Bayesian Estimation of the Mixture of Generalized Exponential Distribution using the Censored Sample under Different Loss Functions

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Constructing a flexible parametric classes of probability distributions is most popular approach in Bayesian analysis for the last few decades. This study is planned in the same direction for two component Mixture of Generalized exponential probability distribution considering heterogeneous population. We have considered censored sample environment due to its popularity in reliability theory. In addition we have worked out expressions for the Maximum Likelihood (ML) estimates along with their variances and constructed components of the information matrix. To examine the performance of these estimators we have evaluated their properties for different sample sizes, censoring rates, proportions of the component of mixture and a variety of loss functions. The Bayes estimates are evaluated under squared error, entropy, squared logarithmic and precautionary loss functions. Hazard rate of generalized exponential distribution graphically and numerically compared with mixture of other life time distributions. To highlight the practical significance we have included an illustrative application example based on a real-life mixture data.

Keywords: Information matrix; Censored sampling; Inverse transformation method; Squared error loss function; Entropy loss function; Squared logarithmic loss function; Precautionary loss function; Fixed test termination time; Maximum Likelihood; Mixture distribution; Posterior risk; Noninformative priors; Hazard rate of mixture distribution.

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Acronyms and Notations

SELF	Squared error loss function
ELF	Entropy loss function
SLLF	Squared logarithmic loss function
PLF	Precautionary loss function
MLEs	Maximum Likelihood Estimates
GE	Generalized exponential
UP	Uniform prior
JP	Jeffreys prior
BEs	Bayes estimates
PR	Posterior risk
LF	Loss function
p	mixing component
T	Termination time
L(.)	Likelihood function
l	log likelihood
det	Determinant
λ	Scale parameter
α	Shape parameter
F(.)	Cumulative distribution function
f(.)	Probability density function
HFM	Hazard Function of Mixture
mEX	Mixture of exponential distribution
mRA	Mixture of Rayleigh distribution
mWE	Mixture of Weibull distribution
mGE	Mixture of GE

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1. Introduction

In current computational age, analysts are able to explain estimates, predict and infer about the multifaceted structure of interest. In many recent studies it is observed that the GE distribution can be used quite effectively in many circumstances, in place of lognormal, generalized Rayleigh and Weibull distributions. Also, it has been shown that for certain ranges of the parameter values, it is extremely difficult to distinguish between GE and Weibull, gamma, log-normal, generalized Rayleigh distributions. The generalized exponential distribution widely used as a life time distribution due to longevity of its survival function.

Censoring is a primary quality of the lifetime data because most of the times it is not possible to continue the testing until the last observation in order to obtain a complete data set, i.e. a data set with exact life times of all the objects. A censored data set contains at least one observation about which only fractional information on the exact failure time is available. Romeu [1] and Gijbels [2] have given an account of censoring.

Mixture distributions have been used in a wide variety of significant realistic situations because they provide a powerful way to extend common parametric families of distributions to fit datasets not adequately fit by single common parametric distributions. Mixture models have been used in the physical, chemical, social science, biological and other fields. As examples, Harris [3] applied mixture distributions to modeling crime and justice data, Kanji [4] described wind shear data using mixture distributions. Jones and McLachlan [5] fit the mixture of Laplace and normal distribution to wind shear data. A finite mixture of some suitable probability distribution is recommended to study a population that is supposed to comprise a number of subpopulations mixed in an unknown proportion. A population of lifetimes of certain electrical elements or medicines may be divided into a number of subpopulations depending upon the possible cause of failure.

Statistical inferences, order statistics, closeness properties of GE with other distributions have been discussed by several authors. It needs to be mentioned here that GE distribution has been considered before in literature. For example, Gupta and Kundu [6] compared the GE distribution with weibull and gamma distributions and found better fit for skewed data, Gupta and Kundu [7] compared different estimation methods including weighted least square, percentile method including other usual available methods while Gupta and Kundu [8] studied the different properties of exponentiated exponential family. Gupta and Kundu [9] point out the discrimination between Weibull and GE distribution using likelihood ratio method. Mitra and Kundu [10] consider the maximum likelihood estimation procedure of the parameters of the GE distribution when the data is left censored, Kundu and Gupta [11] perform the Bayesian analysis using Gibbs sampling and Monte Carlo simulation for generalized exponential distribution. Aslam et al. [12] provide acceptance sampling plans for generalized exponential distribution when the lifetime experiment is truncated at a pre-determined time. Kundu and Pradhan [13] deal with the Bayesian inference of the unknown parameters of the progressively censored generalized exponential distribution and suggest Lindley’s approximation and importance sampling technique to compute the approximate Bayes estimates. Markov Chain Monte Carlo method has been used to compute the approximate Bayes estimates and also to construct the highest posterior density credible intervals. They also provide different criteria to compare two different sampling schemes and hence to find the optimal sampling schemes. Yarmohammadi and Pazira [14] obtain Bayesian and Classical estimators for the shape parameter, reliability and failure rate functions of the GE distribution in the case of complete and type-II censored samples. The Bayes estimators are obtained using symmetric and asymmetric loss functions. Comparisons are made between these estimators using Monte-Carlo simulation study. It is observed that finding the optimum censoring procedure is a computationally expensive process. So they recommended using the sub-optimal censoring procedure, which can be obtained very easily. A comprehensive literature review about GE distribution is available in Nadarajah [15].

The inspiration toward the use of Bayesian mixture model is that it allocate decomposing the complex arrangement of mixture model to the simple model when the number of components are unidentified. In current study, a population of certain objects is assumed to be calm of two subgroups mixed together in an un-known proportion. The random observations taken from this population are supposed to be characterized by one of the two distinct unknown members of a GE distribution. So the two-component mixture of the GE distribution is recommended to model this population. Right censoring is considered and the observations greater than the fixed cut off censor value, T, are taken as censored ones. The inverse transformation method of simulation, the probabilistic mixing and the computations concerned are conducted in Minitab 12.0 and Mathematica 6.0. It is observed that the Bayes estimators cannot be expressed in explicit forms and they can be obtained by two dimensional numerical integrations only using suitable software like Mathematica or SAS. The purpose of this article is to set records as mixture of GE is not considered earlier in literature.

The GE mixture model is defined in Section 2, and its sampling scheme, likelihood and ML estimates are developed in Section 3. In Section 4, the expressions for the joint mixture distribution using uniform and Jeffreys prior are given. BEs and their PRs are evaluated under SELF, ELF, SLLF and PLF, are also discussed in Section 4. A simulation study is performed in section 5, and hazard rates are compared for different life time mixture distributions in Section 6. A real life data is used in Section 7 for the assessment of Bayes estimates. Some concluding remarks and further research proposal are given in the last Section 8.

2. The Population and the Model

A finite mixture distribution function with the two component densities of specified parametric form with unknown mixing weight (p) is defined as follows:

$$f(x) = pf_1(x) + (1-p) f_2(x), \qquad 0 < p < 1. \tag{1}$$

The following GE distribution is assumed for both components of the mixture:

$$f_i(x) = \frac{2}{\lambda_i}\exp\left(-\frac{x}{\lambda_i}\right)\left(1-\exp\left(-\frac{x}{\lambda_i}\right)\right)^{\alpha-1}, \qquad \lambda_i > 0,\ i = 1,2;\ 0 < x < \infty.$$

For simplicity taking $\alpha = 2$, the mixture model (1) takes the following form:

$$f(x) = \frac{2p}{\lambda_1}\exp\left(-\frac{x}{\lambda_1}\right)\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right) + \frac{2q}{\lambda_2}\exp\left(-\frac{x}{\lambda_2}\right)\left(1-\exp\left(-\frac{x}{\lambda_2}\right)\right); \quad q = 1-p,\ 0 < p < 1.$$

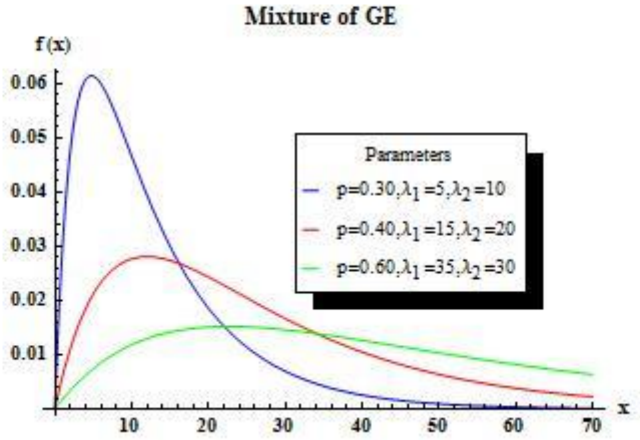

Fig.1

It is clear from figure 1, the curves are unimodal and right skewed shape.
The corresponding mixture distribution function is given by:

$$F(x) = pF_1(x) + qF_2(x) = p\left(1-e^{-\frac{x}{\lambda_1}}\right)^2 + q\left(1-e^{-\frac{x}{\lambda_2}}\right)^2.$$

After little sort of simplification we get distribution function in the following form

$$F(x) = p + q - p\,e^{-\frac{x}{\lambda_1}}\left(2-e^{-\frac{x}{\lambda_1}}\right) - q\,e^{-\frac{x}{\lambda_2}}\left(2-e^{-\frac{x}{\lambda_2}}\right).$$

3. Sampling

Assume n units from the above defined mixture model are engaged to a life testing experiment with a fixed test termination time T. Let the test be conducted and it is observed that out of n, r units failed until the test termination time T is over and the remaining $n-r$ units are still performing. As described in Mendenhall and Hader [16], in many real life situations only the futile items can easily be recognized as member of either subpopulation 1 or subpopulation 2. An engineer, for example, may categorize a failed electronic object as a member of the first or the second subpopulation based on the reason of its failure. So depending upon the cause of failure, it may be observed that r_1 and r_2 failures are from the first and the second subpopulation, respectively. Obviously the remaining $(n-r)$ censored objects provide no information about the subpopulation to which they belong to, and $r = r_1 + r_2$ are the number of uncensored observations. Let define, x_{ij} as the failure time of the j^{th} unit belonging to the i^{th} subpopulation, where $j = 1, 2, 3, ..., r_i,\ i = 1, 2,\ 0 < x_{1j},\ x_{2j} \le T$.

