



# STEP STRESS MODEL

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## OUTLINE OF THE TALK

- ACCELERATED LIFE TEST
- STEP STRESS TEST
- ACCELERATION MODEL
- CUMULATIVE EXPOSURE MODEL
- PARAMETRIC INFERENCE
- BAYESIAN APPROACH
- OPTIMAL TEST PLAN
- OPEN PROBLEMS



## WHAT IS AN ACCELERATED LIFE TEST

ACCELERATED LIFE TEST IS A POPULAR EXPERIMENTAL STRATEGY TO OBTAIN INFORMATION ON LIFE DISTRIBUTIONS OF HIGHLY RELIABLE PRODUCT. THE MAIN IDEA IS TO SUBMIT MATERIALS TO HIGHER THAN USUAL ENVIRONMENTAL CONDITIONS TO ENSURE EARLY FAILURE. DATA OBTAINED FROM SUCH AN EXPERIMENT NEED TO BE EXTRAPOLATED TO ESTIMATE LIFETIME DISTRIBUTION UNDER NORMAL CONDITIONS.



## STRESS FACTORS

- SINGLE STRESS
- MULTIPLE STRESS

THE CLASSICAL STRESSES ARE

- TEMPERATURE
- VOLTAGE
- CURRENT
- PRESSURE
- LOAD



## ADVANTAGES

- (a) IF YOU INCREASE THE STRESS, REASONABLE NUMBER OF FAILURES ARE ENSURED
- (b) REDUCE THE EXPERIMENTAL TIME



## DISADVANTAGES

- (a) NEED TO KNOW THE EXACT RELATION BETWEEN THE STRESS AND LIFE-TIME
- (b) MODEL MUST TAKE INTO ACCOUNT THE EFFECT OF ACCUMULATION OF STRESS
- (c) MODEL BECOMES MORE COMPLEX
- (d) EVEN IN SIMPLE CASE ANALYSIS BECOMES DIFFICULT



## WHAT IS A STEP-STRESS LIFE TEST?

A STEP-STRESS LIFE TEST IS A PARTICULAR ACCELERATED LIFE-TEST. YOU OBSERVE THE FAILURE TIMES OF THE OBJECTS AT A PARTICULAR STRESS LEVEL, THEN CHANGE THE STRESS LEVEL TO A DIFFERENT LEVEL. OBSERVE THE FAILURE TIMES IN THE NEW STRESS LEVEL, AND THEN THE CHANGE THE STRESS LEVEL AGAIN AND SO ON.



SEVERAL VARIATIONS ARE POSSIBLE DEPENDING ON THE NEED

- (a) THE STRESS CAN BE CHANGED AT THE PRE SPECIFIED TIMING. IT IS KNOWN AS STEP-STRESS WITH TYPE-I CENSORING. IN THIS CASE NUMBER OF FAILURES IS RANDOM.
  
- (b) THE STRESS CAN BE CHANGED AT THE PRE SPECIFIED NUMBER OF FAILURES. IT IS KNOWN AS THE STEP-STRESS WITH TYPE-II CENSORING. IN THIS CASE FAILURE TIME IS RANDOM.



## EXAMPLE

IT IS FROM NELSON (1980). THE TEST IS REGARDING CABLE INSULATION. THE STRESS HERE IS VOLTAGE (KILOVOLTS). THE AIM IS TO ESTIMATE LIFE AT CERTAIN DESIGN STRESS.

Step	1	2	3	4	5	6	7	8	9	10	11
Stress	5	10	15	20	26	28.5	31	33.4	36	38.5	41
Time exposure (min)	10	10	10	10	15	15	15	15	15	15	15
	10	10	10	10	60	60	60	60	60	60	60
	10	10	10	10	240	240	240	240	240	240	240
	10	10	10	10	960	960	960	960	960	960	960



## ACCELERATION MODEL

THE MAIN PROBLEM WITH THE STEP-STRESS TESTING IS TO FIND THE RELATIONSHIP BETWEEN THE LIFETIME IN TEST (STRESS CONDITION) AND LIFETIME IN USED (NORMAL CONDITION). A RELATIONSHIP BETWEEN THE LIFETIMES NEED TO BE ESTABLISHED.



