Analysis of Hybrid Censored Competing Risks Data

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Abstract

In this paper we consider analysis of hybrid censored competing risks data, based on Cox's latent failure time model assumptions. It is assumed that lifetime distributions of latent causes of failure follow Weibull distribution with the same shape parameter, but different scale parameters. Maximum likelihood estimators of the unknown parameters can be obtained by solving a one dimensional optimization problem, and we propose a fixed point type algorithm to solve this optimization problem. Approximate maximum likelihood estimators have been proposed based on Taylor series expansion, and they have explicit expressions. Bayesian inference of the unknown parameters are obtained based on the assumption that the shape parameter has a log-concave prior density function, and for the given shape parameter, the scale parameters have Beta-Gamma priors. We propose to use MCMC samples to compute Bayes estimates and also to construct highest posterior density credible intervals. Monte Carlo simulations are performed to investigate the performances of the different estimators, and two data sets have been analyzed for illustrative purposes.

Key Words and Phrases: Maximum likelihood estimators; approximate maximum likelihood estimators; Type-I and Type-II censoring; Fisher information matrix; latent failure rate; Markov Chain Monte Carlo

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1 Introduction

In medical studies or in reliability analysis, an investigator is often interested in the assessment of a specific risk factor in presence of other risks. These risk factors in some sense compete with each other for the failure of the experimental units. Due to this reason, in the statistical literature it is well known as the competing risk problem. Ideally the data which are available from a competing risk problem, consists of a failure time and the associated cause of failure. Different models which are available in the literature to analyze competing risks data are known as the competing risks model. For a basic introduction on competing risks models, readers are referred to the monograph of Crowder [5], where several examples have been provided in which case failures may occur due to more than one cause.

It may be mentioned that analysis of competing risks data can be performed usually by two different methods; (a) latent failure times model as suggested by Cox [4] and (b) cause specific hazard function model as suggested by Prentice [20]. It is observed by Kundu [8] that when life time distributions are exponential or Weibull, the latent failure times model and the cause specific hazard function model lead to the same likelihood function, although the interpretations might be different.

In this paper we use the latent failure time modeling of Cox [4] for analyzing competing risks data. In latent failure time modeling, it is assumed that competing causes of failures are independent random variables. Moreover, here it is assumed that lifetime of the competing causes of failure follow Weibull distribution with the same shape parameter but different scale parameters. Therefore, if T_i denotes the lifetime of the i-th individual, then

$$T_i = \min\{T_{i1}, \cdots, T_{iM}\},$$

where $T_{i1}, \cdots, T_{iM}$ are latent failure times of the M different causes for the i-th individual. According to the latent failure time model assumption, $T_{i1}, \cdots, T_{iM}$ are independently distributed random variables. Moreover $T_{i1}, \cdots, T_{iM}$ are not observable, only T_i is observable, and the indicator J, such that $T_{iJ} = \min\{T_{i1}, \cdots, T_{iM}\}$ is observable. Moreover, T_{ij} has the following probability density function (PDF)

$$f(t; \alpha, \lambda_j) = \begin{cases} \alpha \lambda_j e^{-\lambda_j t^\alpha t^{\alpha-1}} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0, \end{cases} \tag{1}$$

here $\alpha > 0$, $\lambda_j > 0$ are the shape and scale parameters respectively of the Weibull distribution, and it will be denoted by $\text{WE}(\alpha, \lambda_j)$. It may be mentioned that the assumption of common shape parameter of the Weibull distribution in case of competing risk model is not very uncommon in the statistical literature, see for example Mukhopadhyay and Basu [15], Rao *et al.* [21], Park and Kulasekera [18] or Kundu and Pradhan [13]. In this paper it is assumed that $M = 2$, mainly for notational convenience, although all the results provided here are valid for general M also.

In any life testing experiment censoring is inevitable. Type-I and Type-II censoring schemes are the two most important censoring schemes. Consider n items/ individuals under observation in an experimental study. In the conventional Type-I censoring scheme, the experiment continues up to a pre-specified time T. Failures that occur after the time point T are not observed. The termination point T is pre-fixed and it is independent of the failure times. By contrast, the conventional Type-II censoring scheme requires the experiment to continue until a pre-specified number of failures $R \leq n$ occur. In this case, only R smallest lifetimes are observed. In Type-I censoring scheme, the number of failures observed is random, and the endpoint of the experiment is fixed, whereas in Type-II censoring the endpoint is random, while the number of failures observed is fixed. Mixture of Type-I and Type-II censoring scheme is known as the hybrid censoring scheme, and it was introduced by Epstein [7]. In hybrid censoring scheme, the experiment is stopped if the experiment reaches the time point T or the R-th item fails, whichever happens earlier. The hybrid censoring scheme introduced by Epstein [7] is also known as Type-I hybrid censoring scheme.

Therefore, in Type-I hybrid censoring scheme the experimental time and the maximum number of failures are pre-fixed. Recently, different hybrid censoring schemes have been proposed in the literature by different authors. For a recent account of different hybrid censoring schemes, readers are referred to the article by Balakrishnan and Kundu [1].

Let us consider the following experiment from Nelson [16]. In this case, n specimens of a superalloy are being tested at a particular strain range and at high temperature. Failure is defined by the occurrence of a surface defect or an interior defect. Therefore, the data obtained from this experiment is a competing risk data, and it is assumed that latent failure time distributions are independently distributed. Failure time of the specimen and the cause of failure are recorded. The test is terminated when a pre-chosen R out of n items fail or a pre-determined time has been reached.

The main focus of this article is to analyze both from the frequentist and Bayesian points of view, the competing risks data when the data are hybrid censored. The analysis is based on Cox's [4] latent failure time model assumptions with two independent causes of failure. It is observed that the maximum likelihood estimators (MLEs) of the unknown parameters do not always exist, and whenever they exist, they cannot be obtained in explicit forms. The MLEs of the unknown parameters can be obtained by solving a one-dimensional optimization problem. A simple fixed point type iterative method can be used to solve this optimization problem.

Since the MLEs do not have explicit forms, we propose to use approximate maximum likelihood estimators (AMLEs), which can be obtained by first order Taylor series expansion, and they have explicit expressions. It is observed that MLEs and AMLEs behave quite similarly. The exact distributions of the MLEs or AMLEs are not possible to obtain, hence construction of exact confidence intervals is quite difficult. The asymptotic distribution of the MLEs can be used to construct approximate confidence intervals of the unknown parameters. Moreover, the MLEs and AMLEs are asymptotically equivalent, hence similar asymptotic confidence intervals can be constructed using AMLEs also. Alternatively, parametric bootstrap can be used quite conveniently to construct confidence intervals based on small sample sizes.