3.1 The Likelihood Function

The likelihood function for the above circumstances is

$$L(\lambda_1, \lambda_2, p \mid \mathbf{x}) \propto \left\{ \prod_{j=1}^{r_1} p\, f_1(x_{1j}) \right\} \left\{ \prod_{j=1}^{r_2} q\, f_2(x_{2j}) \right\} \{1 - F(T)\}^{n-r} .$$

where $\mathbf{x} = \left(x_{11}, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2} \right)$ are the observed failure times for the non-censored observations.

$$L(\lambda_1, \lambda_2, p \mid \mathbf{x}) \propto \left[\prod_{j=1}^{r_1} \frac{2p}{\lambda_1} \mathrm{e}^{\left(-\frac{x_{1j}}{\lambda_1}\right)} \left(1 - \mathrm{e}^{\left(-\frac{x_{1j}}{\lambda_1}\right)} \right) \right] \left[\prod_{j=1}^{r_2} \frac{2q}{\lambda_2} \mathrm{e}^{\left(-\frac{x_{2j}}{\lambda_2}\right)} \left(1 - \mathrm{e}^{\left(-\frac{x_{2j}}{\lambda_2}\right)} \right) \right]$$

$$\left[1 - \left\{ p + q - p\,\mathrm{e}^{-\frac{x_{1j}}{\lambda_1}} \left(2 - \mathrm{e}^{-\frac{x_{1j}}{\lambda_1}} \right) - q\,\mathrm{e}^{-\frac{x_{2j}}{\lambda_2}} \left(2 - \mathrm{e}^{-\frac{x_{2j}}{\lambda_2}} \right) \right\} \right]^{(n-r)} \qquad (2)$$

After simplifications, the (2) can be represented as:

$$L(\lambda_1, \lambda_2, p \mid \mathbf{x}) \propto \sum_{m=0}^{n-r} \binom{n-r}{m} p^{(n-r_2-m)} q^{(r_2+m)} \left(\frac{1}{\lambda_1} \right)^{r_1} \left(\frac{1}{\lambda_2} \right)^{r_2} e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j} + (n-r-m)T \right]} e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j} + mT \right]}$$

$$\left(r_1 - e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j} \right]} \right) \left(r_2 - e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j} \right]} \right) \left(2 - e^{-\frac{T}{\lambda_1}} \right)^{n-r-m} \left(2 - e^{-\frac{T}{\lambda_2}} \right)^{m} \qquad (3)$$

The generalized form of above defined situation can be written as:

$$L(\boldsymbol{\lambda}, \mathbf{p} \mid \mathbf{x}) \propto \{ \prod_{j=1}^{r_1} p_1\, f_1(x_{1j}) \}\ \{ \prod_{j=1}^{r_2} \mathrm{p}_2\, f_2(x_{2j}) \} \ldots \{ \prod_{j=1}^{r_m} \mathrm{p}_m\, f_m(x_{mj}) \}\ \{ (1 - F(T))^{n-r} \}$$

$$L(\boldsymbol{\lambda}, \mathbf{p} \mid \mathbf{x}) \propto \sum_{i}^{H_{n-r}^{m}} \binom{n-r}{m_1, m_2, ..., m_m} \left\{ p_1^{r_1+m_1}\ p_2^{r_2+m_2} \ldots p_m^{r_m+m_m} \right\} \left\{ \left(\frac{1}{\lambda_1} \right)^{r_1} \left(\frac{1}{\lambda_2} \right)^{r_2} ... \left(\frac{1}{\lambda_m} \right)^{r_m} \right\} e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j} + (m_1)T \right]}$$

$$\times e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j} + m_2 T \right]} ... e^{-\frac{1}{\lambda_m}\left[\sum_{j=1}^{r_m} x_{r_m j} + (m_m)T \right]} \left(r_1 - e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j} \right]} \right) \left(r_2 - e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j} \right]} \right) ...$$

$$\times \left(r_m - e^{-\frac{1}{\lambda_m}\left[\sum_{j=1}^{r_m} x_{r_m j} \right]} \right) \prod_{i=1}^{m} \left(2 - e^{-\frac{T}{\lambda_m}} \right)^{m}$$

Here 'm' denotes the components i.e. m_1 means first component; m_2 2nd component and so on where H_{n-r}^{k} denotes the number of distinct terms in the expansion of the multinomial

$\left(1-\left\{p_1\left\{1-e^{-\frac{T}{\lambda_1}}\right\}+p_2\left\{1-e^{-\frac{T}{\lambda_2}}\right\}+...+p_m\left\{1-e^{-\frac{T}{\lambda_m}}\right\}\right\}\right)^{n-r}$ as discussed in Chuan-Chong and Mhee-Meng [17]. The MLEs of $\boldsymbol{\lambda}$ and $\mathbf{p}$ are obtained by solving the system of $2m-1$ nonlinear equations. But in this paper, we present only for two components mixture estimators and in similar way can be extended to '*m*' components.

3.2 ML Estimates

We gain the MLEs of, λ_1, λ_2 and *p* by solving the system of three non-linear equations (4)- (6), obtaining by differentiating the natural 'log' of likelihood (2) with respect to λ_1, λ_2 and '*p*' respectively. Here, we give the Maple 13 code which is presented in appendix for obtaining the likelihood equations to reduce bulk, let $l=\log L(\lambda_1,\lambda_2,p\,|\,\mathbf{x})$.

$$l \propto r_1\ln p + r_1\ln(2) - r_1\ln\lambda_1 - \frac{1}{\lambda_1}\sum_{j=1}^{r_1}x_{1j} + \ln\left(r_1 - e^{-\frac{\sum_{j=1}^{r_1}x_{1j}}{\lambda_1}}\right) + r_2\ln(1-p) + r_2\ln(2) - r_2\ln\lambda_2$$

$$-\frac{1}{\lambda_2}\sum_{j=1}^{r_2}x_{2j} + \ln\left(r_2 - e^{-\frac{\sum_{j=1}^{r_2}x_{2j}}{\lambda_2}}\right) + (n-r)\ln\left(pe^{-\frac{T}{\lambda_1}}\left\{2-e^{-\frac{T}{\lambda_1}}\right\} + qe^{-\frac{T}{\lambda_2}}\left\{2-e^{-\frac{T}{\lambda_2}}\right\}\right)$$

3.3 Variances of the MLEs

Variances of the MLEs are on the main diagonal of the inverted information matrix. The Maple code for obtaining information matrix component is given by and also the expectation of the negative Hessian as below:

$$I(\lambda_1,\lambda_2,p) = -E\begin{pmatrix} \frac{\partial^2 l}{\partial\lambda_1^2} & \frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} & \frac{\partial^2 l}{\partial\lambda_1\partial p} \\ \frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 l}{\partial\lambda_2^2} & \frac{\partial^2 l}{\partial\lambda_2\partial p} \\ \frac{\partial^2 l}{\partial p\partial\lambda_1} & \frac{\partial^2 l}{\partial p\partial\lambda_2} & \frac{\partial^2 l}{\partial p^2} \end{pmatrix}, \qquad (7)$$

For obtaining determinant we use Maple commands are presented in appendix. We do not write here 2nd derivatives values because equations are very complex and an iterative procedure is required to solve these equations.

4. Posterior Distribution Analysis using UP and JP

Let $\lambda_1 \sim \lambda_1^{-d_1}\ \forall\ \lambda_1 \in (0,\infty), d_1 > 0$, $\lambda_2 \sim \lambda_2^{-d_2}\ \forall\ \lambda_2 \in (0,\infty), d_2 > 0$, for *d=0* this will leads to diffusion (uniform), *d=1* Jeffreys noninformative while mixing component follows uniform distribution i.e. p ~ U (0, 1). Jeffreys [18] proposed a formal rule for obtaining a non-informative prior as: If λ is a k-vector valued parameter, then JP of λ is: $g(\lambda) \propto \sqrt{|\det I(\lambda)|}$, where $I(\lambda)$ is a kxk Fisher's (information) matrix whose *(i, j)* th element is $-E\left[\frac{\partial^2 \ln L(\lambda|\mathbf{x})}{\partial\lambda_i\partial\lambda_j}\right]$; *i, j=1, 2,..., k*. Fisher's information matrix is not directly related to the notation of lack of information. The connection comes from the role of Fisher's matrix in asymptotic theory. Jeffreys non-informative priors based on Fisher's information matrix often lead to a family of improper priors. Let

$h(\lambda_i) \propto \sqrt{|I(\lambda_i)|}, I(\lambda_i) = -E\left[\partial^2 \ln L(\mathbf{x}|\lambda_i)/\partial\lambda_i^2\right]$ with $i=1, 2$ and $p \sim U(0,1)$. Assuming independence**,** the joint prior is included with the likelihood function (3) to yield a joint posterior distribution of λ_1, λ_2 and 'p'.

The joint posterior distribution of λ_1, λ_2 and 'p' is

$$g(\lambda_1, \lambda_2, p \mid \mathbf{x}) \propto \sum_{m=0}^{n-r} \binom{n-r}{m} p^{(n-r_2-m)} q^{(r_2+m)} \left(\frac{1}{\lambda_1}\right)^{r_1+d_1} \left(\frac{1}{\lambda_2}\right)^{r_2+d_2} e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j}+(n-r-m)T\right]} e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j}+mT\right]}$$
$$\left(r_1 - e^{-\frac{1}{\lambda_1}\left[\sum_{j=1}^{r_1} x_{1j}\right]}\right)\left(r_2 - e^{-\frac{1}{\lambda_2}\left[\sum_{j=1}^{r_2} x_{2j}\right]}\right)\left(2-e^{-\frac{T}{\lambda_1}}\right)^{n-r-m}\left(2-e^{-\frac{T}{\lambda_2}}\right)^{m} \quad (14)$$

$$0<\lambda_1<\infty,\ 0<\lambda_2<\infty,\ 0<p<1$$

4.1 Bayes Estimators and PRs under different Loss Functions

The SELF; $L(\lambda, d) = (\lambda - d)^2$ was proposed by Legendre [19] and Gauss [20] to develop least square theory. Later, it was used in estimation problems when unbiased estimators of parameter were evaluated in terms of the risk function which becomes nothing but the variance of the estimators.

