racugno, W. (2011), "Recent advances on Bayesian inference for $P(X < Y)$", *Bayesian Analsis*, vol. 6, 411 - 428.