Further, we consider Bayesian inference for the unknown parameters. For Bayesian inference, we need to assume some priors on the unknown parameters. If the common shape parameter α is known, the most convenient but quite general conjugate priors on the scale parameters are the Beta-Gamma priors, as suggested by Pena and Gupta [19], see also Kundu and Pradhan [13] or Kundu and Gupta [12] in this respect. In this case explicit expressions of the Bayes estimators of the scale parameters can be obtained. When the common shape parameter is also unknown, the conjugate prior on α does not exist. In this case following the idea of Berger and Sun [3] or Kundu [10], we assume that the prior on α has the support on $(0, \infty)$ and it has a log-concave density function. Based on the above prior distributions, we obtain the joint posterior distribution function of the unknown parameters. As expected, the Bayes estimates of the shape and scale parameters cannot be obtained in explicit forms. We propose to use Markov Chain Monte Carlo (MCMC) samples to compute the Bayes estimates, and also to construct highest posterior density (HPD) credible intervals. Extensive simulations are performed to compare the performances of the proposed method, and the analysis of two data sets have been performed for illustrative purpose.

The rest of this paper is organized as follows. Model descriptions and prior assumptions are provided in Section 2. MLEs and AMLEs are provided in Section 3. Bayesian inference are provided in Section 4. Simulation results and the analysis of two data sets are presented in Section 5 and in Section 6 respectively. Finally we conclude the paper in Section 7.

2 Notation and Prior Assumptions

2.1 Notation

In this paper we have assumed that there are only two independent causes of failure. We introduce the following notations:

T_{ji} :		latent failure time of the i-th individual under cause j, for j = 1,2.
T_i :	=	$\min\{T_{1i}, T_{2i}\}$.
$t_{i:n}$:		i-th ordered failure time.
T_* :	=	$\min\{t_{R:n}, T\}$.
d :	=	the total number of failures before T_*.
d_i :	=	the number of failures observed due to cause i; i = 1,2.
δ_i :	=	indicator variable denoting the cause of failure of the i-th ordered individual
$\mathcal{D}$:	=	$\{(t_{1n}, \delta_1), \ldots, (t_{dn}, \delta_d)\}$, the observation before the experiment stops.
I_j :	=	$\{t_{i:n}; \delta_i = j\}; j = 1, 2$.
$\|I_j\|$:	=	cardinality of I_j, we assume $\|I_j\| = d_j$.
$GA(\alpha, \lambda)$:		gamma random variable with PDF; $\dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x};\ x > 0$.
$exp(\lambda)$:		exponential random variable with PDF; $\lambda e^{-\lambda x}; x > 0$.
$WE(\alpha, \lambda)$:		Weibull random variable with PDF; $\alpha\lambda x^{\alpha-1} e^{-\lambda x^\alpha};\ x > 0$.
$Bin(N, p)$:		Binomial random variable with probability mass function $\binom{N}{i} p^i (1-p)^{N-i}$.
$Beta(a, b)$:		Beta random variable with PDF; $\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1};\ 0 < p < 1$.

In this paper it is assumed that $\{(T_{1i}, T_{2i}); i = 1, \cdots, n\}$ are n independent and identically distributed (*i.i.d.*) random variables. It is also assumed that T_{1i} and T_{2i} are independent for all $i = 1, \cdots, n$, moreover, T_{1i} and T_{2i} follow $WE(\alpha, \lambda_1)$ and $WE(\alpha, \lambda_2)$, respectively. In presence of hybrid censoring, we have the following observations

$$\begin{aligned} \text{Case I}: \quad & (t_{1:n}, \delta_1), \cdots, (t_{R:n}, \delta_R), \text{ if } \ t_{R:n} < T \\ \text{Case II}: \quad & (t_{1:n}, \delta_1), \cdots, (t_{J:n}, \delta_J), \text{ if } \ t_{J:n} < T < t_{J+1:n}. \end{aligned}$$

Here J denotes the number of observed failures up to time point T for Case II. For Case I, the experiment stops at $t_{R:n}$ and for Case II it stops at T. For Case II, it is clear that $t_{J+1:n} \cdots < t_{R:n}$ are not observed. For Case I, the experiment stops at $t_{R:n}$, and for Case II, it stops at T. Therefore, $d = R$, for Case I and $d = J$ for Case II.

2.2 Prior Assumption

We make the prior assumptions on the scale parameters and the common shape parameter similarly as in Kundu and Pradhan [13]. Let us denote $\lambda = \lambda_1 + \lambda_2$, then similarly as in Pena and Gupta [19], it is assumed that $\lambda \sim GA(a_0, b_0)$, with $a_0 > 0, b_0 > 0$, and $p = \lambda_1/(\lambda_1 + \lambda_2) \sim Beta(a_1, b_1)$ with $a_1 > 0, b_1 > 0$, and they are independently distributed. After simplification, it can be easily seen, see Pena and Gupta [19] that the joint PDF of λ_1 and λ_2 takes the following form;

$$\begin{aligned} \pi(\lambda_1, \lambda_2 | a_0, b_0, a_1, b_1) = \quad & \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)} (b_0 \lambda)^{a_0 - a_1 - a_2} \times \frac{b_0^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1 - 1} e^{-b_0 \lambda_1} \\ & \times \frac{b_0^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2 - 1} e^{-b_0 \lambda_2}. \end{aligned} \quad (2)$$

The joint PDF (2) is known as the PDF of the Beta-gamma distribution, and it will be denoted by $\text{BG}(b_0, a_0, a_1, a_2)$.

It is a very flexible prior on the scale parameters. First of all it is a conjugate prior if the shape parameter is known. The correlation between λ_1 and λ_2 can be both positive and negative, depending on the values of a_0, a_1 and a_2. If $a_0 = a_1 + a_2$, the prior distributions of

λ_1 and λ_2 become independent. The following result will be useful for further development, whose proof can be easily obtained from Theorem 2 of Pena and Gupta [19].

RESULT: If $(\lambda_1, \lambda_2) \sim \text{BD}(b_0, a_0, a_1, a_2)$, then for $i = 1, 2$,

$$E(\lambda_i) = \frac{a_0 a_i}{b_0(a_1 + a_2)} \quad \text{and} \quad V(\lambda_i) = \frac{a_0 a_i}{b_0^2(a_1 + a_2)} \times \left\{ \frac{(a_i + 1)(a_0 + 1)}{a_1 + a_2 + 1} - \frac{a_0 a_i}{a_1 + a_2} \right\}. \tag{3}$$

Moreover, the generation from a Beta-gamma distribution can be performed very efficiently, see for example Kundu and Pradhan [13], who had suggested a very simple procedure to generate samples from a Beta-Gamma distribution.