The SLLF; is $L(\lambda, d) = (\log d - \log \lambda)^2$. Also in many practical places, it appears to be more sensible to express the loss in terms of the ratio d/l . So entropy loss provides explicit estimators for the estimation of the natural parameter which is canonical form of the exponential family. Calabria and Pulcini [21] defined generalized entropy loss function as:

$L(\lambda, d) = b\left[\left(\frac{d}{\lambda}\right)^c - c\log\frac{d}{\lambda} - 1\right],\ c \neq 0$ as a valid alternative to the modified linex loss function. Taking special case of entropy loss function when b=1, we have

$$L(\lambda, d) = \left[\frac{d}{\lambda} - \log\frac{d}{\lambda} - 1\right],\ c = 1$$

Norstrom [22] introduced an alternative asymmetric PLF, and also presented a general class of precautionary loss functions as a special case which is defined as $L(\lambda, d) = \frac{(\lambda - d)^2}{d}$. According to him these loss functions approach infinitely near the origin to prevent underestimation, thus giving conservative estimators, especially when underestimation may lead to serious consequence. Since in risk analysis, both the potentiality of an undesired event and its consequences are investigated. This potentiality is usually measured by either a probability or a failure rate. The Bayesian approach is widely applied to estimate this failure rate. When dealing with disastrous consequences, it can be worse to underestimate the potentiality of an event than to overestimate it. This is important when risk-level is the basis of a risk-reducing initiative, either by reducing the potentiality or the consequences. An erroneously low estimated risk-level can lead to the absence of necessary initiative to reduce the risk level. It is unreasonable to use a loss function that allows one to estimate a failure probability of zero. A positive loss function at the origin allows estimating zero, and in a risk analysis, estimating zero failure probability simply means that no risk is anticipated. Hence, a precautionary loss function is used. Also optimal policy selection has traditionally been discussed in relation to symmetric and often quadratic loss functions. So by using non-symmetric loss functions one is able to deal with cases where it is more damaging to miss the target on one side than the other. Taking expectation of each parameter with respect to its marginal distributions gives the Bayes estimator of the parameters. The following table shows the Bayes estimators and their respective posterior risks under different loss functions.

Table.1 Bayes Estimators and PRs under different Loss Functions

Loss Function	Bayes Estimator (BE)	Posterior Risk (PR)
SELF	$E(\lambda)$	$E(\lambda^2) - (E(\lambda))^2$
SLLF	$\exp(E(\log\lambda))$	$E(\log\lambda)^2 - (E(\log\lambda))^2$

ELF	$\left(E(\lambda^{-1})\right)^{-1}$	$E(\log\lambda)-\log\left(E(\lambda^{-1})\right)$
PLF	$\sqrt{E(\lambda^2 \mid \mathbf{x})}$	$2\left(\sqrt{E(\lambda^2 \mid \mathbf{x})}-E(\lambda \mid \mathbf{x})\right)$

4.2 Expressions for the Bayes Estimators assuming the UP and JP

Taking expectation of each parameter with respect to its marginal distributions gives the Bayes estimator of the parameters. We cannot get simplified form so we will use numerical integration here for finding Bayes estimators and their respective PRs.

5. Simulation Study

A simulation study was conceded in order to inspect the presentation of the Bayes estimators and the impact of small and large sample sizes and different censoring rate in the fit of the model. Samples of size *n=25, 50* and *200* were generated from the two component mixture of the Generalized exponential distribution with parameters, λ_1, λ_2 and '*p*' such that $(\lambda_1, \lambda_2) \in \{(3,4),(20,15)\}$ and $p \in (0.30, 0.40, 0.60)$. Probabilistic mixing is used here to generate the mixture data. For each observation a random number '*u*' was generated from the uniform on *(0, 1)* distribution. If '*u*<*p*', the observation is taken randomly from F_1 (the GE distribution with parameter λ_1) and if '*u*>*p*', the observation is taken randomly from F_2 (the GE distribution with parameter λ_2). Right censoring was carried out using a fixed censoring time *T*. All observations that are larger than *T* are avowed as censored ones. Different fixed censoring times *T* are chosen in order to evaluate the impact of censoring rate on estimates. For each of the combinations of parameters, sample size, censoring rate, *5000* sample were generated using a routine Minitab v *12.0*. In each case, only failures were recognized to be a member of either subpopulation-1 or subpopulation-2 of the mixture. For each of the *5000* samples, the Bayes estimates are computed using Mathematica 6.0 and the average of the *5000* estimates presented in Tables 2-13.

Table 2: BEs and PRs using UP and JP when **p=0.30** under **SELF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		7.35751 (15.8614)	6.86590 (4.25681)	0.333992 (0.009169)	6.01222 (9.21829)	6.47948 (3.51564)	0.329481 (0.009066)
50		6.01941 (4.46406)	6.44949 (1.71031)	0.32263 (0.004860)	5.46423 (3.52278)	6.28673 (1.565700)	0.319761 (0.004815)
200		3.44483 (1.67611)	4.38942 (1.128976)	0.30967 (0.003674)	3.24567 (1.0756)	4.28657 (1.056473)	0.30532 (0.0032647)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		8.72120 (26.9528)	6.68171 (4.87107)	0.343734 (0.011361)	6.93898 (14.4513)	6.26956 (3.98289)	0.336592 (0.011361)
50		6.23838 (5.07974)	6.34628 (1.92912)	0.338767 (0.005967)	5.53559 (3.89034)	6.17149 (1.75927)	0.334241 (0.005942)
200		3.65711 (1.95240)	4.23609 (1.10453)	0.30981 (0.003892)	3.51081 (1.0023)	4.135801 (1.204393)	0.309432 (0.003276)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		45.8490 (724.571)	26.7024 (62.8451)	0.307948 (0.008938)	32.7665 (163.505)	25.2473 (52.9502)	0.304594 (0.008911)
50		36.2409 (471.170)	24.9966 (25.51300)	0.299903 (0.004769)	26.3742 (131.190)	24.3762 (23.6324)	0.294709 (0.004761)
200		21.4562 (124.5671)	16.8430 (10.54904)	0.30091 (0.004088)	20.2378 (109.2610)	15.875323 (9.564820)	0.301457 (0.003931)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		48.2115 (1139.12)	27.9808 (79.6223)	0.308023 (0.010928)	36.3798 (490.714)	26.4232 (66.4929)	0.300749 (0.010934)

50		37.0895 (223.266)	25.4884 (28.9688)	0.29733 (0.01090)	32.7665 (163.505)	24.8655 (26.7027)	0.302289 (0.005639)
200		22.3459 (99.2870)	21.5276 (16.3569)	0.29910 (0.009872)	21.6749 (88.5688)	16.5428 (12.364535)	0.302134 (0.004321)

Table 3: BEs and PRs using UP and JP when **p=0.40** under **SELF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		6.79442 (9.1575)	6.88747 (5.27892)	0.425258 (0.009981)	5.90354 (6.29712)	6.39405 (4.2158)	0.412696 (0.009974)
50		5.41766 (2.34529)	6.48119 (2.10797)	0.4164 (0.005389)	5.08912 (2.00422)	6.25187 (1.89169)	0.424885 (0.005388)
200		3.33764 (0.91204)	4.11087 (1.09815)	0.405641 (0.004521)	3.10345 (1.01230)	4.14672 (1.014373)	0.40562 (0.004245)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		7.39703 (12.3824)	6.94192 (6.68619)	0.436902 (0.012283)	6.35221 (8.33175)	6.36751 (5.21537)	0.427017 (0.012470)
50		5.88853 (3.23437)	6.42630 (2.49759)	0.432127 (0.006572)	5.49763 (2.76936)	6.18310 (2.24059)	0.429683 (0.006528)
200		3.56701 (1.21078)	4.12419 (1.329801)	0.4106536 (0.005565)	3.19456 (0.96723)	4.10468 (1.210734)	0.40728 (0.005217)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		41.4878 (367.564)	27.4675 (81.1227)	0.390088 (0.009851)	32.9433 (253.509)	25.5381 (66.1102)	0.388921 (0.009846)
50		32.3303 (88.4109)	26.6074 (35.4946)	0.399974 (0.005389)	30.2911 (74.6216)	25.7316 (35.2600)	0.398016 (0.005342)
200		22.0456 (10.1238)	16.5425 (12.56438)	0.400456 (0.003891)	21.2678 (9.58041)	16.5436 (6.523020)	0.40214 (0.003786)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		39.2914 (411.135)	30.1189 (114.037)	0.394325 (0.012009)	32.9433 (253.509)	27.8701 (91.3981)	0.390491 (0.012001)
50		33.5003 (119.44)	26.6292 (38.6594)	0.3980 (0.006365)	30.9300 (97.9865)	25.7316 (35.2600)	0.39870 (0.006165)
200		23.4578 (30.9806)	16.86356 (11.5087)	0.3993 (0.0054879)	18.48720 (22.7081)	15.2187 (8.123490)	0.40514 (0.005218)

Table 4: BEs and PRs using UP and JP when **p=0.60** under **SELF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		5.47483 (3.19511)	8.2100 (14.3484)	0.610407 (0.009887)	5.09008 (2.60089)	7.07253 (9.72385)	0.611581 (0.009871)
50		5.08997 (1.28716)	7.01766 (4.39266)	0.615918 (0.005328)	4.92158 (1.17672)	6.55349 (3.66758)	0.610716 (0.005327)
200		3.11456 (0.86541)	4.09134 (2.24348)	0.602738 (0.004219)	3.15629 (0.95317)	4.06372 (2.019738)	0.60536 (0.004037)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		5.90121 (4.27895)	7.69548 (15.4789)	0.606317 (0.011882)	5.45392 (3.40836)	6.46064 (9.47851)	0.609831 (0.010031)

50		5.45232 (1.71024)	6.98545 (5.39239)	0.617709 (0.006525)	5.27232 (1.56943)	6.44511 (4.4549)	0.619908 (0.006504)
200		3.28907 (1.10564)	4.163526 (1.310283)	0.603428 (0.005534)	3.30765 (1.01398)	4.20693 (1.204329)	0.60032 (0.005258)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		34.353 (130.437)	33.8922 (225.700)	0.584746 (0.009986)	31.8797 (104.033)	29.4652 (155.044)	0.587243 (0.009982)
50		32.8311 (54.7148)	27.1287 (60.5068)	0.597434 (0.005402)	31.6859 (49.1556)	25.4584 (51.7148)	0.595772 (0.005399)
200		21.4876 (17.6590)	16.45837 (14.67383)	0.600543 (0.004532)	21.44532 (14.4587)	15.4873 (11.875439)	0.59644 (0.004276)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		34.3351 (162.076)	36.4331 (298.0689)	0.596639 (0.012232)	31.4842 (126.150)	31.2950 (200.045)	0.573266 (0.012208)
50		32.0732 (62.2899)	29.6345 (81.4730)	0.596727 (0.006574)	30.8538 (55.8767)	27.6777 (69.7666)	0.596638 (0.006530)
200		22.6782 (20.5642)	16.3297 (14.38372)	0.598543 (0.0060432)	21.7638 (20.45678)	15.5487 (10.652788)	0.59643 (0.005367)

Table 5: BEs and PRs using UP and JP when **p=0.30** under **SLLF**.

Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		7.35751 (0.226007)	6.89590 (0.0823503)	0.5000 (1.0000)	6.01222 (0.203991)	6.47948 (0.077156)	0.329481 (0.099484)
50		6.01941 (0.110877)	6.44949 (0.0396277)	0.3500 (1.000)	5.46423 (0.106235)	6.28673 (0.038277)	0.31972 (0.091723)
200		3.4562 (0.08715)	4.28731 (0.037810)	0.32067 (0.65781)	3.3761 (0.08601)	4.15738 (0.032647)	0.30546 (0.089753)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		8.7212 (0.263213)	6.68171 (0.0993677)	0.5000 (1.0000)	6.93898 (0.241889)	6.26956 (0.093844)	0.336592 (0.116269)
50		6.23838 (0.11952)	6.34628 (0.0460907)	0.4500 (0.89329)	5.63712 (0.11696)	6.17149 (0.044627)	0.319669 (0.110432)
200		3.18674 (0.08654)	4.17801 (0.032901)	0.31095 (0.7632)	3.43129 (0.07567)	4.16542 0 (0.031875)	0.302345 (0.106538)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		45.849 (0.247046)	26.7024 (0.0809774)	0.307948 (0.068790)	36.6700 (0.217735)	25.2473 (0.077077)	0.304594 (0.610493)
50		36.2460 (1.12518)	24.9966 (0.0394579)	0.3065 (0.058970)	32.7517 (1.21447)	24.3762 (0.038543)	0.303676 (0.052541)
200		21.6548 (1.00156)	15.8943 (0.029843)	0.3016 (0.029087)	21.4579 (0.98650)	16.4328 (0.027641)	0.301875 (0.026554)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		48.2117 (0.315147)	27.9808 (0.0937238)	0.30934 (0.011872)	36.3798 (0.27836)	26.4232 (0.089114)	0.300749 (0.011256)
50		37.0895	25.4884	0.29873	32.7665	24.8655	0.302829

		(0.142111)	(0.0430873)	(0.005644)	(0.136138)	(0.042035)	(0.005032)
200		23.5642 (0.104532)	20.3729 (0.03576)	0.29880 (0.0051054)	21.80931 (0.09731)	15.4218 (0.027645)	0.301028 (0.004862)

Table 6: BEs and PRs using UP and JP when **p=0.40** under **SLLF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		6.79442 (0.166117)	6.88747 (0.099961)	0.41621 (0.0099811)	5.90354 (0.154586)	6.39405 (0.093958)	0.412695 (0.006567)
50		5.41766 (0.129978)	6.48119 (0.047807)	0.41231 (0.007910)	5.08912 (0.12777)	6.25187 (0.046272)	0.414885 (0.007623)
200		3.40567 (0.103289)	4.09630 (0.038510)	0.402313 (0.006912)	3.201827 (0.098452)	4.10687 (0.036768)	0.40321 (0.006134)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		7.39703 (0.189637)	6.94192 (0.123928)	0.423071 (0.027826)	6.35221 (0.180009)	6.36751 (0.117636)	0.427017 (0.018199)
50		5.88853 (0.088025)	6.42630 (0.0578999)	0.420818 (0.021765)	5.49763 (0.0869579)	6.18310 (0.056444)	0.41267 (0.021652)
200		3.56783 (0.08764)	4.23179 (0.0053186)	0.4087428 (0.011654)	3.306457 (0.08572)	4.26180 (0.005147)	0.40284 (0.009741)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		39.2914 (0.211832)	27.4675 (0.097224)	0.402374 (0.056432)	32.9433 (0.196911)	25.5381 (0.092683)	0.38892 (0.053968)
50		32.3303 (0.171803)	26.6074 (0.048044)	0.401267 (0.042176)	30.2911 (0.15656)	25.7316 (0.041623)	0.39879 (0.038551)
200		22.6754 (0.108753)	16.6547 (0.0375858)	0.39990 (0.0031517)	21.56429 (0.107621)	15.3240 (0.037176)	0.40184 (0.003017)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		39.2914 (0.211832)	30.1189 (0.113806)	0.410276 (0.043276)	32.9433 (0.196911)	27.8701 (0.108660)	0.394901 (0.042305)
50		33.5003 (0.0980897)	26.6292 (0.0525956)	0.398061 (0.038943)	30.9300 (0.0954387)	25.7316 (0.051623)	0.39801 (0.037181)
200		23.4589 (0.09654)	15.6537 (0.0421790)	0.3990 (0.0316589)	21.23890 (0.094610)	15.8729 (0.041901)	0.40127 (0.030181)

Table 7: BEs and PRs using UP and JP when **p=0.60** under **SLLF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		6.56466 (2.274531)	8.2100 (0.172197)	0.610642 (0.09871)	6.09300 (1.20304)	7.07253 (0.156814)	0.611581 (0.029071)
50		5.08997 (2.61618)	7.01766 (0.0824459)	0.610023 (0.089027)	4.92158 (2.26626)	6.55349 (0.079543)	0.60735 (0.080673)
200		3.14702 (1.81760)	4.23937 (0.0810762)	0.608366 (0.073219)	2.149852 (1.567812)	4.12748 (0.079264)	0.60325 (0.071634)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		5.90121 (0.1106)	7.69548 (0.206059)	0.61234 (0.096325)	5.45392 (0.105101)	6.46064 (0.189693)	0.609831 (0.036498)

50		5.45232 (0.0558446)	6.98545 (0.102743)	0.610973 (0.0876520)	5.23272 (0.055113)	6.44511 (0.100852)	0.609908 (0.080536)
200		3.12450 (0.052398)	4.12892 (0.098755)	0.6032456 (0.0715647)	3.116542 (0.051028)	4.10648 (0.095325)	0.60321 (0.065320)
Prior	10% Censoring	**UP**			**JP**		
n		**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**
		λ_1=20, λ_2=15					
25		34.3530 (0.964198)	33.8922 (0.164323)	0.584746 (0.032055)	31.8797 (0.817926)	29.4652 (0.152667)	0.587243 (0.032043)
50		32.8311 (0.9013425)	27.12870 (0.076510)	0.596539 (0.028562)	31.6859 (0.813255)	25.4584 (0.074386)	0.594650 (0.025318)
200		21.3451 (0.801265)	15.428764 (0.0585262)	0.5992435 (0.024375)	21.05678 (0.756234)	16.5785 (0.055327)	0.59734 2 (0.021647)
Prior	20% Censoring	**UP**			**JP**		
n		**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**
		λ_1=20, λ_2=15					
25		34.3351 (0.826598)	36.4331 (0.187614)	0.596639 (0.042645)	31.4842 (0.809655)	31.2950 (0.177537)	0.573266 (0.042561)
50		32.0732 (0.79261)	29.6345 (0.087530)	0.596953 (0.036321)	29.05621 (0.71654)	27.6777 (0.086427)	0.58763 (0.033245)
200		22.3476 (0.75234)	15.54638 (0.0652734)	0.597843 (0.030542)	22.08752 (0.68901)	16.6567 (0.058358)	0.59643 (0.025781)

Table 8: BEs and PRs using UP and JP when **p=0.30** under **ELF**.

Prior	10% Censoring	**UP**			**JP**		
n		**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(p \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(p \mid x)$**
		λ_1=3, λ_2=4					
25		5.86682 (3.64742)	6.32278 (3.72891)	0.36000 (0.80910)	4.90254 (3.4574)	5.99805 (3.62099)	0.33507 (0.446550)
50		5.38786 (3.4228)	6.19886 (3.66845)	0.34000 (0.72187)	4.90077 (3.10540)	6.05059 (3.61937)	0.32146 (0.685432)
200		3.28431 (1.09876)	4.12098 (1.12678)	0.310854 (0.49615)	3.27840 (1.07541)	4.09283 (1.018273)	0.31567 (0.474528)
n	20% Censoring	**λ_1=3, λ_2=4**					
		$E(\lambda_1 \mid x)$	**$E(\lambda_2 \mid x)$**	**$E(p \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(p \mid x)$**
25		6.69494 (3.93032)	6.04906 (3.64898)	0.3700 (0.85470)	5.44681 (3.71467)	5.90205 (3.57285)	0.32960 (0.54368)
50		5.53559 (3.48166)	6.06033 (3.62648)	0.34050 (0.6500)	5.01516 (3.38166)	5.70744 (3.53018)	0.312840 (0.461027)
200		3.24657 (1.340348)	4.12800 (2.01678)	0.310560 (0.400147)	3.44210 (1.30765)	4.208751 (1.887210)	0.306382 (0.345754)
Prior	10% Censoring	**UP**			**JP**		
n		**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**
		λ_1=20, λ_2=15					
25		35.7655 (7.27217)	24.6245 (6.44746)	0.31045 (0.076541)	29.4796 (7.06587)	23.3739 (6.34137)	0.328306 (0.071298)
50		32.2849 (3.4746)	24.0294 (6.3782)	0.3087 (0.06987)	29.3294 (3.4746)	23.4545 (6.32932)	0.316521 (0.061653)
200		22.5689 (1.13568)	17.4189 (2.56878)	0.3029 (0.048901)	21.9925 (1.11836)	16.435721 (1.862423)	0.3014174 (0.041748)
Prior	20% Censoring	**UP**			**JP**		
n		**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**	**$E(\lambda_1 \mid x)$**	**$E(\lambda_2 \mid x)$**
		λ_1=20, λ_2=15					
25		35.0749 (7.26555)	25.4755 (6.52192)	0.300198 (0.00981)	27.516 (7.0075)	24.1686 (6.414490)	0.32504 (0.006611)
50		32.1706	24.4132	0.30017839	28.5940	23.8417	0.314190

		(7.01211)	(6.41175)	(0.00917)	(6.89155)	(6.36387)	(0.00856)
200		21.6745 (3.46522)	21.3401 (2.81031)	0.299456 (0.00910)	21.5689 (2.11564)	15.3290 (2.110564)	0.305239 (0.00846)

Table 9: BEs and PRs using UP and JP when **p=0.40** under **ELF**.