## SOME DEFINITIONS

- $\theta(s)$ : STRESS FUNCTION AT THE STRESS LEVEL  $s$
- $X$  : THE TIME TO FAILURE UNDER STRESS
- $Y$  : THE TIME TO FAILURE UNDER NORMAL CONDITION
- $F_X$ : THE CUMULATIVE DISTRIBUTION FUNCTION OF  $X$
- $F_Y$ : THE CUMULATIVE DISTRIBUTION FUNCTION OF  $Y$
- $\phi_\theta$  : THE ACCELERATION FUNCTION

THE ACCELERATION FUNCTION

$$\phi_\theta : [0, \infty) \rightarrow [0, \infty).$$

AND

$$Y = \phi_\theta(X)$$



## RELATIONS BETWEEN $X$ AND $Y$

- THE DISTRIBUTION FUNCTIONS

$$F_Y(y) = F_X(\phi_\theta^{-1}(y))$$

- THE DENSITY FUNCTIONS

$$f_Y(y) = f_X(\phi_\theta^{-1}(y)) \left| \frac{d}{dy} \phi_\theta^{-1}(y) \right|$$

- HAZARD FUNCTIONS

$$\lambda_Y(y) = \left| \frac{d}{dy} \phi_\theta^{-1}(y) \right| \lambda_X(\phi_\theta^{-1}(y))$$



## DIFFERENT FORMS OF $\phi_\theta$

- LINEAR ACCELERATION FUNCTION

$$\phi_\theta = \theta(s)X; \quad \theta(s) \geq 1.$$

IN THIS CASE

$$\lambda_Y(y) = \frac{1}{\theta(s)}\lambda_X(x).$$

THIS IS A PROPORTIONAL HAZARD MODEL



- POWER TRANSFORM

$$Y = AX^{\theta(s)}$$

WITH SUCH AN ACCELERATION FUNCTION, A WEIBULL( $\alpha, \beta$ ) IS TRANSFORMED TO ANOTHER WEIBULL( $A\alpha^\theta, \frac{\beta}{\theta}$ ).



## LIFETIME DISTRIBUTION UNDER STEP-STRESS

- CUMULATIVE EXPOSURE MODEL

CLASSICAL ASSUMPTION FOR THE ANALYSIS OF STEP-STRESS DATA. THE MAIN IDEA IS TO ASSUME THAT THE REMAINING LIFETIME OF THE SPECIMENS DEPENDS ONLY ON THE CURRENT ACCUMULATED STRESS, REGARDLESS OF HOW IT HAS ACCUMULATED.

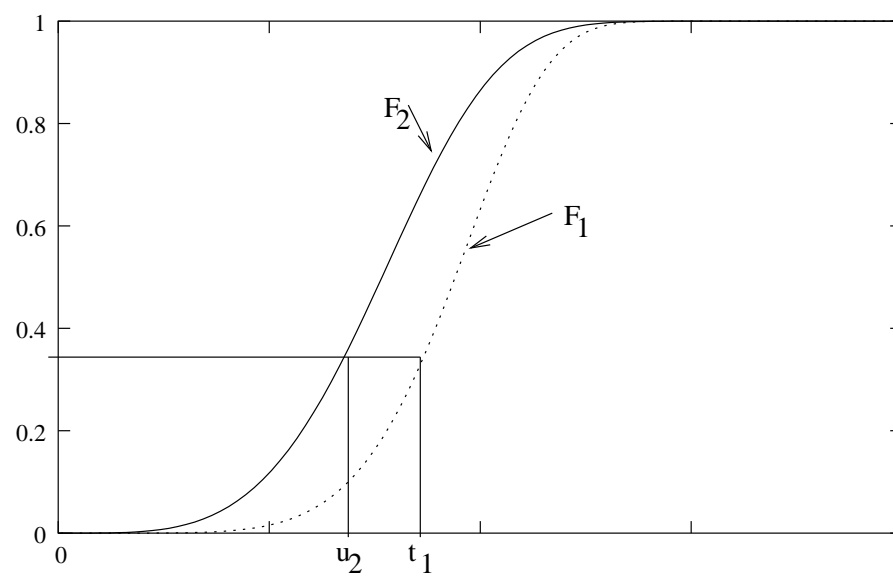


- $F_1$  : THE CUMULATIVE DISTRIBUTION FUNCTION UNDER STRESS  $s_1$
- $F_2$  : THE CUMULATIVE DISTRIBUTION FUNCTION UNDER STRESS  $s_2$
- $G$  : THE CUMULATIVE DISTRIBUTION FUNCTION UNDER A STEP-STRESS PATTERN

$G$  CAN BE OBTAINED FROM  $F_1$  AND  $F_2$  UNDER THE ASSUMPTIONS THAT THE LIFETIME  $F_1$  (UNDER STRESS  $s_1$ ) AT THE TIME  $t_1$  HAS AN EQUIVALENT TIME  $u_2$  OF THE DISTRIBUTION FUNCTION  $F_2$  (UNDER STRESS  $s_2$ ).