3 MLEs and AMLEs

3.1 MLEs

Note that the likelihood contribution of the observation $(t, 1)$ and $(t, 2)$ become

$$\alpha \lambda_1 t^{\alpha - 1} e^{-(\lambda_1 + \lambda_2) t^\alpha} \quad \text{and} \quad \alpha \lambda_2 t^{\alpha - 1} e^{-(\lambda_1 + \lambda_2) t^\alpha},$$

respectively. Similarly, the likelihood contribution of any censored observation is $e^{-(\lambda_1+\lambda_2)T_*^\alpha}$. Therefore, based on the observations $\mathcal{D}$, the likelihood function can be written as

$$L(\mathcal{D}|\alpha, \lambda_1, \lambda_2) \propto \alpha^d \lambda_1^{d_1} \lambda_2^{d_2} \prod_{i=1}^{d} t_{i:n}^{\alpha - 1} \times e^{-(\lambda_1 + \lambda_2) w_2(\alpha)}, \tag{4}$$

where $w_1 = \sum_{i=1}^{d} \ln t_{i:n}$. Hence, the log-likelihood function without the additive constant becomes:

$$l(\alpha, \lambda_1, \lambda_2 | \mathcal{D}) = d \ln \alpha + d_1 \ln \lambda_1 + d_2 \ln \lambda_2 + (\alpha - 1) w_1 - (\lambda_1 + \lambda_2) w_2(\alpha), \tag{5}$$

where $w_2(\alpha) = \sum_{i=1}^{d} t_{i:n}^\alpha + (n - d) T_*^\alpha$. The MLEs of the unknown parameters can be obtained by maximizing the log-likelihood function (5) with respect to the unknown parameters. It is clear that if $d_1 = 0$ or $d_2 = 0$, the MLEs of the unknown parameters do not exist. Therefore, it is assumed that $d_1 > 0$ and $d_2 > 0$. Taking derivatives of (5) with respect to λ_1 and λ_2 and equating them to zero, we obtain the MLEs of λ_1 and λ_2 for fixed α as

$$\widehat{\lambda}_1(\alpha) = \frac{d_1}{w_2(\alpha)} \quad \text{and} \quad \widehat{\lambda}_2(\alpha) = \frac{d_2}{w_2(\alpha)}. \tag{6}$$

Now replacing $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$ in α, we obtain the profile log-likelihood function of α without the additive constant, as

$$p(\alpha) = l(\alpha, \widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha)) = d \ln \alpha - d \ln w_2(\alpha) + \alpha w_1, \tag{7}$$

and the MLE of α can be obtained by maximizing (7) with respect to α. The following result will be helpful for further development.

LEMMA 1: $p(\alpha)$ is an unimodal function.

PROOF: See in the appendix. ∎

From Lemma 1, it is clear that the MLE of α can be obtained by maximizing $p(\alpha)$ with respect to α, and it is unique. For a given α, $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$ maximize (5), and they are unique, therefore, for $d_1 > 0$ and $d_2 > 0$, the MLEs exist and they are unique. Since, $p(\alpha)$ is a smooth differentiable function, the MLE of α can be obtained by solving the equation $p'(\alpha) = 0$. If $\widehat{\alpha}$ is the MLE of α, then it can be obtained as the fixed point solution of the following equation

$$\alpha = h(\alpha) = \frac{d w_2(\alpha)}{d w_2'(\alpha) - w_1 w_2(\alpha)}. \tag{8}$$

A very simple iterative procedure similarly as in Pareek *et al.* [17] can be used to compute $\widehat{\alpha}$. Start with an initial guess of α, say $\alpha^{(0)}$, and at the k-th stage compute $\alpha^{(k+1)} = h(\alpha^{(k)})$. The iteration continues until the convergence takes place, say for example $|\alpha^{(k+1)} - \alpha^{(k)}| < \epsilon$, for some pre-assigned $\epsilon > 0$. Once $\widehat{\alpha}$ is obtained, the MLEs of λ_1 and λ_2 can be obtained as $\widehat{\lambda}_1 = \widehat{\lambda}_1(\widehat{\alpha})$ and $\widehat{\lambda}_2 = \widehat{\lambda}_2(\widehat{\alpha})$ respectively.

3.2 AMLEs

It is observed that the MLEs cannot be obtained in explicit forms. We propose the approximate maximum likelihood estimators (AMLEs), which can be obtained in explicit forms. If the random variable $T \sim WE(\alpha, \lambda)$, then $Y = \ln T$ has the extreme value distribution with the PDF

$$f_Y(y; \mu, \sigma) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma} - e^{\frac{y-\mu}{\sigma}}}; \quad -\infty < y < \infty, \tag{9}$$

where $\mu = -\dfrac{\ln \lambda}{\sigma}$ and $\sigma = \dfrac{1}{\alpha}$. The PDF (9) is known as the PDF of an extreme value distribution with location and scale parameters as μ and σ, respectively. In this case it has been assumed that $T_{1i} \sim WE(\alpha, \lambda_1)$, $T_{2i} \sim WE(\alpha, \lambda_2)$ and they are independently distributed. The main idea of the AMLEs is to expand the normal equations by first order Taylor series expansion, hence obtain the solutions of the normal equations in explicit forms. It has been observed by several authors, see for example Balasooriya and Balakrishnan [2] or Kundu [9] and see the references cited therein, that in case of Weibull lifetime distribution, this approximation works very well.

We use the following notations to define the AMLEs; $p_i = \dfrac{i}{n+1}$, $q_i = 1 - p_i$ for $i = 1, \cdots, n$, $p_j^* = \dfrac{p_j + p_{j+1}}{2}$, $q_j^* = 1 - p_j^*$, for $j = 1, \cdots, n-1$. Further, $\alpha_i = 1 + \ln q_i (1 - \ln(-\ln q_i))$ and $\beta_i = -\ln q_i$ for $i = 1, \cdots, d$.

Now ignoring the cause of failures, and following the same approach as Balasooriya and Balakrishnan [2], the AMLEs of μ and σ can be obtained as follow;

$$\widetilde{\mu} = A - B\widetilde{\sigma}, \tag{10}$$

where $\widetilde{\sigma}$ can be obtained as the positive solution of

$$d\sigma^2 + F\sigma - G = 0. \tag{11}$$

Here for Case I

$$A = \frac{\sum_{i=1}^{d} \beta_i t_{i:n} + (n-d)\beta_d t_{d:n}}{\sum_{i=1}^{d} \beta_i + (n-d)\beta_d} \quad \text{and} \quad B = \frac{\sum_{i=1}^{d} \alpha_i - (n-d)(1-\alpha_d)}{\sum_{i=1}^{d} \beta_i + (n-d)\beta_d},$$

$$\begin{aligned} F &= \sum_{i=1}^{d} \alpha_i (t_{i:n} - A) - (n-d)(1-\alpha_d)(t_{d:n} - A) - 2B \sum_{i=1}^{d} \beta_i (t_{i:n} - A) \\ &\quad -2B(n-d)\beta_d (t_{d:n} - A) \\ G &= \sum_{i=1}^{d} \beta_i (t_{i:n} - A)^2 + (n-d)\beta_d (t_{d:n} - A)^2 \geq 0. \end{aligned}$$