Prior		**UP**			**JP**		
n	10% Censoring	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**
		$\lambda_1=3,\ \lambda_2=4$					
25		5.75372 (3.58059)	6.23179 (3.70853)	0.408643 (0.0099809)	5.05796 (3.44624)	5.82028 (3.56909)	0.42309 (0.008129)
50		5.02946 (3.10968)	6.17857 (3.6659)	0.407219 (0.008965)	4.73475 (3.10917)	5.96911 (3.59618)	0.41489 (0.007976)
200		3.20754 (2.10736)	4.15398 (1.32765)	0.403761 (0.006951)	3.21097 (1.68910)	4.34657 (1.235670)	0.40582 (0.006140)
n	20% Censoring	**$\lambda_1=3,\ \lambda_2=4$**					
		$E(\lambda_1\|x)$	**$E(\lambda_2\|x)$**	**$E(p\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**
25		6.11708 (3.71536)	6.13181 (3.68825)	0.416743 (0.08642)	5.30538 (3.42642)	5.66018 (3.52539)	0.413843 (0.076931)
50		5.39233 (3.41379)	6.06467 (3.63381)	0.412878 (0.06543)	5.03984 (3.2781)	5.84366 (3.55889)	0.41027 (0.051523)
200		3.29764 (1.65784)	4.12490 (1.34264)	0.403218 (0.035773)	3.201467 (1.559873)	4.12474 (1.201701)	0.40213 (0.032917)
Prior	10% Censoring	**UP**			**JP**		
n		**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**
		$\lambda_1=20,\ \lambda_2=15$					
25		31.7682 (7.02028)	25.3592 (6.49016)	0.405647 (0.043285)	27.0479 (6.69214)	23.2772 (6.32068)	0.415722 (0.035061)
50		29.8883 (6.77436)	24.9219 (6.47936)	0.402156 (0.0290657)	28.0751 (4.71321)	24.4367 (6.41796)	0.41043 (0.029017)
200		21.5687 (2.45821)	16.3217 (2.47618)	0.401109 (0.023166)	22.34762 (2.21693)	15.5401 (2.310657)	0.40124 (0.022701)
Prior	20% Censoring	**UP**			**JP**		
n		**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**
		$\lambda_1=20,\ \lambda_2=15$					
25		31.7682 (7.25631)	26.8760 (6.63882)	0.409287 (0.027817)	27.0479 (6.69214)	24.9983 (6.491680)	0.407321 (0.026416)
50		30.3686 (6.87546)	25.2645 (6.48504)	0.406415 (0.023901)	28.1137 (6.71995)	24.4367 (6.417960)	0.40173 (0..023623)
200		22.5310 (2.38566)	15.4585 (2.43671)	0.40256 (0.0175490)	21.60120 (2.10459)	16.2372 (2.301813)	0.40123 (0.015378)

Table 10: BEs and PRs using UP and JP when **p=0.60** under **ELF**.

Prior		**UP**			**JP**		
n	10% Censoring	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**
		$\lambda_1=3,\ \lambda_2=4$					
25		6.09273 (3.65056)	6.90822 (3.94894)	0.6100456 (0.105443)	5.68455 (3.15373)	6.04461 (3.67486)	0.63511 (0.091505)
50		4.85293 (2.78512)	6.46208 (3.77269)	0.61056 (0.095632)	4.69670 (2.81398)	6.05227 (3.64028)	0.61735 (0.090516)
200		3.15701 (1.54084)	4.12098 (1.65782)	0.605642 (0.087424)	3.190370 (1.32189)	4.15836 (1.458371)	0.60152 (0.083256)
n	20% Censoring	**$\lambda_1=3,\ \lambda_2=4$**					
		$E(\lambda_1\|x)$	**$E(\lambda_2\|x)$**	**$E(p\|x)$**	**$E(\lambda_1\|x)$**	**$E(\lambda_2\|x)$**	**$E(p\|x)$**
25		5.28271 (2.38352)	6.25783 (3.76786)	0.60973 (0.097652)	4.90936 (2.23444)	5.34257 (3.44449)	0.63980 (0.017421)

50		5.15612 (2.30830)	6.30306 (3.73320)	0.608076 (0.081027)	4.98956 (2.04230)	5.82668 (3.5752)	0.62174 (0.015683)
200		3.12540 (1.20654)	4.1062537 (1.08363)	0.6054267 (0.061524)	3.105641 (1.005643)	4.16840 (1.003746)	0.60845 (0.010373)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		31.1080 (8.83740)	28.7519 (6.7974)	0.611230 (0.041898)	29.0432 (8.68090)	25.2927 (6.53552)	0.607287 (0.015402)
50		31.2780 (6.19034)	25.1307 (6.486010)	0.609263 (0.036738)	30.23360 (5.217741)	23.6338 (6.36213)	0.607362 (0.028087)
200		22.8091 (2.348760)	15.32746 (2.363910)	0.603238 (0.028647)	21.34569 (1.80652)	15.3237 (2.302745)	0.604328 (0.023764)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		30.3722 (1.10944)	30.1896 (6.90090)	0.61506 (0.043621)	28.0325 (1.07276)	26.2017 (6.619320)	0.612890 (0.041678)
50		26.2891 (0.987610)	27.1507 (6.64639)	0.612167 (0.039561)	25..8910 (0.9654190)	25.3863 (6.511500)	0.60764 (0.035782)
200		23.3458 (0.901287)	15.7364 (2.645722)	0.604743 (0.032677)	21.2568 (0.882610)	15.6438 (2.128750)	0.60542 (0.028467)

Table 11: BEs and PRs using UP and JP when **p=0.30** under **PLF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		8.366263 (2.017606)	7.169197 (0.606595)	0.347447 (0.026909)	6.735360 (1.446281)	6.745317 (0.531674)	0.342963 (0.026964)
50		6.379448 (0.720076)	6.580747 (0.262514)	0.330076 (0.014892)	5.777594 (0.626727)	6.410045 (0.246629)	0.327203 (0.014885)
200		3.680076 (0.470493)	4.516191 (0.253543)	0.315546 (0.011753)	3.407341 (0.323342)	4.408078 (0.243017)	0.310620 (0.010601)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		10.149489 (2.856578)	7.036783 (0.710146)	0.359881 (0.032293)	7.912063 (1.946166)	6.579534 (0.619949)	0.353065 (0.032947)
50		6.633033 (0.789306)	6.496490 (0.300421)	0.347462 (0.017391)	5.876487 (0.681793)	6.312413 (0.281846)	0.343015 (0.017547)
200		3.914952 (0.515685)	4.364515 (0.256849)	0.316029 (0.012438)	3.650765 (0.279910)	4.278930 (0.286258)	0.314681 (0.010498)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		53.166736 (14.635472)	27.54322 (2.303844)	0.322133 (0.028371)	35.173122 (4.813244)	26.275014 (2.055428)	0.318886 (0.028585)
50		42.244205 (12.006609)	25.501824 (1.010447)	0.307751 (0.015696)	28.753929 (4.759458)	24.856217 (0.960034)	0.302679 (0.015939)
200		24.185442 (5.458484)	17.153299 (0.620599)	0.307628 (0.013435)	22.777830 (5.080060)	16.173766 (0.596886)	0.307908 (0.012902)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		58.851242 (21.279485)	29.369158 (2.776717)	0.325279 (0.034511)	42.593472 (12.427344)	27.662374 (2.478349)	0.318408 (0.035319)
50		39.986210 (5.793420)	26.050477 (1.124153)	0.315127 (0.035594)	35.173122 (4.813244)	25.396767 (1.062535)	0.311476 (0.018375)

200		24.466839 (4.241879)	21.904211 (0.753222)	0.315171 (0.032142)	23.629856 (3.909912)	16.912385 (0.739170)	0.309202 (0.014136)

Table 12: BEs and PRs using UP and JP when **p=0.40** under **PLF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		7.437852 (1.286864)	7.260589 (0.746239)	0.436836 (0.023155)	6.414741 (1.022402)	6.715629 (0.643158)	0.424608 (0.023824)
50		5.629949 (0.424579)	6.641821 (0.321263)	0.422821 (0.012843)	5.282363 (0.386487)	6.401372 (0.299005)	0.431179 (0.012588)
200		3.471582 (0.267884)	4.242334 (0.262929)	0.411176 (0.011069)	3.262469 (0.318037)	4.267278 (0.241115)	0.410819 (0.010399)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		8.191364 (1.588669)	7.407864 (0.931887)	0.450740 (0.027676)	6.977272 (1.250124)	7.764655 (0.794289)	0.441377 (0.028719)
50		6.157041 (0.537022)	6.617773 (0.382946)	0.439665 (0.015077)	5.743979 (0.492698)	6.361707 (0.357214)	0.437213 (0.015061)
200		3.732873 (0.331726)	4.282376 (0.316372)	0.417374 (0.013441)	3.342520 (0.295921)	4.249604 (0.289847)	0.413635 (0.012710)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		45.703408 (8.431217)	28.906509 (2.878018)	0.402517 (0.024857)	36.589206 (7.291812)	26.801208 (2.526216)	0.401379 (0.024917)
50		33.669856 (2.679112)	27.266249 (1.317698)	0.406655 (0.013362)	31.498767 (2.415334)	26.407863 (1.352526)	0.404671 (0.013310)
200		22.274027 (0.456854)	16.917999 (0.750998)	0.405285 (0.009658)	21.491852 (0.448105)	16.739585 (0.391971)	0.406820 (0.009360)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		44.214806 (9.846813)	31.955987 (3.674175)	0.409269 (0.029888)	36.589206 (7.291812)	29.464225 (3.188251)	0.405566 (0.030151)
50		35.237907 (3.475214)	27.345451 (1.432502)	0.405917 (0.015835)	32.475397 (3.090794)	26.407863 (1.352526)	0.406358 (0.015316)
200		24.109106 (1.302611)	17.201406 (0.675691)	0.406114 (0.013627)	19.091481 (1.208563)	15.483292 (0.529183)	0.411529 (0.012779)

Table 13: BEs and PRs using UP and JP when **p=0.60** under **PLF**.