# GRAPHICAL REPRESENTATION





## STEP-STRESS: TYPE-I CENSORING

- $[0, \tau_1) \rightarrow$  STRESS IS  $s_1$
- $[\tau_1, \tau_2) \rightarrow$  STRESS IS  $s_2$
- $[\tau_2, \tau_3) \rightarrow$  STRESS IS  $s_2$
- $\downarrow$



- STEP 1:  $G(t) = F_1(t)$ ; for  $0 \leq t \leq \tau_1$

BY ASSUMPTION THERE EXISTS A  $u_1$  SUCH THAT

$$F_1(\tau_1) = F_2(u_1)$$

- STEP 2:  $G(t) = F_2(t - \tau_1 + u_1)$ ; for  $\tau_1 \leq t \leq \tau_2$

BY ASSUMPTION THERE EXISTS A  $u_2$  SUCH THAT

$$F_2(\tau_2 - \tau_1 + u_1) = F_3(u_2)$$

- STEP 3:  $G(t) = F_3(t - \tau_2 + u_2)$ ; for  $\tau_2 \leq t \leq \tau_3$



## STEP STRESS - TYPE II CENSORING

- PUT  $n$  ITEMS ON TEST
- CONTINUE WITH THE STRESS  $s_1$  UNTIL, THE FIRST  $r_1$  ITEMS FAIL
- CHANGE THE STRESS LEVEL TO  $s_2$
- CONTINUE WITH THE STRESS  $s_2$  UNTIL TOTAL  $r_2$  ITEMS FAIL
- CHANGE THE STRESS LEVEL OF  $s_3$  AND SO ON



## STEP-STRESS: TYPE-II CENSORING

IF  $t_{1:n} < \dots < t_{r:n}$  ARE THE FAILURE TIMES

- $[0, t_{r_1:n}) \rightarrow$  STRESS IS  $s_1$
- $[t_{r_1:n}, t_{r_2:n}) \rightarrow$  STRESS IS  $s_2$
- $[t_{r_2:n}, t_{r_3:n}) \rightarrow$  STRESS IS  $s_2$
- $\downarrow$



## DISTRIBUTION FUNCTION UNDER STEP-STRESS WITH TYPE-II CENSORING

- $G(t) = F_1(t)$  IF  $0 \leq t \leq t_{r_1:n}$ .
- $G(t) = F_2(t - t_{r_1:n} + u_1)$  IF  $0 \leq t_{r_1:n} \leq t_{r_2:n}$ .
- $G(t) = F_3(t - t_{r_2:n} + u_2)$  IF  $0 \leq t_{r_2:n} \leq t_{r_3:n}$ .

WHERE  $u_i$  SATISFIES THE FOLLOWING RELATIONS



- $F_1(t_{r_1:n}) = F_2(u_1)$
- $F_2(t_{r_2:n} - t_{r_1:n} + u_1) = F_3(u_2)$
- $F_3(t_{r_3:n} - t_{r_2:n} + u_2) = F_4(u_3)$
- $\downarrow$



## EXPONENTIAL DISTRIBUTION, TYPE-I CENSORING

LET US CONSIDER SIMPLE STEP STRESS MODEL ONLY. THE STRESS CHANGES AT  $\tau$ .

- $F_1$  FOLLOWS EXPONENTIAL WITH MEAN  $\theta_1$
- $F_2$  FOLLOWS EXPONENTIAL WITH MEAN  $\theta_2$

IN THIS CASE  $u = \frac{\theta_2}{\theta_1}\tau$



## THE DISTRIBUTION FUNCTION $G(t)$ UNDER STEP-STRESS CONDITION

- $G(t) = F_1(t) = 1 - e^{-\frac{t}{\theta_1}}$  if  $0 < t < \tau$ .
- $G(t) = F_2\left(\frac{\theta_2}{\theta_1} + t - \tau\right) = 1 - e^{-\frac{\tau}{\theta_2} + \frac{\tau-t}{\theta_2}}$  if  $\tau < t < \infty$ .



SUPPOSE  $n_1$  IS THE NUMBER OF FAILURES BEFORE TIME  $\tau$  AND  $t_{1:n} < \dots < t_{r:n}$  ARE THE OBSERVED FAILURES.