For Case II

$$A = \frac{\sum_{i=1}^{d} \beta_i t_{i:n} + (n-d)\beta_d t_{d:n}}{\sum_{i=1}^{d} \beta_i + (n-d)\beta_d} \quad \text{and} \quad B = \frac{\sum_{i=1}^{d} \alpha_i - (n-d)(1-\alpha_d)}{\sum_{i=1}^{d} \beta_i + (n-d)\beta_d},$$

$$\begin{aligned} F &= \sum_{i=1}^{d} \alpha_i (t_{i:n} - A) - (n-d)(1-\alpha_d)(t_{d:n} - A) - 2B \sum_{i=1}^{d} \beta_i (t_{i:n} - A) \\ &\quad -2B(n-d)\beta_d (t_{d:n} - A) \\ G &= \sum_{i=1}^{d} \beta_i (t_{i:n} - A)^2 + (n-d)\beta_d (t_{d:n} - A)^2 \geq 0. \end{aligned}$$

In both cases $\widetilde{\sigma} = (-F + \sqrt{F^2 + 4dG})/(2d)$ is the only positive root. Once we obtain $\widetilde{\sigma}$ we can obtain $\widetilde{\mu}$ using (10). Finally we can obtain the AMLEs of α, λ_1 and λ_2 as

$$\widetilde{\alpha} = \frac{1}{\widetilde{\sigma}}, \quad \widetilde{\lambda}_1 = e^{-\widetilde{\mu}_1/\widetilde{\sigma}}, \quad \widetilde{\lambda}_2 = e^{-\widetilde{\mu}_2/\widetilde{\sigma}}, \tag{12}$$

where

$$\widetilde{\mu}_1 = \widetilde{\mu} + \widetilde{\sigma} \ln(1 + d_2/d_1), \quad \text{and} \quad \widetilde{\mu}_2 = \widetilde{\mu} + \widetilde{\sigma} \ln(1 + d_1/d_2).$$

It is not possible to obtain the exact distributions of the MLEs or AMLEs. Therefore, constructing the exact confidence intervals may not be possible. Due to this reason we have used the observed Fisher information matrix to construct confidence intervals of the unknown parameters.

4 Bayesian Inference

In this section we provide the Bayesian inference of the unknown parameters. First we obtain the Bayes estimates and the associated credible set of the scale parameters, when the common shape parameter is known. Then we consider the case when the common shape parameter is also unknown. In developing the Bayes estimates, we have assumed the squared error loss function, although any other loss function also can be easily incorporated.

4.1 Common Shape Parameter Known

Based on the observed sample, the likelihood function has been provided in (4). Therefore, for known α, when (λ_1, λ_2) has the joint prior $BG(b_0, a_0, a_1, a_2)$, it can be easily observed that the joint posterior of (λ_1, λ_2) given the *data* is as follows;

$$\pi(\lambda_1, \lambda_2 | data, \alpha) \sim BG(b_0 + w_2(\alpha), a_0 + d_1 + d_2, a_1 + d_1, a_2 + d_2). \tag{13}$$

Since, with respect to the squared error loss function, the Bayes estimates of λ_1 and λ_2 are posterior means, therefore, using (3), they are as follows:

$$\begin{aligned} E(\lambda_1) &= \widehat{\lambda}_{1B} = \frac{(a_0 + d_1 + d_2)(a_1 + d_1)}{(b_0 + w_2(\alpha))(a_1 + a_2 + d_1 + d_2)} \\ E(\lambda_2) &= \widehat{\lambda}_{2B} = \frac{(a_0 + d_1 + d_2)(a_2 + d_2)}{(b_0 + w_2(\alpha))(a_1 + a_2 + d_1 + d_2)}. \end{aligned} \tag{14}$$

The posterior variances are of the following form:

$$V(\lambda_1) = A_1 \times B_1 \quad \text{and} \quad V(\lambda_2) = A_2 \times B_2, \tag{15}$$

where

$$\begin{aligned} A_1 &= \frac{(a_0 + d_1 + d_2)(a_1 + d_1)}{(b_0 + w_2(\alpha))^2(a_1 + a_2 + d_1 + d_2)} \\ A_2 &= \frac{(a_0 + d_1 + d_2)(a_2 + d_2)}{(b_0 + w_2(\alpha))^2(a_1 + a_2 + d_1 + d_2)} \end{aligned}$$

$$
\begin{aligned}
B_1 &= \frac{(a_1+d_1+1)(a_0+d_1+d_2+1)}{a_1+a_2+d_1+d_2+1} - \frac{(a_0+d_1+d_2)(a_1+d_1)}{a_1+a_2+d_1+d_2} \\
B_2 &= \frac{(a_2+d_2+1)(a_0+d_1+d_2+1)}{a_1+a_2+d_1+d_2+1} - \frac{(a_0+d_1+d_2)(a_2+d_2)}{a_1+a_2+d_1+d_2}.
\end{aligned}
$$

Under the assumptions of non-informative priors, *i.e.* $a_0 = b_0 = a_1 = a_2 = 0$, the Bayes estimates of λ_1 and λ_2 are

$$\widehat{\lambda}_{1B} = \frac{d_1}{w_2(\alpha)} \quad \text{and} \quad \widehat{\lambda}_{2B} = \frac{d_2}{w_2(\alpha)}, \tag{16}$$

and they coincide with the corresponding MLEs.

Although, when the shape parameter is known the Bayes estimates can be obtained in explicit form, the HPD credible intervals of the unknown parameters cannot be obtained explicitly. One way to construct the HPD credible intervals is by direct sampling from the joint posterior density function as it was suggested by Kundu and Pradhan [13]. Using the method suggested by Kundu and Pradhan [13], the samples from the posterior distribution function can be easily generated, and using those generated samples, HPD credible intervals can be easily constructed.

Now we describe how to construct a 100(1-γ)% credible set of (λ_1, λ_2) for known α. Note that a set $C_{\alpha,1-\gamma}$ is said to be a 100(1-γ)% credible set of (λ_1, λ_2), if

$$P((\lambda_1, \lambda_2) \in C_{\alpha,1-\gamma}) = 1 - \gamma, \tag{17}$$

when $(\lambda_1, \lambda_2) \sim \pi(\lambda_1, \lambda_2 | data, \alpha)$. We need the following lemma for further development;

Lemma 2: If

$$(X, Y) \sim BG(b_0 + w_2(\alpha), a_0 + d_1 + d_2, a_1 + d_1, a_2 + d_2),$$

then

$$X + Y \sim GA(a_0 + d_1 + d_2, b_0 + w_2(\alpha), \quad \frac{X}{X+Y} \sim Beta(a_1 + d_1, a_2 + d_2)$$

and they are independently distributed.