Prior		**UP**			**JP**		
n	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		λ_1=3, λ_2=4					
25		5.759242 (0.568825)	9.041709 (1.663418)	0.618414 (0.016013)	5.339457 (0.498755)	7.729459 (1.313857)	0.619598 (0.016035)
50		5.214878 (0.249816)	7.323948 (0.612576)	0.619964 (0.008091)	5.039709 (0.236258)	6.827577 (0.548174)	0.615062 (0.008692)
200		3.250522 (0.271924)	4.356896 (0.612576)	0.606227 (0.006979)	3.303837 (0.295093)	4.305062 (0.482684)	0.608685 (0.006650)
n	20% Censoring	**λ_1=3, λ_2=4**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
25		6.253257 (0.704095)	8.642876 (1.894793)	0.616038 (0.019441)	5.757916 (0.607993)	7.156702 (1.392123)	0.618001 (0.016339)

50		5.606963 (0.309286)	7.361311 (0.751722)	0.622968 (0.010518)	5.419113 (0.293586)	6.781913 (0.673606)	0.625132 (0.010448)
200		3.453060 (0.327979)	4.318012 (0.308973)	0.607996 (0.009136)	3.457532 (0.299764)	4.347712 (0.281561)	0.604683 (0.008727)
Prior	10% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		36.201735 (3.697469)	37.072648 (6.360896)	0.593223 (0.016955)	33.471699 (3.183998)	31.988154 (5.045908)	0.595681 (0.016877)
50		33.654062 (1.645925)	28.221856 (2.186312)	0.601938 (0.009008)	32.452301 (1.532802)	26.454582 (1.992365)	0.600286 (0.009028)
200		21.894656 (0.814111)	16.898277 (0.879814)	0.604304 (0.007523)	21.779817 (0.668995)	15.866061 (0.757522)	0.600014 (0.007148)
Prior	20% Censoring	**UP**			**JP**		
n		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		λ_1=20, λ_2=15					
25		36.619327 (4.568453)	40.316742 (7.767284)	0.606803 (0.020328)	33.427606 (3.886812)	34.342714 (6.095428)	0.583817 (0.021101)
50		33.029987 (1.913574)	30.978647 (2.688295)	0.602210 (0.010966)	31.746396 (1.785192)	28.910581 (2.465761)	0.602085 (0.010895)
200		23.127147 (0.897895)	16.764332 (0.869264)	0.603570 (0.010054)	22.228805 (0.930009)	15.887569 (0.677739)	0.600912 (0.008965)

It is immediate from Tables 2-13, the posterior risks of the estimates reduce as the sample size increases. As censoring rate increased the parameter values are over-estimated as compared to small censoring rate. For small mixing component, parameters are more over-estimated as compared to large mixing component. So here we can say that increase in mixing component provides surety of decreasing BEs and PRs. By making judgment between different loss functions we see that SLLF posterior risk is smaller than the other three loss functions. One interesting thing is notable here, is that BEs under SELF and SLLF are equal up to five decimal places and there only exists difference after sixth or seventh decimal place. By making comparison between noninformative priors, one can easily observes that the Jeffreys prior posterior risk is less than the Uniform prior. For large censoring rate posterior risk increases and reason is that we gained less information from data than the small censoring rate.

6. Hazard Rate of Mixture Distribution

In reliability theory, classification of lifetime models is defined in terms of their reliability function and hazard rate. Since hazard rate function is ratios of life time model and survival function. Now there are two possibilities, one is if survival function value less which mean item or component have less survival time then hazard rate will be a large value (chance of failure will increase) while on contrary if survival function value large, then hazard rate will decrease (chance of failure will decrease). Since GE distribution belongs to the family which has minimum chance of failure as compared to usual life time distributions. We compare the hazard rate of mixture of GE distribution with exponential, Rayleigh and Weibull distributions. Following expressions are hazard function of mixture distribution (HFM) for the said distributions.
HFM for exponential distribution mixture:

$$h(t)=\frac{p\lambda_1\exp(-t/\lambda_1)+q\lambda_2\exp(-t/\lambda_2)}{p\exp(-t/\lambda_1)+q\exp(-t/\lambda_2)} \tag{15}$$

HFM for Rayleigh distribution mixture:

$$h(t)=\frac{\left(2pt/\lambda_1\right)\exp(-t^2/\lambda_1^{\,2})+\left(2q\lambda_2/t\right)\exp(-t^2/\lambda_2^{\,2})}{p\exp(-t^2/\lambda_1^{\,2})+q\exp(-t^2/\lambda_2^{\,2})} \tag{16}$$

HFM for Weibull distribution mixture:

$$h(t)=\frac{\left(2pt/\lambda_1^{\,2}\right)\exp(-t/\lambda_1)^2+\left(2qt/\lambda_2^{\,2}\right)\exp(-t/\lambda_2)^2}{p\exp(-t/\lambda_1)^2+q\exp(-t/\lambda_2)^2} \tag{17}$$

and HFM for GE distribution:

$$h(t)=\frac{\left(2p/\lambda_1\right)\exp(-t/\lambda_1)\{1-\exp(-t/\lambda_1)\}+\left(2q/\lambda_2\right)\exp(-t/\lambda_2)\{1-\exp(-t/\lambda_2)\}}{p\exp(-t/\lambda_1)\{2-\exp(-t/\lambda_1)\}+q\exp(-t/\lambda_2)\{1-\exp(-t/\lambda_2)\}} \tag{18}$$

where $q=1-p$

The cumulative hazard function $H(x)$ is defined as:

$$H(x) = -\{1\text{-logF(x)}\} = -\left[1-\log\left\{1-p\,e^{-\frac{x}{\lambda_1}}\left(2\text{–}e^{-\frac{x}{\lambda_1}}\right)-q\,e^{-\frac{x}{\lambda_2}}\left(2\text{–}e^{-\frac{x}{\lambda_2}}\right)\right\}\right] \qquad (19)$$

The failure rate average (fra) is defined as $FRA(x)=\left(\frac{\text{H(x)}}{x}\right),\ x>0$. The conditional survival of 't' is:

$$\text{R(x|t)}=\left(\frac{\text{R(x+t)}}{\text{R(x)}}\right),\quad \text{t,x,R(.)>0} \qquad (20)$$

Similarly to h(x) and FRA(x), the distribution of X belongs to the new better than used (NBU), exponential, or new worse than used (NWU) classes, when

R(x|t)<R(x), R(x|t)=R(x) or R(x|t)>R(x)

respectively. In our case

$$R(x\,|\,t)=\frac{p\exp(-(t+x)/\lambda_1)\{2-\exp(-(t+x)t/\lambda_1)\}+q\exp(-(t+x)/\lambda_2)\{1-\exp(-(t+x)/\lambda_2)\}}{p\exp(-x/\lambda_1)\{2-\exp(-x/\lambda_1)\}+q\exp(-x/\lambda_2)\{1-\exp(-x/\lambda_2)\}} \qquad (21)$$

so solving numerically, one can easily observes that mixture distribution belongs to NBU class.

Theorm: h(t) is increasing failure rate (IFR).
Proof: By lemma 5.9, P.77 Barlow and Proschan [23], it is sufficient to show that the pdf of mixture is concave. Let Ψ(x)=log f(x), taking twice derivative in (0,∞) with respect to x is –ve ∀ x, hence Ψ(x)=log f(x) is concave in x, and hence the proof.

In reliability theory, classification of lifetime models is defined in terms of their reliability function and other reliability characteristics. For example, the increasing failure rate/decreasing failure rate (IFR/DFR) class, increasing failure rate average/decreasing failure rate average (IFRA/DFRA) class. the new better than used/new worse than used (NBUE/NWUE) class and increasing mean residual lifetime/decreasing mean residual lifetime (IMRL/DMRL) class. Since GE distribution belongs to IFR class, so the following chain of implication holds for this distribution (p. 159 Barlow and Proschan [23]):

IFR⇒IFRA⇒NBU⇒NBUE⇒DMRL

The graphical presentation of hazard function of simple and mixture distribution is as follows:

$\lambda_1=3, \lambda_2=4, p=0.60$

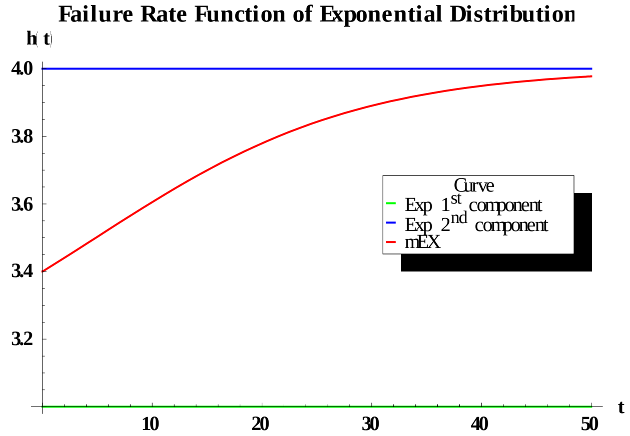

Fig. 2

$$\lambda_1 = 20, \lambda_2 = 15, p = 0.60$$

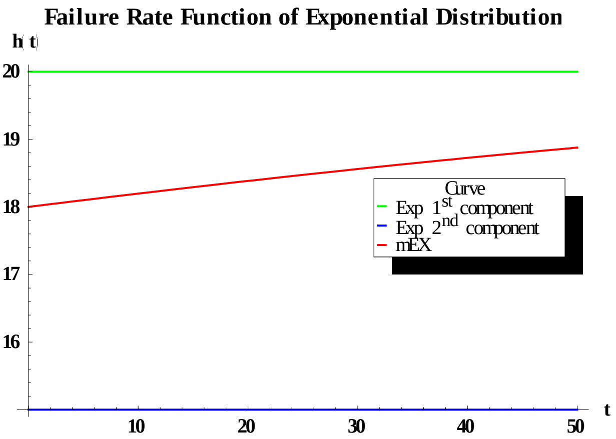

Fig.3

$$\lambda_1 = 3, \lambda_2 = 4, p = 0.60$$

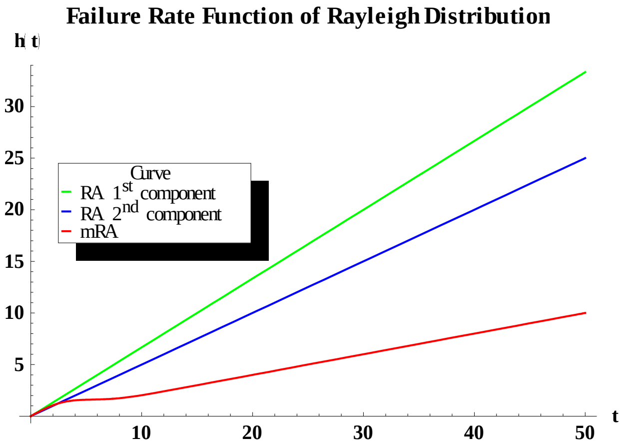

Fig.4

$$\lambda_1 = 20, \lambda_2 = 15, p = 0.60$$

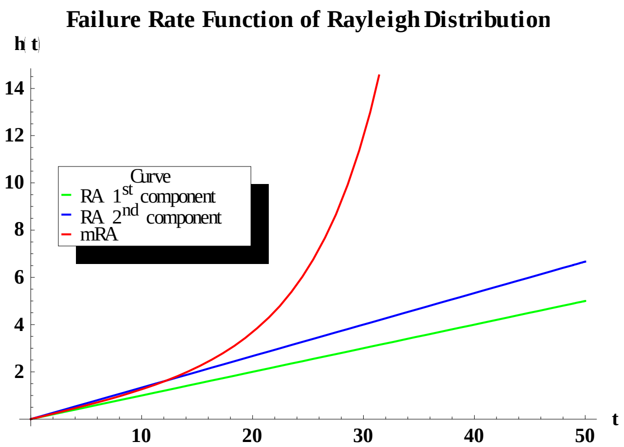

Fig.5

$\lambda_1 = 3, \lambda_2 = 4, p = 0.60$

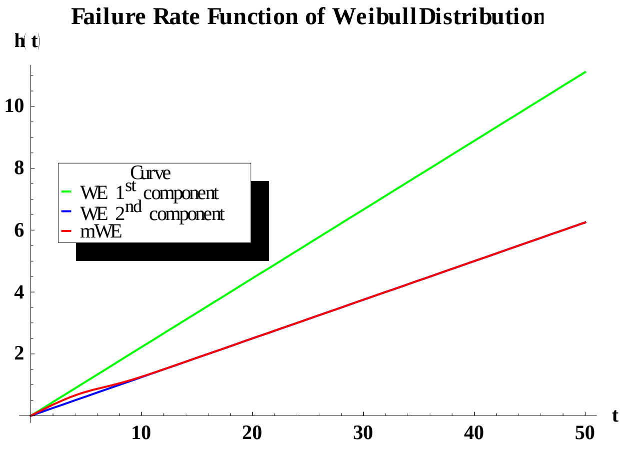

Fig.6

$\lambda_1 = 20, \lambda_2 = 15, p = 0.60$

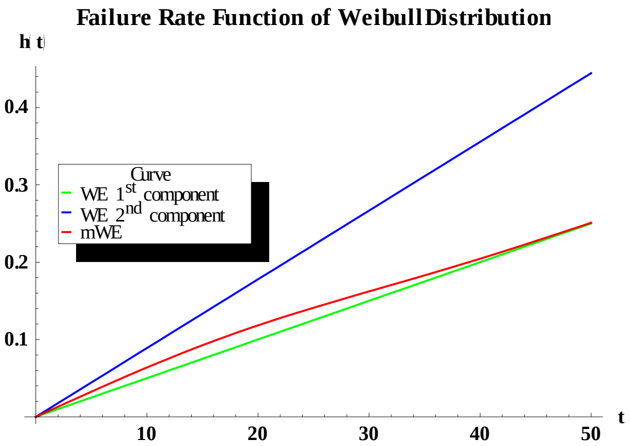

Fig.7

$\lambda_1 = 3, \lambda_2 = 4, p = 0.60$

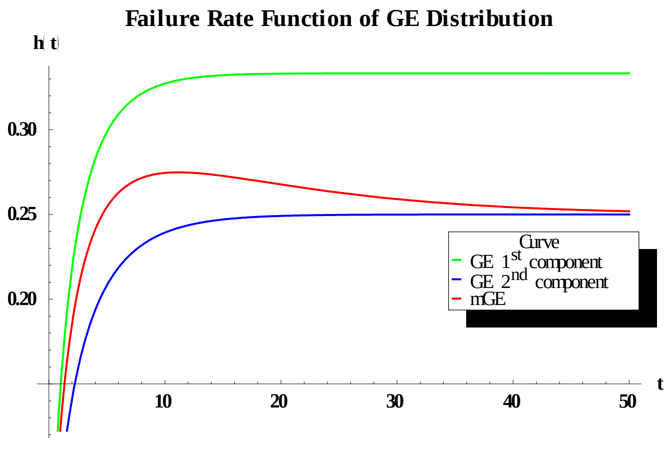

Fig.8

$\lambda_1 = 20, \lambda_2 = 15, p = 0.60$

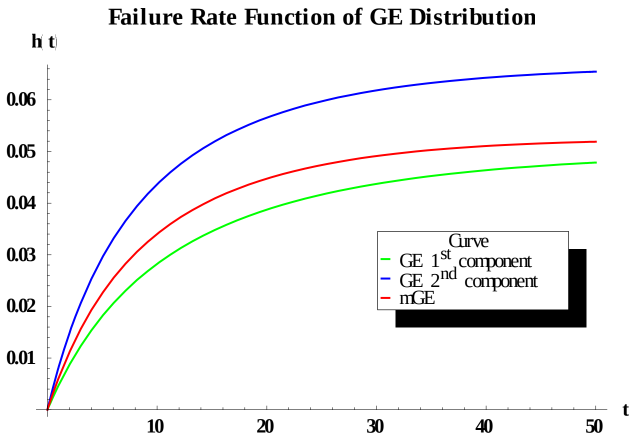

Fig.9

$\lambda_1 = 3, \lambda_2 = 4, p = 0.30$

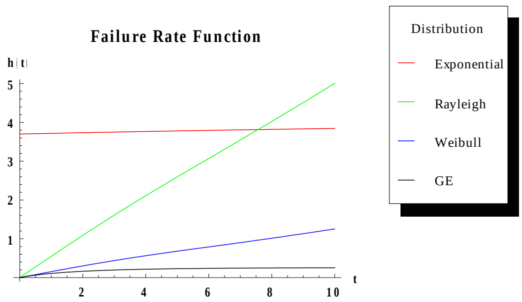

Fig.10

$\lambda_1 = 20, \lambda_2 = 15, p = 0.30$

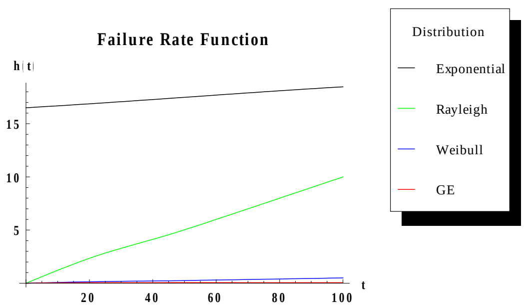

Fig.11

$\lambda_1 = 3, \lambda_2 = 4, p = 0.40$

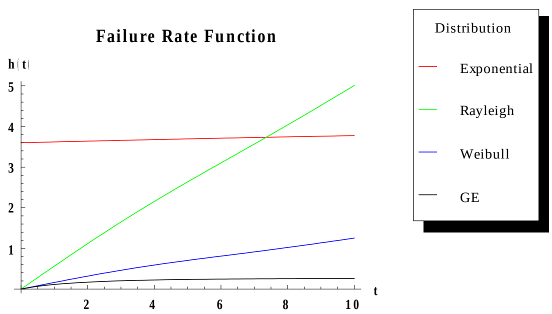

Fig.12

$\lambda_1 = 20, \lambda_2 = 15, p = 0.40$

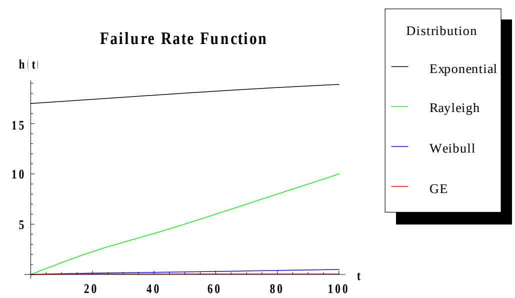

Fig.13

$\lambda_1 = 3, \lambda_2 = 4, p = 0.60$

$\lambda_1 = 20, \lambda_2 = 15, p = 0.60$

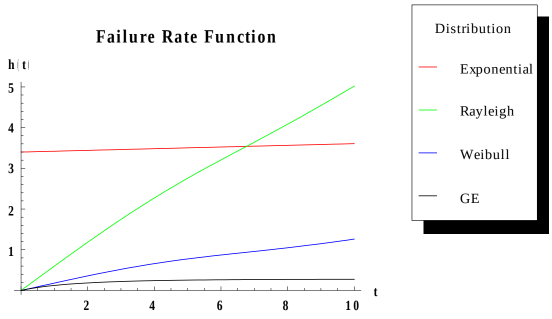

Fig.14

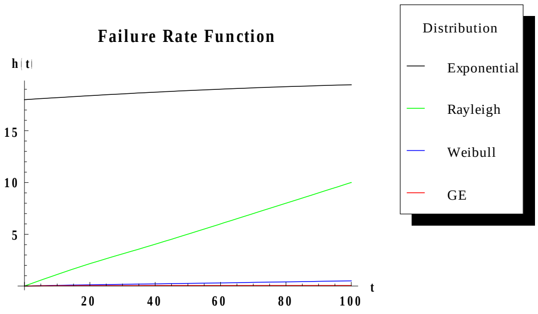

Fig.15

It is clear from Fig. 2-15 (on line color graphs), mixture of GE distribution have minimum failure rate as compared to mixture of Weibull, Rayleigh and simple exponential distribution. Also from Fig. 2-9, one can easily observe that mixture distribution is more flexible than simple distribution and has minimum increasing failure rate as compared to mWE and mRA. Hazard rate (numerical values) are given in the following Tables 14-16 for different mixing component and parameter values.

Table 14: Hazard Rate when mixing component (p) is **0.30**.

t	h(t)							
	$\lambda_1=3,\lambda_2=4$				$\lambda_1=20,\lambda_2=15$			
	mEX	mRA	mWE	mGE	mEX	mRA	mWE	mGE
1	3.717200	0.548315	0.153184	0.107172	16.517600	0.123320	0.007721	0.007048
5	3.779710	2.593990	0.679827	0.228981	16.588900	0.614949	0.038411	0.025967
10	3.842990	5.005510	1.253220	0.253395	16.680600	1.219220	0.075575	0.038684
50	3.993400	25.00000	6.250000	0.250550	17.482500	5.029570	0.253450	0.056825

Table 15: Hazard Rate when mixing component (p) is **0.40**.

t	h(t)							
	$\lambda_1=3,\lambda_2=4$				$\lambda_1=20,\lambda_2=15$			
	mEX	mRA	mWE	mGE	mEX	mRA	mWE	mGE
1	3.619820	0.564732	0.162760	0.112781	17.020000	0.119984	0.007331	0.006705
5	3.694690	2.637590	0.705258	0.236864	17.100800	0.598046	0.036439	0.024822
10	3.775350	5.008560	1.254990	0.259433	17.202900	1.184190	0.071488	0.037099
50	3.989770	25.00000	6.250000	0.250852	18.026800	5.019130	0.252232	0.054905

Table 16: Hazard Rate when mixing component (p) is **0.60**.

t	h(t)							
	$\lambda_1=3,\lambda_2=4$				$\lambda_1=20,\lambda_2=15$			
	mEX	mRA	mWE	mGE	mEX	mRA	mWE	mGE
1	3.420160	0.598046	0.182194	0.124113	18.020000	0.113318	0.006554	0.006020
5	3.502800	2.756610	0.774688	0.254384	18.099100	0.564732	0.032552	0.022556
10	3.60536	5.019130	1.261160	0.274524	18.196300	1.118120	0.063781	0.034026
50	3.977270	25.00000	6.250000	0.251893	18.876800	5.008560	0.250998	0.051887

One can easily observe from above tables that mixture of GE has small failure rate as compared to other mixture distributions failure rate. Also, as we increase mixing component, our failure increases which confirms the theory because as we increase mixing proportion, there are greater chances of failure. For large time, hazard rate decreases for mixture of GE for small parameter values. The results presented in tables 14-16 are well matched with graphical results.