CLEARLY,  $0 \leq n_1 \leq r$ . IF  $1 \leq n_1 \leq r - 1$  THEN THE MLES OF  $\theta_1$  AND  $\theta_2$  EXIST

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n_1} t_{i:n} + (n - n_1)\tau}{n_1}$$

$$\hat{\theta}_2 = \frac{\sum_{i=n_1+1}^r (t_{i:n} - \tau) + (n - r)(t_{r:n} - \tau)}{(r - n_1)}$$



THE PROBLEM OCCURS IN COMPUTING THE DISTRIBUTION OF  $\hat{\theta}_1$  AND  $\hat{\theta}_1$ . XIONG (1998) COMPUTED THE THE ORDER STATISTICS AND USED TO COMPUTE THE DISTRIBUTIONS. HE HAS CLAIMED THE FOLLOWING RESULTS:

- THE MINIMUM ORDER STATISTICS OF  $G(t)$  HAS AN EXPONENTIAL DISTRIBUTION THEREFORE,  $nt_{1:n}$  HAS AN EXPONENTIAL DISTRIBUTION
- HE HAS USED THE SPACINGS OF  $t_{2:n} - t_{1:n}, \dots, t_{n_1:n} - t_{n_1-1:n}$  HAVE INDEPENDENT EXPONENTIAL DISTRIBUTIONS



BOTH ARE NOT CORRECT

- NOTE THAT FOR FIXED  $\tau$ ,  $nt_{1:n}$  HAS BOUNDED SUPPORT AND IT CAN NOT EXCEED  $n\tau$ .
- $t_{2:n} - t_{1:n}, \dots, t_{n_1:n} - t_{n_1-1:n}$  ARE ORDER STATISTICS FROM A RIGHT CENSORED DISTRIBUTION



THE EXACT DENSITY FUNCTIONS OF  $\hat{\theta}_1$  AND  $\hat{\theta}_2$  CAN BE OBTAINED USING THE CONDITIONAL MOMENT GENERATING FUNCTION APPROACH. THEY ARE QUITE COMPLICATED.

- THE DENSITY FUNCTION OF  $\hat{\theta}_1$  CAN BE WRITTEN AS A DOUBLE SUMMATION OF WEIGHTED SHIFTED GAMMA. THE WEIGHT FUNCTIONS MAY BE NEGATIVE ALSO.
- THE DENSITY FUNCTION OF  $\hat{\theta}_1$  IS A MIXTURE OF GAMMA DISTRIBUTION



- ALL THE MOMENTS CAN BE CALCULATED FOR BOTH  $\hat{\theta}_1$  AND  $\hat{\theta}_2$
- $\hat{\theta}_1$  IS BIASED,  $\hat{\theta}_2$  IS UNBIASED.
- CONFIDENCE INTERVALS OF  $\theta_1$  AND  $\theta_2$  CANNOT BE OBTAINED DIRECTLY
- UNDER THE ASSUMPTIONS:  $P_{\theta_1}(\hat{\theta}_1 \geq b)$  AND  $P_{\theta_2}(\hat{\theta}_2 \geq b)$  ARE INCREASING FUNCTIONS OF  $\theta_1$  AND  $\theta_2$  RESPECTIVELY, CONFIDENCE INTERVALS OF  $\theta_1$  AND  $\theta_2$  CAN BE OBTAINED.



## CONFIDENCE INTERVALS

- EXACT CONFIDENCE INTERVALS
- ASYMPTOTIC CONFIDENCE INTERVALS
- BOOTSTRAP CONFIDENCE INTERVALS



## STEP STRESS MODEL UNDER TYPE II CENSORING

THE MLES OF  $\theta_1$  AND  $\theta_2$  ALWAYS EXIST.

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{r_1} t_{i:n} + (n - r_1)\tau}{r_1}$$

$$\hat{\theta}_2 = \frac{\sum_{i=r_1+1}^r (t_{i:n} - t_{r_1:n}) + (n - r_1)(t_{r:n} - t_{n_1:n})}{(r - r_1)}$$



- ALL THE MOMENTS CAN BE CALCULATED FOR BOTH  $\hat{\theta}_1$  AND  $\hat{\theta}_2$
- $\hat{\theta}_1$  AND  $\hat{\theta}_2$  BOTH ARE UNBIASED.
- CONFIDENCE INTERVALS OF  $\theta_1$  AND  $\theta_2$  CAN BE OBTAINED DIRECTLY



# OPTIMAL DESIGN

- TYPE-I CENSORING: OPTIMAL CHOICE OF  $\tau$
- TYPE-II CENSORING: OPTIMAL CHOICE OF  $n_1$



# BAYESIAN ANALYSIS

- TYPE-I CENSORING: NOT MUCH WORK HAS BEEN DONE
- TYPE-II CENSORING: POSSIBLE TO DO



## OPEN PROBLEMS

- OTHER DISTRIBUTIONS
- COMPETING RISKS SET UP
- LINK FUNCTION APPROACH



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THANK YOU

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