PROOF: It can be obtained by simple transformation technique. ∎

Now using Lemma 2, $C_{\alpha,1-\gamma}$ can be constructed as follows:

$$C_{\alpha,1-\gamma} = \{(\lambda_1, \lambda_2); \lambda_1 > 0, \lambda_2 > 0, A \leq \lambda_1 + \lambda_2 \leq B, C \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq D\}. \tag{18}$$

Here A, B, C and D are such that

$$P(A < U < B) = 1 - \gamma_1 \quad \text{and} \quad P(C < V < D) = 1 - \gamma_2, \tag{19}$$

$U \sim GA(a_0+d_1+d_2, b_0+w_2(\alpha))$ and $V \sim Beta(a_1+d_1, a_2+d_2)$, are independently distributed. Moreover, γ_1 and γ_2 are two constants such that $1 - \gamma = (1 - \gamma_1)(1 - \gamma_2)$.

It can be easily seen that $C_{\alpha,1-\gamma}$ is a trapezoid enclosed by the following four straight lines

$$(i)\lambda_1 + \lambda_2 = A, \quad (ii)\lambda_1 + \lambda_2 = B, \quad (iii)\lambda_1(1 - D) = \lambda_2 D, \quad (iv)\lambda_1(1 - C) = \lambda_2 C \tag{20}$$

Simple calculation shows that the area of $C_{\alpha,1-\gamma}$ is $(B^2 - A^2)(D - C)/2$. One possible way to choose the hyper parameters might be to choose them so that the area of the posterior credible range becomes minimum. It becomes a purely numerical problem, it has not been pursued further in this paper.

4.2 Common Shape Parameter Unknown

Now we will discuss the more important case namely when the common shape parameter α is unknown. In this case based on the priors as discussed in Section 2, the joint posterior density function of α, λ_1 and λ_2 can be written as

$$\pi(\alpha, \lambda_1, \lambda_2 | data) = \pi(\lambda_1, \lambda_2 | \alpha, data) \times \pi(\alpha | data), \tag{21}$$

here $\pi(\lambda_1, \lambda_2 | data, \alpha)$ is same as defined in (13), and

$$\pi(\alpha | data) \propto \pi(\alpha) \alpha^d \prod_{i=1}^{d} t_{i:n}^{\alpha} \times \frac{1}{(w_2(\alpha) + b_0)^{a_0+d_1+d_2}}. \tag{22}$$

Therefore, the Bayes estimate of any function α, λ_1 and λ_2 cannot be obtained in explicit form, and we use the following lemma for further development.

LEMMA 3: $\pi(\alpha|data)$ is log-concave.

PROOF: Proof can be similarly as the proof of Lemma 2 of Kundu [10] ∎.

Although, the Bayes estimate cannot be obtained in explicit form, using Lemma 2 and the posterior distribution of λ_1 and λ_2 given α, *i.e.* using (13) it is possible to generate samples from the joint posterior PDF of $\pi(\lambda_1, \lambda_2, \alpha|data)$ and that can be used to compute Bayes estimates of the unknown parameters and also to construct associated credible intervals.

Note that Devroye [6] proposed a method to generate samples from a distribution function with any log-concave PDF. Moreover, the approximation of a log-concave PDF with two-parameter gamma PDF as suggested by Kundu [10] can also be used to generate MCMC samples from $\pi(\alpha|data)$. It is observed in extensive simulation experiments that the later method works very well to generate samples from log-concave PDF. Once the samples from $\pi(\alpha|data)$ are drawn, the generation from $\pi(\lambda_1, \lambda_2|data, \alpha)$, *i.e.* from a Beta-Gamma distribution can be performed as suggested by Kundu and Pradhan [13]. The following algorithm can be used to compute the Bayes estimate of any function of λ_1, λ_2 and α, say $g(\alpha, \lambda_1, \lambda_2)$, and the associated highest posterior density (HPD) credible interval.

ALGORITHM:

- Step 1: Generate α from $\pi(\alpha|data)$ as given in (22).
- Step 2: For a given α, generate λ_1 and λ_2 from $\pi(\lambda_1, \lambda_2|data, \alpha)$ as given in (13).
- Step 3: Repeat Step 1 and Step 2, to generate $(\alpha_1, \lambda_{11}, \lambda_{21}), \cdots, (\alpha_M, \lambda_{1M}, \lambda_{2M})$.
- Step 4: The Bayes estimate of $g(\alpha, \lambda_1, \lambda_2)$ and the corresponding posterior variance

can be obtained as

$$\widehat{g}(\alpha, \lambda_1, \lambda_2) = \frac{1}{M} \sum_{i=1}^{M} g(\alpha_i, \lambda_{1i}, \lambda_{2i}), \quad \text{and}$$

$$\widehat{V}(g(\alpha, \lambda_1, \lambda_2)) = \frac{1}{M} \sum_{i=1}^{M} \left(g(\alpha_i, \lambda_{1i}, \lambda_{2i}) - \widehat{g}(\alpha, \lambda_1, \lambda_2) \right)^2,$$

respectively.

- Step 5: To construct the HPD credible of $g(\alpha, \lambda_1, \lambda_2)$, first order g_i as $g_{(1)} < g_{(2)} < \cdots < g_{(M)}$, where $g_i = g(\alpha_i, \lambda_{1i}, \lambda_{2i})$. Then 100(1-2$\beta$)% credible interval of $g(\alpha, \lambda_1, \lambda_2)$ becomes

  $$(g_{(j)}, g_{(j+M-2\beta)}), \quad \text{for} j = 1, \cdots, 2M\beta.$$

  Therefore, 100(1-2β)% HPD credible interval becomes $(g_{(j^*)}, g_{(j^*+M-2\beta)})$, where j^* is such that

  $$g_{(j^*+M-2\beta)} - g_{(j^*)} \leq g_{(j+M-2\beta)} - g_{(j)}$$

  for all $j = 1, \cdots, 2M\beta$.

We have already mentioned how a credible set of (λ_1, λ_2) for a given α can be constructed. Similarly, it is possible to construct a 100(1-γ)% credible set of $(\lambda_1, \lambda_2, \alpha)$. Note that a set $S_{1-\gamma}$ is said to be a 100(1-γ)% credible set of $(\lambda_1, \lambda_2, \alpha)$ if

$$P((\lambda_1, \lambda_2, \alpha) \in S_{1-\gamma}) = 1 - \gamma,$$

where $(\lambda_1, \lambda_2, \alpha) \sim \pi(\lambda_1, \lambda_2, \alpha | data)$. Now choose β and δ such that $(1-\beta) \times (1-\delta) = (1-\gamma)$. Then a 100(1-$\gamma$)% credible interval of $(\lambda_1, \lambda_2, \alpha)$ can be obtained as follows:

$$S_{1-\gamma} = (\alpha_L, \alpha_U) \times C_{\alpha, 1-\delta},$$

here α_L and α_U are such that,

$$\int_{\alpha_L}^{\alpha_U} \pi(\alpha | data) d\alpha = 1 - \beta,$$

and $C_{\alpha, 1-\delta}$ is same as defined in Section 4.1.