7. Real Life Application

Taking the real data set for GE used by Gupta and Kundu [6] about the number of million revolutions before failure for each of the 23 ball bearing in the life test, the results are given below used for mixture distribution analysis.

Descriptive Statistics

```
Variable          N      Mean    Median  Tr Mean     StDev  SE Mean
GE               23     72.22     67.80    69.99     37.49     7.82

Variable        Min       Max        Q1        Q3
GE            17.88    173.40     45.60     98.64
```

We divide the real data for mixture requirements as follows:

Table 17: Real Data for Mixture of GE

10% censoring					
p	T	r_1	r_2	$\sum_{i=1}^{r_1} x_{1i}$	$\sum_{j=1}^{r_2} x_{1j}$
0.30	128	7	13	383.04	794.88
0.40	105	9	10	439.44	560.64
0.60	128	11	9	637.36	522.00
20% censoring					
0.30	93	5	12	304.32	644.80
0.40	105	9	8	491.92	435.12
0.60	84	9	8	451.96	408.76

Table 18: BEs and PRs using UP and JP under Different Loss Functions.

Prior		**UP**			**JP**		
p	10% Censoring	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
		SELF					
0.30		99.6392 (2716.74)	90.8398 (874.4440)	0.365017 (0.010345)	82.4378 (1592.53)	84.4147 (694.5490)	0.361458 (0.010296)
0.40		84.2838 (1306.08)	93.3646 (1302.130)	0.470773 (0.011758)	73.9146 (906.651)	84.0776 (951.3850)	0.468958 (0.011823)
0.60		89.9660 (1103.42)	95.9250 (1671.610)	0.583514 (0.011066)	81.6450 (830.881)	84.2355 (1153.230)	0.544467 (0.011104)
p	20% Censoring	**SELF**					
		E(λ_1|x)	**E(λ_2|x)**	**E(p|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(p|x)**
0.30		144.8440 (9827.78)	91.2426 (947.0170)	0.331468 (0.011969)	110.576 (4143.22)	84.4400 (744.4310)	0.326273 (0.011992)
0.40		105.784 (1971.87)	110.0780 (2552.750)	0.524688 (0.013524)	94.0398 (1406.70)	95.5310 (1710.490)	0.525824 (0.013469)
0.60		92.0750 (1442.28)	97.4334 (1919.250)	0.592327 (0.013256)	81.7064 (1013.62)	84.6240 (1266.530)	0.572464 (0.013236)
Prior	10% Censoring	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		SLLF					
0.30		99.6392 (0.210043)	90.8398 (0.094424)	0.365017 (0.088559)	82.4378 (1.6460)	84.4147 (0.088026)	0.361457 (0.089770)
0.40		84.2838 (4.03606)	93.3646 (0.127661)	0.470773 (0.059985)	73.9146 (2.94926)	84.0776 (0.117849)	0.468958 (0.060780)
0.60		89.9660 (7.08920)	95.9250 (0.151255)	0.543514 (0.041877)	81.6450 (6.31834)	84.2355 (0.139148)	0.544467 (0.041900)
Prior	20% Cens	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		SLLF					

0.30		144.8440 (0.290246)	91.2426 (0.100843)	0.331468 (0.129247)	110.576 (0.245689)	84.4400 (0.094134)	0.326273 (0.133410)
0.40		105.784 (7.79621)	110.0780 (0.173639)	0.524688 (0.056363)	94.0398 (6.86484)	95.5310 (0.161264)	0.525824 (0.056287)
0.60		92.0750 (5.569260)	97.4334 (0.163893)	0.592327 (0.055259)	81.7064 (4.65143)	84.6240 (0.149487)	0.584627 (0.055961)
Prior	10% Censoring	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		ELF					
0.30		80.7117 (8.88280)	82.6472 (8.875490)	0.409527 (0.109799)	68.2592 (8.36539)	77.2959 (8.738600)	0.376658 (0.094236)
0.40		72.2260 (0.396048)	82.1568 (8.879580)	0.427896 (0.305839)	64.0564 (0.27584)	74.7193 (8.685280)	0.413239 (0.288826)
0.60		79.9022 (0.812626)	82.4374 (8.897630)	0.587970 (0.167273)	73.1054 (0.730923)	73.2822 (8.65660)	0.583666 (0.200389)
Prior	20% Censoring	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		ELF					
0.30		108.0008 (9.50116)	82.4793 (8.87460)	0.598083 (0.315920)	86.3442 (9.03538)	76.8460 (8.73000)	0.290643 (0.061747)
0.40		91.0450 (0.854407)	92.4887 (9.13899)	0.488930 (0.273560)	81.7574 (0.759565)	81.1826 (8.87526)	0.490178 (0.027080)
0.60		79.8136 (0.589107)	82.6636 (8.90909)	0.584682 (0.275935)	71.6333 (0.492587)	72.8520 (8.64996)	0.609050 (0.026487)
Prior	10% Censoring	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		PLF					
0.30		112.204851 (25.681862)	95.531739 (9.383879)	0.378923 (0.027811)	91.588869 (18.302137)	88.432972 (8.036544)	0.375430 (0.027944)
0.40		91.705174 (14.842748)	100.095347 (13.461494)	0.483099 (0.024653)	79.812399 (11.795599)	89.556841 (10.958482)	0.481398 (0.024881)
0.60		95.902561 (11.873121)	104.274712 (16.699425)	0.592920 (0.018813)	86.583988 (9.877976)	90.823177 (13.175354)	0.554570 (0.020207)
Prior	20% Censoring	**UP**			**JP**		
p		**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**	**E(λ_1|x)**	**E(λ_2|x)**
		PLF					
0.30		175.520837 (61.3536751)	96.292414 (10.099629)	0.349056 (0.035176)	127.946363 (34.740726)	88.738631 (8.597262)	0.344160 (0.035774)
0.40		114.726303 (17.884606)	111.220124 (2.284249)	0.537421 (0.025466)	101.243197 (14.406794)	104.099289 (17.136578)	0.538479 (0.025310)
0.60		99.599627 (15.049253)	106.829385 (18.791969)	0.603413 (0.022172)	87.690112 (11.967425)	91.802785 (14.357570)	0.583910 (0.022892)

The above table results are compatible with simulation study, however there are some exceptional cases in which the results are underestimated and ELF posterior risk less than SLLF. But overall we can say that JP has less posterior risk under SLLF. The large censoring rate cause greater posterior risk. Mixing proportion increase does not guaranty that posterior risk will be decreased.

8. Final Remarks

The simulation study has displayed some interesting properties of the Bayes estimates. The posterior risks of the estimates of the parameters seem to be quite large for the relatively larger values of the parameters and vice versa. However, in each case, the posterior risk of parameters is reduced as the sample size increases. Another interesting remark concerning the posterior risk of the estimates of the parameters is that increasing (decreasing) the proportion of the component in the mixture reduces (increases) posterior risk of the estimate of the corresponding parameter.

The effect of censoring on the estimates of parameters is in the form of over-estimation. To be more specific, larger degree of censoring results in large size of over-estimation for small sample size. However, as we increase the sample size, the effect of censoring reduces. On the other hand, the proportion parameter is either under-estimated or over-estimated depending upon the values of the parameters or censoring degree and sample size. The large censoring rate causes greater posterior risk.

The proportion parameter is over-estimated (some degree under-estimated) whenever the parameter of the first subgroup is smaller (greater) than the parameter of the second subgroup. Here we can say the size of over or under-estimation is directly proportional to the amount of censoring rates and inversely proportional to the sample size. Also, the extent of over or under-estimation is more intensive for larger parameter values of the proportion parameter. Furthermore, increasing the sample size reduces the posterior risk of the estimate of the proportion parameter. The increase in proportion of a component in the mixture does not guaranty the reduction in variance of estimate of the proportion parameter.

Further, estimates of the mixing proportion parameter are over-estimated but the degree of over-estimation is quite smaller in case of Bes using JP but when p=0.60 for small sample size, this is under-estimated especially in case of real life data set. One thing which is common here can be observed, as we increase sample size the posterior risk decreases and degree of censoring increasing does not provide surety that risk will decrease.

In the real life application, PRs using UP are greater than the Jeffreys prior. As question of choice of loss function based on priors, one can easily observe that the SLLF has minimum posterior risk than other (SELF, ELF and PLF) loss functions. However, there are some exceptional cases in real life example for some mixing proportion where ELF or PLF performance better especially in case of first parameter but overall SELF is best for GE distribution. Hazard rate of GE is less than the other mixture distributions hazard rate. In future this work can be extended using mixture of truncated GE distribution using different informative and noninformative priors and also by considering mixture of record statistics.

Appendix

$$(simplify(diff(l,(p))=0,(p))) \tag{4}$$

$$(simplify(diff(l,\lambda_1)=0,(\lambda_1))) \tag{5}$$

$$(simplify(diff(l,\lambda_2)=0,(\lambda_2))) \tag{6}$$

For obtaining simplified parameters values use these commands

$$solve(diff(l,(p))=0,(p));$$

$$solve(diff(l,(\lambda_2)=0,(\lambda_2));$$

$$solve(diff(l,(\lambda_1)=0,(\lambda_1));$$

The 2nd derivative of parameters using Maple code as follows

with (linalg);

det (matrix name);

$$(simplify(diff(l,\lambda_1\$2),(\lambda_1))) \tag{8}$$

$$(simplify(diff(l,\lambda_2\$2),(\lambda_2))) \tag{9}$$

$$(simplify(diff(lnL,\lambda_2,\lambda_1),(\lambda_2,\lambda_1))) \tag{10}$$

$$(simplify(diff(lnL,P\$2),(P))) \tag{11}$$

$$(simplify(diff(l,\lambda_1,p),(\lambda_1,p))) \tag{12}$$

$$(simplify(diff(l,\lambda_2,p),(\lambda_2,p))) \tag{13}$$

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