5 Simulation Study

In this section we present the results of simulation study to show how the different methods perform for different sampling schemes. For the simulation study we have considered $\alpha = 2$, $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$. We have taken different sampling schemes, *i.e.* different (n, R, T), namely; Scheme 1: (25, 15, 1.5), Scheme 2: (25, 20, 2), Scheme 3: (50, 30, 1.5), Scheme IV: (50, 40, 2). The main idea to take different schemes is to see how the MLEs and Bayes estimators perform for different n, R and T.

For a particular choice of n, R and T, competing risk hybrid censored data are generated. It may be mentioned that the generation from a hybrid censored competing risk data is quite straight forward. First we generate hybrid censored Weibull samples with shape and scale parameters as α and λ, respectively as in Kundu [9]. Then with each failure time, with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2}$ and $\frac{\lambda_1}{\lambda_1 + \lambda_2}$ cause 1 or cause 2 are assigned.

Based on a particular set of observations, we compute the MLEs, AMLEs and Bayes estimates of α, λ_1 and λ_2. To compute the MLEs we have used the iterative method suggested in (8). In each case we have started the iteration process with the initial guess of α as the true value of α and the iteration stops whenever the absolute difference of two consecutive iterates is less than 10^{-5}. Both non-informative and informative priors are considered to compute the Bayes estimates mainly to see the effect of the hyper parameters on the estimators. For non-informative priors, $a = b = a_0 = b_0 = a_1 = a_2 = 0$, and for informative priors, hyper parameters are taken as follows: $a = 5$, $b = 2.5$, $a_0 = b_0 = 2$, $a_1 = 0.4$ and $a_2 = 0.6$. The hyper parameter are chosen such that the prior expectations match with the corresponding true parameter values.

The Bayes estimators are computed by direct sampling method (See Section 4.2) based on 10000 samples. Based on 1000 replications, we compute average estimates and the associated

MSEs. The different estimates are reported under MLE, AMLE, BayesNI (Bayes estimates under non-informative prior) and BayesI (Bayes estimates under informative prior). The MSEs are reported in the parentheses along with the average estimates. All the results are reported in Table 1.

Some of the points are quite clear from the simulation experiments. Comparing Schemes 1,2 and Schemes 3,4 it is clear that for fixed n as T or R increases the biases and the MSEs of all the estimators decrease. Since as sample size increases the biases and MSEs decrease in all cases. The performances of the MLEs and AMLEs are very similar. Both these estimators behave very similarly as the Bayes estimators with non-informative priors. In this case Bayesian estimators under informative prior perform better than the corresponding MLEs, AMLEs and BayesNIs, as expected. The choice of the hyper parameters is an important issue, it is not pursued here further. It needs more attention.

Table 1: Estimated values & MSEs for $\alpha = 2, \lambda_1 = 0.4, \lambda_2 = 0.6$

Scheme	Estimator	(n, R, T)	α	λ_1	λ_2
	MLE	(25, 15, 1.5)	2.2724 (0.2905)	0.4755 (0.0659)	0.7099(0.1127)
I	AMLE	(25, 15, 1.5)	2.2510(0.2915)	0.4772(0.0659)	0.7124(0.1269)
	BayesNI	(25, 15, 1.5)	2.2049(0.2814)	0.4671(0.0577)	0.6961(0.1018)
	BayesI	(25, 15, 1.5)	2.1328(0.1685)	0.4402(0.0311)	0.6561(0.0488)
	MLE	(25, 20, 2)	2.1549 (0.2149)	0.4255(0.0296)	0.6378(0.0460)
II	AMLE	(25,20, 2)	2.1371(0.2062)	0.4300(0.0303)	0.6445(0.0470)
	BayesNI	(25, 20, 2)	2.1314(0.1937)	0.4231(0.0289)	0.6341(0.0445)
	BayesI	(25, 20, 2)	2.0953(0.1373)	0.4179(0.0231)	0.6264(0.0337)
	MLE	(50,30, 1.5)	2.1022(0.1393)	0.4232(0.0178)	0.6470(0.0330)
III	AMLE	(50,30,1.5)	2.0922(0.1361)	0.4241(0.0178)	0.6483(0.0331)
	BayesNI	(50, 30,1.5)	2.0897(0.1465)	0.4222(0.0178)	0.6455(0.0335)
	BayesI	(50,30,1.5)	2.0666(0.1128)	0.4166(0.0146)	0.6364(0.0262)
	MLE	(50,40, 2)	2.0687(0.0843)	0.4163(0.0119)	0.6162(0.0174)
IV	AMLE	(50,40, 2)	2.0610(0.0830)	0.4184(0.0120)	0.6194(0.0175)
	BayesNI	(50,40, 2)	2.0618(0.1041)	0.4150(0.0118)	0.6141(0.0172)
	BayesI	(50,40, 2)	2.0537(0.0901)	0.4137(0.0107)	0.6127(0.0155)

6 Illustrative Data Analysis

6.1 Simulated Data Set:

Here we present, the analysis of one simulated data set. We generate hybrid censored competing risks data for $\alpha = 2$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, n =50, $R = 30$ and $T = 1.5$. The generated observations are presented in Table 2.

Table 2: Generated hybrid censored competing risk data with $\alpha = 2$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, n =50, $R = 30$ and $T = 1.5$.

(0.055660, 0)	(0.198134, 1)	(0.205648, 0)	(0.353072, 0)	(0.382692, 1)
(0.386015, 0)	(0.404080, 1)	(0.409272, 0)	(0.421276, 0)	(0.462808, 0)
(0.466898, 0)	(0.474780, 0)	(0.500283, 0)	(0.521199, 0)	(0.552813, 0)
(0.567324, 0)	(0.573163, 1)	(0.606657, 0)	(0.675424, 0)	(0.684682, 1)
(0.686781, 0)	(0.699642, 1)	(0.716803, 1)	(0.866542, 1)	(0.874578, 0)
(0.901971, 1)	(0.923157, 1)	(0.942409, 1)	(0.954552, 0)	(1.077902, 0)

To compute the MLE of α, first we plot the profile log-likelihood function of α, (7) in Figure 1, which helps us to provide an initial estimate of α. We start the iterative process with the initial estimate of α as 1.5, and use the same stopping criterion as it has been used in the previous section. We compute the MLEs of λ_1 and λ_2, and also obtain the associated 95% confidence intervals based on the observed Fisher information matrix.

For this data set, we compute MLEs, AMLEs, BayesNI and BayesI. For the computation of Bayes estimates, we have considered both the informative and non-informative priors. For informative priors, hyper parameter values are taken as follows: $a = 5$, $b = 2.5$, $a_0 = b_0 = 2$, $a_1 = 0.4$ and $a_2 = 0.6$. 10,000 posterior samples are generated for both the cases. The posterior PDFs of α and the histogram of the generated samples obtained using gamma approximation are provided in Figure 2 and Figure 3 respectively. The approximation works very well. We compute different estimates along with confidence/credible intervals and the

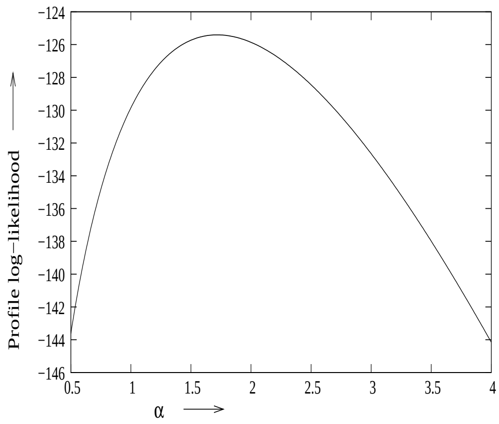

Figure 1: Profile log-likelihood function of α for the simulated data set

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results are reported in Table 3. Assuming $\alpha = 2$, the joint 95% credible region of (λ_1, λ_2) is provided in Figure 4.

Table 3: Estimated values along with confidence/credible intervals for the simulated data in Example 1

Estimator	α	λ_1	λ_2
MLE	1.85197	0.40113	0.60170
	[1.25429, 2.44965]	[0.17031, 0.63196]	[0.31666, 0.88674]
AMLE	1.84570	0.40223	0.60334
	[1.30665, 2.66222]	[0.18270, 0.75112]	[0.33651, 1.50425]
BayesNI	1.87291	0.48827	0.55611
	[1.17333, 2.58552]	[0.23804, 0.75697]	[0.29271, 0.84927]
BayesI	1.88826	0.48384	0.55928
	[1.24160, 2.63599]	[0.25311, 0.75129]	[0.30296, 0.84398]

6.2 Real Data Set:

In this section we present the analysis of a real hybrid data set for illustrative purposes. The data under consideration was from an experiment in which small electrical appliances were being tested, and it has been taken from Lawless [14]. The appliances were operated

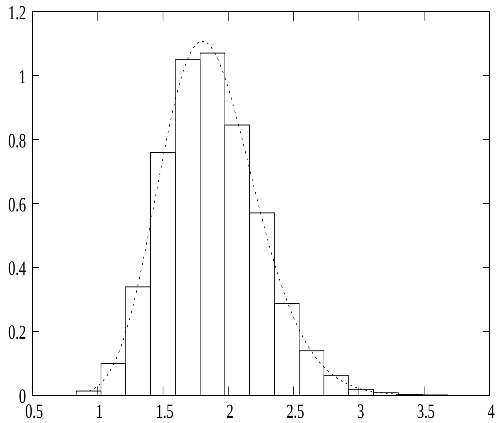

Figure 2: Posterior PDF and histogram of the generated samples of α from the approximate gamma PDF, for the simulated data set in case of non-informative priors.

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repeatedly by an automatic testing machine; the lifetimes given were the number of cycles of use until the appliances failed. Total 36 appliances were used, and there were 18 different modes according to which the appliance could have failed. Failure due to 9^{th} mode was considered as cause 1 and the remaining as cause 0. The complete data set has already been analyzed by Kundu and Basu [11] using Weibull latent failure time distributions with equal shape parameter, and the performance was quite satisfactory. We have created a hybrid data set with the following sampling scheme $T = 3000$ and $R = 25$. The hybrid data set is presented below: (11, 0), (35, 0), (49, 0), (170, 0), (329, 0), (381, 0), (708, 0), (958, 0), (1062, 0), (1167, 1), (1594, 0), (1925, 1), (1990, 1), (2223, 1), (2327, 0), (2400, 1), (2451, 0), (2471, 1), (2551, 1), (2565, 0), (2568, 1), (2694, 1), (2702, 0), (2761, 0), (2831, 0).

In this case $d_1 = 16$, $d_2 = 9$, $d = 25$, $T_* = 2831$. Before progressing further, first we would like to check whether the assumption of the equality of the shape parameters of the two Weibull distributions is reasonable or not. It is assumed that T_{1i} and T_{2i} follow $WE(\alpha_1, \lambda_1)$ and $WE(\alpha_2, \lambda_2)$ respectively. We perform the following testing procedure

$$H_0 : \alpha_1 = \alpha_2 \quad vs. \quad H_1 : \alpha_1 \neq \alpha_2.$$

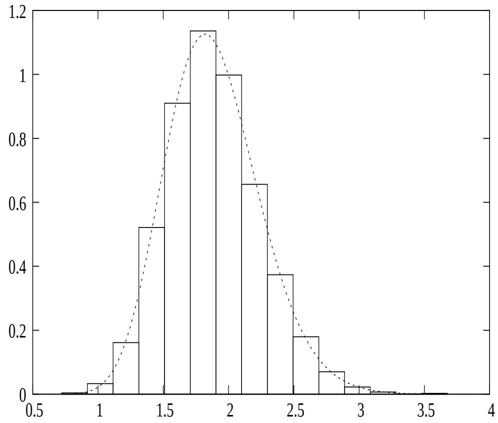

Figure 3: Posterior PDF and histogram of the generated samples of α from the approximate gamma PDF, for the simulated data set in case of informative priors.

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It is observed -2$(L_0 - L_1) = 2.753$, where L_0 and L_1 are the maximum log-likelihood values under H_0 and H_1, respectively. In this case, the associated $p-$value is greater than 0.05, and hence we cannot reject the null hypothesis. Therefore, the assumption of equality of the shape parameters is justified in this case.

Based on the above data, we compute MLEs and AMLEs of the unknown parameters, along with the associated 95% confidence intervals. The profile log-likelihood of α is provided in Figure 5. Since we do not have any prior informations on the unknown parameters, we have assumed non-informative priors, and based on the non-informative priors, we compute Bayes estimates and the associated HPD credible intervals. For computation of Bayes estimates, we have generated 10000 observations from the posterior distribution by direct sampling method as before and the posterior PDF and the histogram of the generated samples based on approximate gamma distribution is provided in Figure 7. The results are reported in Table 4. Based on the assumption that $\alpha = 1$, we have provided a 95% credible set of λ_1 and λ_2 in Figure 7.

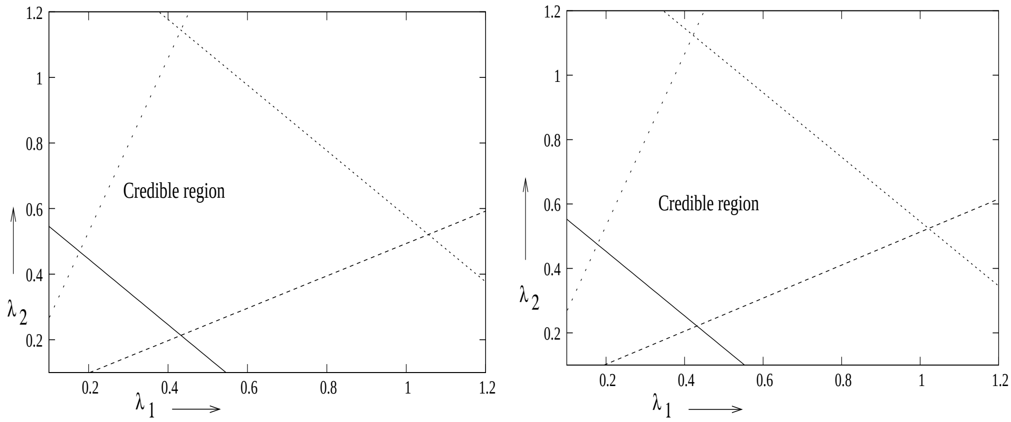

Figure 4: 95% credible set for λ_1 and λ_2 when α is known to be 2 for the simulated data set in case of (a) non-informative and (b) informative priors

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Table 4: Estimated values of parameters along with confidence/credible intervals for electrical appliances data in Example 2

Estimator	α	λ_1	λ_2
MLE	1.04531 [0.66195, 1.42867]	0.00009 [0, 0.00036]	0.00016 [0, 0.00063]
AMLE	1.01633 [0.65162, 2.00470]	0.00011 [0, 0.00170]	0.00020 [0, 0.00387]
BayesNI	0.99570 [0.61103, 1.40595]	0.00037 [0, 0.00149]	0.00066 [0, 0.00263]

7 Conclusions

In this paper we have considered the frequentist and Bayesian inference of the unknown parameters of the competing risk model based on the hybrid censored data. The analysis is based on the Cox's latent failure time model assumptions, when the latent failure time distributions follow Weibull distribution with the same shape parameters but different scale parameters. In this case also, it may be observed similarly as in Kundu [8] that the latent failure time model and the cause specific hazard function model lead to the same likelihood function, but their interpretations are different. We have provided the MLEs and Bayes estimates of the unknown parameters. One natural question arises how the estimation can

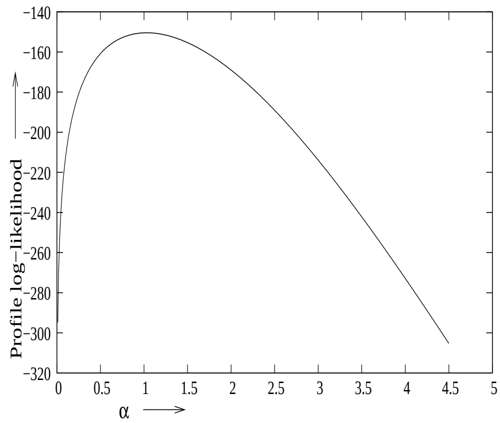

Figure 5: Profile log-likelihood function of α for the electrical data set

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be performed, if the common shape parameters assumption does not hold. The MLEs and AMLEs can be obtained similarly, although Bayes estimates may not be obtained along the same line. More work is needed to develop efficient Bayesian estimation technique in this situation. It may be mentioned that although the results have been provided for the Type-I hybrid censored data, the results may be extended for different other hybrid censored data also. The work is in progress, and it will be reported later.

Acknowledgements:

The authors would like to thank two unknown referees for their valuable suggestions.

Appendix

Proof of Lemma 1: First we will show that $p(\alpha)$ is log-concave. Note that

$$p''(\alpha) = -\frac{d}{\alpha^2} - \frac{w_2(\alpha)w_2''(\alpha) - (w_2'(\alpha))^2}{(w_2(\alpha))^2}.$$

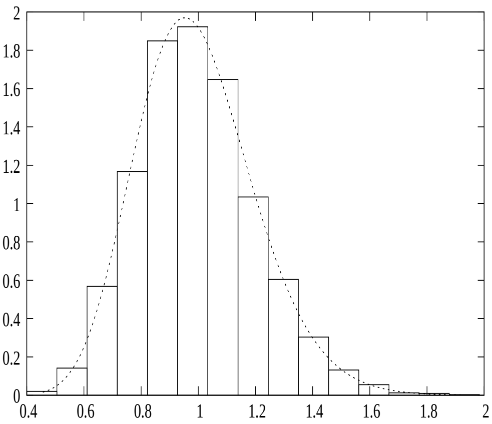

Figure 6: Posterior PDF and histogram of the generated samples of α from the approximate gamma PDF, for the real data set in case of informative priors.

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Let us define for $i = 1, \cdots, d$ $v_i = t_{i:n}^{\alpha/2}$ and $w_i = t_{i:n}^{\alpha/2} \ln t_{i:n}$. For $i = d+1$, $v_i = T_*^{\alpha/2}\sqrt{n-d}$ and $w_i = T_*^{\alpha/2}\sqrt{n-d}\ln T_*$. Therefore, $w_2(\alpha) = \sum_{i=1}^{d+1} v_i^2$, $w_2'(\alpha) = \sum_{i=1}^{d+1} v_i w_i$, and $w_2''(\alpha) = \sum_{i=1}^{d+1} w_i^2$. Applying Cauchy-Schwartz inequality, we get $(w_2(\alpha)w_2''(\alpha) - (w_2'(\alpha))^2)/((w_2(\alpha))^2) \geq 0$. It implies that $p_2(\alpha)$ is log-concave. Now the result follows by observing the fact that $\lim_{\alpha \to 0^+} p(\alpha) = \lim_{\alpha \to \infty} p(\alpha) = -\infty$ ∎

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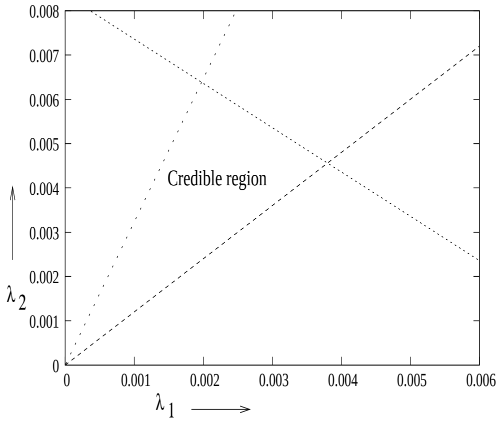

Figure 7: 95% credible set for λ_1 and λ_2 when α is known to be 1 for the real data set based on non-informative priors

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